MINIMIZATION OF LINEAR GROWTH FUNCTIONALS OF MEASURE

By

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J. Christopher Tweddle
I dedicate this work to my friends and family
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MINIMIZATION OF LINEAR GROWTH FUNCTIONALS OF MEASURE

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There are many applications that involve the minimization of a convex, linear growth function of a measure. For example, image restoration models, Plateau’s problem and deformation of a thin plate (the plasticity problem) involve minimizing such functions. In order to understand the theory of these problems, we must understand how to give meaning $F(\mu)$, where $\mu$ is a vector valued measure and $F$ is a convex function with linear growth.

In this dissertation, we use the space of continuous, bounded functions to define the Fenchel transform of a function of measure. We then show that under this definition, the double Fenchel transform coincides with the definition given by Anzellotti and Giaquinta and used throughout the literature. The lower semi-continuity of the functional $\int F(\mu)$ is a direct result of properties of the Fenchel transform.

We use this formulation to establish a partial regularity result for the elastic-plastic deformation problem. We show that the domain $\Omega$ may be decomposed into an open elastic region $E$ and a closed plastic region $P$. On $E$, the solution $u$ satisfies the related Poisson equation and is regular. We use a decay
estimate to establish the desired regularity on a ball. Finally, we show that the elastic region is nonempty for small load.
CHAPTER 1
INTRODUCTION

There are many application problems involving variational integrals of the form

$$\min \int_{\Omega} F(x, u, Du),$$

(1–1)

for open $\Omega \subset \mathbb{R}^n$, where $u = (u^1(x), \ldots, u^N(x))$ is a vector valued function and $F(x, u, p)$ is convex in $p$. For example, such minimization problems are used in image denoising and edge detection, modeling the deformation of a thin plate and determining a surface of minimal area with prescribed boundary conditions. In fact, Hilbert’s 19th and 20th problems deal with these “regular problems in the calculus of variations,” for $n = 2$ and $N = 1$; see Giaquinta [12] for a comprehensive overview of these types of problems as well as extensive references. In 1912, Bernstein [4] used the calculus of variations method to establish existence and regularity results for the 2-dimensional real-valued Dirichlet problem. Serrin [25] applied similar methods to extend these results for $n$-dimensions. We wish to explore the minimization problem for $u \in BV(\Omega, \mathbb{R}^N)$; in this case $Du$ is a Radon measure and we need give meaning to the variational integral (1–1).

For example, Giusti [17] considers a minimal surface area problem (Plateau’s Problem), where

$$F(p) := \sqrt{1 + |p|^2}.$$ 

(1–2)

In this case,

$$\int_{\Omega} F(Du) \, dx = \int_{\Omega} \sqrt{1 + |Du|^2} \, dx$$

is the area of the surface of the graph of $u$. 

1
Anzellotti and Giaquinta [1, 2], Hardt and Kinderlehrer [18] and Zhou [28] study anti-planar shear of a thin plate (the plasticity problem). Here one seeks solutions to the minimization problem (1–1), where

\[
F(p) := \begin{cases} 
\frac{1}{2\beta^2} |p|^2 & \text{if } |p| \leq \beta, \\
|p| - \frac{1}{2\beta} & \text{if } |p| > \beta,
\end{cases}
\]  

(1–3)

for some threshold \( \beta > 0 \). In this case, \( u : \Omega \to \mathbb{R} \) is the displacement of the plate.

Chambolle and Lions [5] proposed a model to recover an image, \( u \), from an observed noisy image \( I = u + \text{noise} \) by

\[
\min \int_\Omega F(Du) + \int_\Omega \frac{\lambda}{2} (u - I)^2 \, dx,
\]

for \( F(p) \) in (1–3). The diffusion from this minimization model is strictly perpendicular to the gradient when \( |Du| > \beta \), where edges are likely to be present, and isotropic when \( |Du| \leq \beta \). Thus the model preserves edges and eliminates noise.

Additionally, Chen, Levine and Rao [6] considered a function of \( q(x) \) growth, for \( q(x) \geq 1 \), which is used as a model for image denoising, enhancement and restoration. In this paper they proposed a new model for image restoration. The proposed model incorporates the strengths of the various types of diffusion arising from the minimization problem

\[
\min \int_\Omega |Du|^p + \frac{\lambda}{2} (u - I)^2,
\]

for \( 1 \leq p \leq 2 \). In particular, they considered the minimization problem (1–1) with

\[
F(x, p) := \begin{cases} 
\frac{1}{q(x)} |p|^{q(x)} & \text{for } |p| \leq \beta, \\
|p| - \frac{\beta q(x)}{q(x)} & \text{for } |p| > \beta,
\end{cases}
\]  

(1–4)
where $\beta > 0$ is fixed and $1 < \alpha \leq q(x) \leq 2$. One may choose

$$q(x) = 1 + \frac{1}{1 + k |\nabla G_\sigma \ast I(x)|^2},$$

where $G_\sigma(x) = \frac{1}{\sigma} \exp\left(-\frac{|x|^2}{4\sigma^2}\right)$ is the Gaussian filter and $k, \sigma > 0$ are fixed.

In this case, the model utilizes a total variation approach when the gradient is large (thus preserving edges) and $L^2$ smoothing when the gradient is small (thus removing noise). Furthermore, it employs anisotropic diffusion ($1 < p < 2$) in regions which may be piecewise smooth or in which the difference between noise and edges is difficult to distinguish.

Models based on minimization of a convex, linear growth functional of measures have shown promising results in numerical implementation. The development of PDE methods in image analysis is dependent upon answering fundamental mathematical questions. In particular, the meaning of such a functional of a measure and its first variation is not trivial to establish or understand. While a definition for a convex, linear growth functional has been given by Anzellotti and Giaquinta [1] and Giaquinta, Modica and Souček [14], we explore the motivation for this formulation and its relation to previous results in convex analysis.

The functional

$$\int_\Omega F(Du)$$

is well defined on the Sobolev space $W^{1,1}(\Omega, \mathbb{R}^N)$. However, it has been shown that the minimization problem for plasticity ($F$ as in (1–3)), [1, 18, 28], and for the image processing problem ($F$ as in (1–4)) [6] has solutions $u \in BV(\Omega)$. This is consistent with our intuition, as both models should allow for discontinuities (where there is shear or edges, respectively). Since the solution is $BV$, its derivative, $Du$, is a Radon measure. Thus we need to understand the meaning of (1–5) in such a case. Indeed, the study of existence and partial regularity of solutions to these
problems depends on how one defines
\[ \int_\Omega F(x, u, Du), \]
for \( u \in BV(\Omega) \). Anzellotti and Giaquinta [2] derive a formula for the special case of the plasticity problem, with \( F \) as in (1–3). We take a different approach; we use the double Fenchel transform to get a formula for the variational integral
\[ \int_\Omega F(Du), \]
for a general convex function \( F \) with linear growth. Our approach has the advantage of “naturalness” in the context of convex analysis. Additionally, we establish some interesting properties of the double Fenchel transform.

We wish to establish some regularity results for solutions to functionals with linear growth. In general, the minimum of a convex, linear growth functional may have singularities, even if the integrand is smooth [12, 13, 14, 16]. In applications, numerical results have shown the effectiveness of various convex, linear growth functionals in feature-preserving image processing. Images restored with these models are smoother in regions where the gradient is small and the edges correspond to places where the gradient is large—a desirable property for image denoising.

In particular, we shall focus on a problem that arises from anti-planar shear for elastic-plastic materials,
\[ \inf_{v \in A} \left\{ \int_\Omega F(Dv) - \int_\Omega fv \right\}, \quad (1–6) \]
where \( v: \Omega \subset \mathbb{R}^n \to \mathbb{R} \), the vertical displacement of material, belongs to an appropriate set \( A \) and \( f \) is a given external force. \( F \) is defined in (1–3). Note that \( F \) is a \( C^1 \) convex function with linear growth. The minimization problem is taken over the function space \( BV(\Omega) \), with either Dirichlet or Neumann boundary conditions.
From numerical results in image processing (or deformation of a thin plate), we see that the solution appears to be smooth when the gradient is small but not when the gradient is large. Our goal is to verify this observation theoretically. That is to say, suppose $u$ is a minimizer of the functional \((1-6)\). We wish to decompose $\Omega$ into sets $E$ and $P$, where $E$ is the elastic region, defined $L^n$-a.e. by

$$E = \{ x \in \Omega : |\nabla u(x)| < 1 \}$$

and $P$ is the plastic region

$$P = \{ x \in \Omega : |\nabla u(x)| \geq 1 \}.$$

We expect $u$ to be free of discontinuity and regular in some sense on $E$, but not so on $P$. Furthermore, we note that it is not evident that we can remove a set $B$ of Lebesgue measure zero so that the set $E \setminus B$ is open, nor is it clear that $u$ has no singular measure on $E$. However, if $E$ is open, and $u$ is free of singular part on $E$, it follows from the standard theory that $-\Delta u = f$ on $E$. However, on $P$, the minimizer $u$ may have discontinuities, or behave like a Cantor function, even if it is continuous on $P$ (see Hardt and Kinderlehrer \([18]\) for simple one-dimensional examples).

We have followed the scheme set forth in an unpublished work by Tonegawa \([26]\). While the general outline of the proof follows this work, we have provided many missing details. In particular, we have provided a detailed proof for the estimates of the Hölder norm of a solution to an auxiliary problem, established the first variation formula, and filled in details for the proofs of the main propositions. Additionally, we have corrected a mistake in the handling of an estimate involving the integral on the boundary by using the trace theorem for $BV$ functions.
CHAPTER 2
CONVEX FUNCTIONS OF A MEASURE

2.1 Introduction.

We explore the meaning of $\int_{\Omega} F(m)$, for a bounded $\mathbb{R}^n$-valued measure. To such ends, Giaquinta, Modica and Souček [14] defined a function

$$\bar{F}(x, p_0, p) := F \left( x, \frac{p}{p_0} \right) p_0,$$

where $x \in \Omega$, $p_0 > 0$ and $p \in \mathbb{R}^n$, and remarked that $\bar{F}$ is continuous on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^n$, convex in $(p_0, p)$ and homogeneous of degree 1 in $(p_0, p)$. They then proved that

$$\lim_{p_0 \to 0^+} \bar{F}(x, p_0, p)$$

exists under appropriate conditions (given below). Let $u \in BV(\Omega)$ and choose a positive Radon measure $\mu$ so that the total variation, $|Du|$, and the Lebesgue measure, $\mathcal{L}^n$, are absolutely continuous with respect to $\mu$. Denote the Radon-Nikodym derivatives of $\mathcal{L}^n$ and the vector-valued measure $Du$ with respect to $\mu$ by

$$\frac{d\mathcal{L}^n}{d\mu}$$

and

$$\frac{dDu}{d\mu},$$

respectively. They defined

$$\int_{\Omega} F(x, Du) := \int_{\Omega} \bar{F} \left( x, \frac{d\mathcal{L}^n}{d\mu}, \frac{dDu}{d\mu} \right) d\mu.$$

By the homogeneity of $\bar{F}$, it follows that this definition is independent of our choice of $\mu$. The authors then applied a result of Reshetnyak [22] to this definition to establish the lower semi-continuity of the integral.
Anzellotti and Giaquinta [1] applied this definition to the plasticity problem with $F$ as in (1–3) with $\beta = 1$. Here, the authors defined a function on $\mathbb{R} \times \mathbb{R}^n$ by

$$\bar{F}(t, p) := \begin{cases} F \left( \frac{p}{t} \right) t & t > 0, \\ \lim_{t \downarrow 0} F \left( \frac{p}{t} \right) t & t = 0. \end{cases}$$

For any $\mathbb{R}^n$-valued measure $m$, consider the $\mathbb{R}^{n+1}$-valued measure $\alpha = (\alpha_0, m)$, where $\alpha_0 = \mathcal{L}^n$ is the Lebesgue measure in $\Omega$. The authors then defined

$$\int_{\Omega} F(m) := \int_{\Omega} \bar{F} \left( \frac{d\alpha_0}{d|\alpha|}, \frac{dm}{d|\alpha|} \right) d|\alpha|,$$

(2–1)

where $|\alpha|$ is the total variation of $\alpha$, and $\frac{d\alpha_0}{d|\alpha|}$ and $\frac{dm}{d|\alpha|}$ are the Radon-Nikodym derivatives. In Lemma 1.1 [2], the authors used (2–1) to show that

$$\int_{\Omega} F(Du) = \int_{\Omega} F(\nabla u) dx + \int_{\Omega} |D^s u|,$$

(2–2)

for $u \in BV(\Omega)$. Here $Du$ is decomposed into its absolutely continuous and singular parts with respect to Lebesgue measure, i.e.,

$$Du = \nabla u dx + D^s u.$$

To see that (2–2) follows from (2–1), we decompose $\Omega$ with respect to the measures $\nabla u dx$ and $D^s u$. That is to say, since the measures are mutually singular, there exists a set $A \subset \Omega$ on which $\nabla u$ is not zero and $D^s u$ is identically zero. Thus on $A$, $Du = \nabla u dx$ and on the complement, $Du = D^s u$. Splitting the integral in (2–1) into the sum of integrals over $A$ and $\Omega \setminus A$ gives (2–2).

In a more general setting, Anzellotti and Giaquinta [3] provided a unified approach to the partial regularity to solutions of the minimization problem (1–1) for a general convex function $F$ with growth $m \geq 1$; that is to say, there are positive constants $\alpha$ and $\beta$ so that

$$\alpha |p|^m \leq F(x, p) \leq \beta (1 + |p|^m).$$

(2–3)
In this case, the authors defined

$$\int_{\Omega} F(Du) = \int_{\Omega} F(\nabla u) \, dx + \int_{\Omega} F^\infty \left( \frac{D^s u}{|D^s u|} \right) |D^s u|, \quad (2-4)$$

where

$$F^\infty(p) := \lim_{t \to \infty} \frac{1}{t} F(tp).$$

Notice that this agrees with the definition taken above for the plasticity problem, as $F^\infty(p) = |p|$ in that case. In fact, one may use (2–1) and these techniques to establish (2–4) in the general case. The definition (2–4) is used by Demengel, Hardt, Kinderlehrer, Temam, Tonegawa and Zhou among others throughout the literature in the study of existence and partial regularity of solutions to the minimization problem with $F$ as in (1–3) [8, 18, 19, 20, 21, 28].

While the definition (2–4) is sufficient for the study of functionals of this type, there are still unsettled questions. In particular, what motivates this definition, how does it relate to previous results and what is the relation to convex analysis? The purpose of this dissertation is to take a very different route to the definition of $F(m)$ where $F$ is a convex function on $\mathbb{R}^n$ and $m$ is a vector measure. Our approach is to use Fenchel transforms to define $\int_{\Omega} F(m)$ in (2–4). Briefly, given a convex function $F$ on $\mathbb{R}^n$, consider the convex functional on $L^1(\Omega, \mathbb{R}^n)$ defined by

$$f \mapsto \int_{\Omega} F(f) \, dx.$$ We wish to extend this functional to the space of bounded vector-valued measures, $\mathcal{M}$. Additionally, we would like this extension to be lower semi-continuous on $\mathcal{M}$ in the topology induced by the space of bounded, $C(\Omega, \mathbb{R}^n)$ functions, denoted by $C_B$. The Fenchel transform (or conjugate) of $F$ on $C_B$ is defined by

$$F^*(\phi) := \sup_{f \in L^1} \int_{\Omega} (f \cdot \phi - F(f)) \, dx.$$
Since bounded vector-valued Radon measures are linear maps on $C_B$, we can naturally repeat this procedure on the space of bounded vector measures $\mathcal{M}$, and define

$$F^{**}(m) := \sup_{\phi \in C_B} \int_{\Omega} \phi \, dm - F^*(\phi).$$

Since $L^1(\Omega, \mathbb{R}^n)$ is a subspace of $\mathcal{M}$, $F^{**}$ is an extension of $F$. In this dissertation we prove that the double Fenchel transform, $F^{**}(m)$ thus defined, is indeed given by the formula

$$F^{**}(m) = \int_{\Omega} F(m^a) \, dx + \int_{\Omega} F^\infty \left( \frac{dm^s}{d|m^s|} \right) \, d|m^s|.$$ 

This result justifies and shows the “naturalness” of the definition of $\int_{\Omega} F(m)$ in (2–4) in the context of convex analysis. One immediate consequence is the lower semi-continuity of this functional. Additionally, this technique may be applied to define convex functionals of objects more general than measures (e.g., certain types of operators). Along the way we also find very interesting properties of $F^{**}$. Just to name one:

$$F^{**}(m + n) = F^{**}(m) + F^{**}(n),$$

whenever the vector measures $m$ and $n$ are mutually singular.

2.2 Notation and Preliminaries.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Let $f \in L^1(\Omega, \mathbb{R}^n)$ and let $F: \mathbb{R}^n \to \mathbb{R}$ be a continuous, non-negative, convex function with $F(0) = 0$, satisfying the linear growth condition

$$\alpha |p| - \gamma \leq F(p) \leq \beta |p| + \gamma,$$  \hspace{1cm} (2–5)

for some $\alpha, \beta, \gamma > 0$. Denote

$$\mathcal{M} = \{ \text{bounded } \mathbb{R}^n\text{-valued measures on } \Omega \}. $$
For any $m \in \mathcal{M}$, let $|m|$ denote the total variation. We may decompose $m$ into its absolutely continuous and singular parts with respect to Lebesgue measure:

$$m = m^a + m^s.$$ 

Denote

$$C_B = \{ \phi \in C(\Omega, \mathbb{R}^n) : \phi \text{ is bounded} \}.$$ 

Endow $L^1$ with the coarsest topology such that the mapping

$$L_\phi(f) = \int_\Omega f \cdot \phi \, dx$$

is continuous for all $\phi \in C_B$. Note that this topology separates points in $L^1$ as $\int f \cdot \phi = \int g \cdot \phi$, for all $\phi \in C_B$ if and only if $f = g$ $\mathcal{L}^n$-a.e. With this topology, $L^1(\Omega, \mathbb{R}^n)$ becomes a locally convex, Hausdorff, topological vector space. $(L^1(\Omega, \mathbb{R}^n), C_B)$ is a dual pair for which we may define the Fenchel transform [7].

Suppose $F$ satisfies the conditions above. If $f \in L^1(\Omega, \mathbb{R}^n)$, the linear growth condition on $F$ gives

$$\int_\Omega (\alpha |f| - \gamma) \, dx \leq \int_\Omega F(f) \, dx \leq \int_\Omega (\beta |f| + \gamma) \, dx;$$

i.e., $F \in L^1$. The Fenchel transform, $F^*$, of $F$ is defined on $C_B$ by

$$F^*(\phi) := \sup_{f \in L^1} \int_\Omega f \cdot \phi \, dx - \int_\Omega F(f).$$

Observe that for fixed $f \in L^1$, the mapping $L_\phi$ is a continuous, affine map. Hence $F^*(\phi)$ is convex and lower semi-continuous on $C_B$. Thus the desired lower semi-continuity follows automatically from the Fenchel transform approach. See van Tiel [27] and Ekeland and Temam [9] for more on the general theory of Fenchel transforms.

We observe that

**Proposition 2.1.** Let $\phi \in C_B$. If $\|\phi\|_\infty > 1$, then $F^*(\phi) = \infty$. 
Proof. Suppose that $\|\phi\|_\infty > 1$ and let $f = |\phi|^{k-2} \phi$, for $k \geq 2$. Since $\Omega$ is bounded, we have $f \in L^1$. Then

$$
\int (f \cdot \phi - F(f)) \, dx = \int \left(|\phi|^k - F(\phi |\phi|^{k-2})\right) \, dx \\
\geq \int |\phi|^k - (\beta |\phi|^{k-1} + \gamma) \, dx \to \infty
$$
as $k \to \infty$.

For $m \in \mathcal{M}$, define the double Fenchel transform

$$
F^{**}(m) := \sup_{\phi \in C_B} \int_\Omega \phi \, dm - F^*(\phi).
$$

(2–7)

By Proposition 2.1, we may define

$$
F^{**}(m) = \sup_{\phi \in C_B \atop |\phi| \leq 1} \int_\Omega \phi \, dm - F^*(\phi).
$$

(2–8)

If we endow $C_B$ with the uniform norm, then $\mathcal{M} \subset C^*_B$. The coarsest topology on $\mathcal{M}$ so that the mapping $m \mapsto \int \phi \, dm$ is continuous is called the weak topology on $\mathcal{M}$. By (2–7), $F^{**}(m)$ is lower semi-continuous in the weak topology and convex on $\mathcal{M}$.

For each $f \in L^1$, $f \, dx \in \mathcal{M}$. It is easy to see

**Proposition 2.2.** Let $f \in L^1(\Omega, \mathbb{R}^n)$. Then

$$
F^{**}(f \, dx) \leq \int_\Omega F(f) \, dx.
$$

Proof. By (2–8),

$$
F^{**}(f \, dx) = \sup_{\phi \in C_B \atop |\phi| \leq 1} \int_\Omega \phi \cdot f \, dx - F^*(\phi),
$$

where $F^*(\phi) = \sup_{f \in L^1} \int (f \cdot \phi - F(f)) \, dx$. For any $f \in L^1$ and every $\phi \in C_B$, we have

$$
F^*(\phi) \geq \int (f \cdot \phi - F(f)) \, dx,
$$
so
\[ \int F(f) \, dx \geq \int f \cdot \phi \, dx - F^*(\phi). \]
Thus taking the supremum, we have
\[ \int F(f) \, dx \geq \sup_{\phi \in \mathcal{C}_B, \|\phi\| \leq 1} \int f \cdot \phi \, dx - F^*(\phi) = F^{**}(f \, dx). \]

Using the duality we started with, we can show that if \( F \) is convex and lower semi-continuous, then
\[ F^{**}(f \, dx) = \int_{\Omega} F(f) \, dx, \]
for all \( f \in L^1 \).

Our goal is to derive an explicit formula for the double Fenchel transform \( F^{**} \). This we hope will justify the definition given in (2-4).

### 2.3 Main Result.

We wish to show

**Theorem 2.1.** Let \( m \in \mathcal{M} \) be a vector measure. Decompose \( m \) into its absolutely continuous part, \( m^a \), and its singular part, \( m^s \), with respect to Lebesgue measure. Then
\[ F^{**}(m) = \int_{\Omega} F(m^a) \, dx + \int_{\Omega} F^{\infty} \left( \frac{dm^s}{d|m^s|} \right) \, d|m^s|, \]
where
\[ F^{\infty}(p) = \lim_{t \to \infty} \frac{1}{t} F(tp). \]

Note that since \( F \) is convex and \( F(0) = 0 \), the limit in the definition of \( F^{\infty} \) exists.

**Proposition 2.3.** Let \( m, n \in \mathcal{M} \) be mutually singular, denoted \( m \perp n \). Then
\[ F^{**}(m + n) = F^{**}(m) + F^{**}(n). \]
We present the proof as a series of claims:

**Claim** (#1). $F^{**}(m)$ is lower semi-continuous in the sense that if $m_k \rightharpoonup m$ weakly in $\mathcal{M}$ (i.e., $\int_\Omega \phi dm_k \to \int_\Omega \phi dm$, for all $\phi \in C_B$), then $F^{**}(m) \leq \liminf F^{**}(m_k)$.

**Proof.** For fixed $\phi \in C_B$, the map $m \mapsto \int_\Omega \phi dm$ is continuous. Since $F^{**}(m)$ is defined to be the supremum of a family of such maps (for $|\phi| \leq 1$), we conclude that $F^{**}$ is lower semi-continuous. □

**Claim** (#2). For any set $K \subset \Omega$, we have

$$F^{**}(m 1_K) \leq F^{**}(m) + |m 1_{K^c}|(\Omega) = F^{**}(m) + |m|(K^c),$$

where we denote $m 1_K = m|_K$.

**Proof.** Indeed, let $\langle m, \phi \rangle$ denote the pairing $\int_\Omega \phi dm$. We have

$$\langle m, \phi \rangle = \langle m 1_K + m 1_{K^c}, \phi \rangle = \langle m 1_K, \phi \rangle + \langle m 1_{K^c}, \phi \rangle.$$ 

Therefore, for $\phi \in C_B$ with $|\phi| \leq 1$, we have

$$\langle m 1_K, \phi \rangle - F^*(\phi) = \langle m, \phi \rangle - F^*(\phi) - \langle m 1_{K^c}, \phi \rangle$$

$$\leq \langle m, \phi \rangle - F^*(\phi) + |m 1_{K^c}|(\Omega),$$

as $|\phi| \leq 1$. The claim follows immediately from (2–8). □

**Claim** (#3). Let $\phi \in C_B$. For $f \in L^1$, denote

$$A = A(f) := \{x \in \Omega : \phi \cdot f \neq 0\}.$$ 

Then

$$F^*(\phi) := \sup_{f \in L^1} \int_\Omega \phi \cdot f - F(f) \, dx$$

$$= \sup_{f \in L^1} \int_\Omega \phi \cdot (f 1_{A(f)}) - F(f 1_{A(f)}) \, dx.$$
Proof. Indeed, since \(F(0) = 0\) and \(F(f) \geq 0\), we see that for \(x \in A(f)\) we have \(F(f) = F(f1_{A(f)})\) and for \(x \not\in A(f)\) we have \(F(f1_{A(f)}) = F(0) = 0\). Thus \(F(f) \geq F(f1_{A(f)})\). Therefore,

\[
\int_{\Omega} \phi \cdot f - F(f) \, dx \leq \int_{\Omega} \phi \cdot (f1_{A(f)}) - F(f1_{A(f)}) \, dx.
\]

Thus

\[
F^*(\phi) = \sup_{f \in L^1} \int_{\Omega} \phi \cdot f - F(f) \, dx \leq \sup_{f \in L^1} \int_{\Omega} \phi \cdot (f1_{A(f)}) - F(f1_{A(f)}) \, dx.
\]

On the other hand,

\[
\sup_{f \in L^1} \int_{\Omega} \phi \cdot (f1_{A}) - F(f1_{A}) \, dx \leq \sup_{f \in L^1} \int_{\Omega} f \cdot \phi - F(f) \, dx
\]

\[
\leq \sup_{f \in L^1} \int_{\Omega} f \cdot \phi - F(f) \, dx
\]

\[
= F^*(\phi),
\]

where \(L^1_A = \{f \in L^1 : \text{spt}(f) \subset A\}\).

Claim (4). Let \(\phi_1, \phi_2 \in C_B\). If \(|\phi_1| ||\phi_2| = 0\), then

\[
F^*(\phi_1 + \phi_2) = F^*(\phi_1) + F^*(\phi_2).
\]

Proof. Let

\[
A = A(f) := \{x \in \Omega : \phi_1 \cdot f \neq 0\} \quad \text{and} \quad B = B(f) := \{x \in \Omega : \phi_2 \cdot f \neq 0\}.
\]

Then for any \(x \in \Omega\), we have

\[
|(\phi_1 \cdot f)(\phi_2 \cdot f)(x)| \leq |\phi_1| |\phi_2| |f|^2 = 0.
\]

Hence, for each \(x \in \Omega\) we cannot have both \((\phi_1 \cdot f)(x) \neq 0\) and \((\phi_2 \cdot f)(x) \neq 0\). Therefore, \(A(f) \cap B(f) = \emptyset\), for any \(f \in L^1\).
Since $A(f)$ and $B(f)$ are disjoint for each $f \in L^1$, we have by claim 3 that

$$F^*(\phi_1) + F^*(\phi_2) = \sup_{f \in L^1} \int_\Omega \phi_1 \cdot (f1_A) - F(f1_A) \, dx$$

$$+ \sup_{f \in L^1} \int_\Omega \phi_2 \cdot (f1_B) - F(f1_B) \, dx$$

$$= \sup_{f \in L^1} \left[ \int_\Omega \phi_1 \cdot (f1_A) - F(f1_A) \, dx \right.$$

$$+ \int_\Omega \phi_2 \cdot (f1_B) - F(f1_B) \, dx \left. \right]$$

$$= \sup_{f \in L^1} \int_\Omega \phi_1 \cdot (f1_A) + \phi_2 \cdot (f1_B)$$

$$- (F(f1_A) + F(f1_B)) \, dx$$

$$= \sup_{f \in L^1} \int_\Omega (\phi_1 + \phi_2) \cdot (f1_{A\cup B}) - F(f1_{A\cup B}) \, dx$$

$$= F^*(\phi_1 + \phi_2),$$

as $\phi_1 \cdot (f1_{A\cup B}) = \phi_1 \cdot (f1_A + f1_B) = \phi_1 \cdot (f1_A) + \phi_1 \cdot (f1_B) = \phi_1 \cdot (f1_A)$. Similarly, $\phi_2 \cdot (f1_{A\cup B}) = \phi_2 \cdot (f1_B)$. Also note that since $A \cap B = \emptyset$ and $F(0) = 0$, that $F(f1_A) + F(f1_B) = F(f1_{A\cup B})$. Therefore, $F^*(\phi_1 + \phi_2) = F^*(\phi_1) + F^*(\phi_2)$ as desired.

\[ \square \]

**Claim** (#5). Let $\phi \in C_B$. Suppose that $\rho \in C(\Omega, \mathbb{R})$ such that $0 \leq \rho \leq 1$. Then

$$F^*(\rho \phi) \leq F^*(\phi).$$

**Proof.** We have

$$F^*(\phi) = \sup_{f \in L^1} \int_\Omega \phi \cdot f - F(f) \, dx$$

$$\geq \sup_{g=\rho f \in L^1} \int_\Omega \phi \cdot g - F(g) \, dx$$

$$= \sup_{f \in L^1} \int_\Omega \phi \cdot (\rho f) - F(\rho f) \, dx$$
\[ \geq \sup_{f \in L^1} \int_{\Omega} (\rho \phi) \cdot f - F(f) \, dx \]
\[ = F^*(\rho \phi). \]

For the second inequality, we note that for \( 0 \leq \alpha \leq 1 \), the convexity of \( F \) gives

\[ F(\alpha x) \leq (1 - \alpha) F(0) + \alpha F(x) \]
\[ = 0 + \alpha F(x) \]
\[ \leq F(x). \]

\[ \square \]

**Claim** (#6). Suppose there exists a compact set \( K \) such that \( |m|(K^c) = 0 \). Let \( U \) be an open set such that \( K \subset U \). Then

\[ F^{**}(m) = \sup_{|\phi| \leq 1, \text{spt}(\phi) \subset U} \int_{\Omega} \phi \, dm - F^*(\phi). \]

**Proof.** From (2–8) we have

\[ F^{**}(m) := \sup_{|\phi| \leq 1} \int_{\Omega} \phi \, dm - F^*(\phi) \]
\[ \geq \sup_{|\phi| \leq 1} \int_{\text{spt}(\phi) \subset U} \phi \, dm - F^*(\phi). \]

On the other hand, let \( \rho \in C(\Omega, \mathbb{R}) \) such that \( 0 \leq \rho \leq 1 \), \( \rho \equiv 1 \) on \( K \) and \( \rho = 0 \) on \( U^c \). Then from claim 5, we have

\[ \int_{\Omega} \phi \, dm - F^*(\phi) = \int_{\Omega} (\phi 1_K + \phi 1_{K^c}) \, dm - F^*(\phi) \]
\[ \leq \int_{\Omega} (\phi 1_K + \phi 1_{K^c}) \, dm - F^*(\rho \phi) \]
\[ = \int_{\Omega} \phi 1_K \, dm - F^*(\rho \phi) \]
\[ = \int_{\Omega} \rho \phi \, dm - F^*(\rho \phi), \]
as \(|m|(K^c) = 0\). Therefore, we have

\[
F^{**}(m) = \sup_{|\phi| \leq 1} \int_{\Omega} \phi \, dm - F^*(\phi)
\]

\[
\leq \sup_{|\phi| \leq 1} \int_{\Omega} \rho \phi \, dm - F^*(\rho \phi)
\]

\[
\leq \sup_{|\phi| \leq 1} \int_{\Omega} \phi \, dm - F^*(\phi),
\]

as \(\{\rho \phi : \phi \in C_B\} \subset \{\psi \in C_B : \text{spt}(\psi) \subset U\}\). Thus the claim holds. Observe that from the proof, we may also write

\[
F^{**}(m) = \sup_{|\phi| \leq 1} \int_{\Omega} \rho \phi - F^*(\rho \phi).
\]

\[
\square
\]

**Claim** (#7). Let \(m, n \in M\). Suppose there exist disjoint, compact sets \(K\) and \(L\) so that \(|m|(K^c) = 0\) and \(|n|(L^c) = 0\). Then

\[
F^{**}(m + n) = F^{**}(m) + F^{**}(n).
\]

**Proof.** Let \(U\) and \(V\) be disjoint, open sets such that \(K \subset U\) and \(L \subset V\). Let \(\rho_1, \rho_2 \in C(\Omega, \mathbb{R})\) such that \(0 \leq \rho_1, \rho_2 \leq 1\) with \(\rho_1 \equiv 1\) on \(K\) and \(\rho_1 = 0\) on \(U^c\), and \(\rho_2 \equiv 1\) on \(L\) and \(\rho_2 = 0\) on \(V^c\). By claims 4, 5 and 6, we have

\[
F^{**}(m) + F^{**}(n) = \sup_{|\phi| \leq 1} \left[ \int_{\Omega} \phi \, dm - F^*(\phi) \right]
\]

\[
+ \sup_{|\phi| \leq 1} \left[ \int_{\Omega} \phi \, dn - F^*(\phi) \right]
\]

\[
= \sup_{|\phi| \leq 1} \left[ \int_{\Omega} \rho_1 \phi \, dm - F^*(\rho_1 \phi) \right]
\]

\[
+ \sup_{|\phi| \leq 1} \left[ \int_{\Omega} \rho_2 \phi \, dn - F^*(\rho_2 \phi) \right]
\]
\[
\begin{align*}
= \sup_{|\phi| \leq 1} \left[ \int_{\Omega} \rho_1 \phi \, dm + \int_{\Omega} \rho_2 \phi \, dn \right. \\
- \left( F^*(\rho_1 \phi) + F^*(\rho_2 \phi) \right) \\
= \sup_{|\phi| \leq 1} \left[ \int_{\Omega} \rho_1 \phi \, dm + \int_{\Omega} \rho_2 \phi \, dn - (F^*(\rho_1 \phi + \rho_2 \phi)) \right] \\
= \sup_{|\phi| \leq 1} \left[ \int_{\Omega} (\rho_1 + \rho_2) \phi \, dm + \rho_2 \phi \, dn - (F^*((\rho_1 + \rho_2)\phi)) \right] \\
= \sup_{spt(\phi) \subset U \cup V} \left[ \int_{\Omega} \phi \, d(m + n) - F^*(\phi) \right] \\
= F^{**}(m + n),
\end{align*}
\]

as desired. \qed

**Claim (\#8).** Suppose now that \( m, n \in \mathcal{M} \) are mutually singular. Then

\[ F^{**}(m + n) = F^{**}(m) + F^{**}(n). \]

**Proof.** For each \( j \in \mathbb{N} \), we can choose compact disjoint sets \( K_j \) and \( L_j \) such that \( |m| (L_j) = 0, \ |n| (K_j) = 0, \ |m| (K_j^c) \to 0, \ |n| (L_j^c) \to 0 \) with \( m_{1K_j} \to m \) and \( n_{1L_j} \to n \), weakly in the sense of measure. By lower semi-continuity, we have

\[ F^{**}(m) \leq \liminf F^{**}(m_{1K_j}). \]

On the other hand, from claim 2 we see that

\[ \lim F^{**}(m_{1K_j}) \leq \lim \left[ F^{**}(m) + |m| (K_j^c) \right] \]

\[ = F^{**}(m). \]

Whence, \( F^{**}(m) = \lim F^{**}(m_{1K_j}) \). Similarly, \( F^{**}(n) = \lim F^{**}(n_{1L_j}) \). Furthermore, since \( m_{1K_j} \to m \) and \( n_{1L_j} \to n \), lower semi-continuity gives

\[ F^{**}(m + n) \leq \liminf F^{**}(m_{1K_j} + n_{1L_j}). \]
Conversely, by claim 2 we have

\[ F^{**}(m1_{K_j} + n1_{L_j}) = F^{**}((m + n)1_{K_j \cup L_j}) \]
\[ \leq F^{**}(m + n) + |m + n|( (K_j \cup L_j)^c) \]
\[ \leq F^{**}(m + n) + |m|((K_j \cup L_j)^c) + |n|((K_j \cup L_j)^c) . \]

Letting \( j \to \infty \), we have
\[ \lim_{j \to \infty} F^{**}(m1_{K_j} + n1_{L_j}) \leq F^{**}(m + n). \]

Thus \( F^{**}(m + n) = \lim F^{**}(m1_{K_j} + n1_{L_j}). \)

Therefore, by claim 7 we have

\[ F^{**}(m + n) = \lim F^{**}(m1_{K_j} + n1_{L_j}) \]
\[ = \lim [F^{**}(m1_{K_j}) + F^{**}(n1_{L_j})] \]
\[ = \lim F^{**}(m1_{K_j}) + \lim F^{**}(n1_{L_j}) \]
\[ = F^{**}(m) + F^{**}(n) \]

as desired. Thus Proposition 2.3 is proved.

Proposition 2.4.

Let \( m \in M \). Then
\[ F^{**}(m) \leq \int_{\Omega} F^{\infty} \left( \frac{dm}{d|m|} \right) d|m|. \]
Proof. Fix \( x_0 \in \Omega \) and \( \alpha > 0 \) and let \( \sigma : \Omega \to S^{n-1} \) be measurable. Consider the measure \( \alpha \sigma(x_0) \delta_{x_0} = m \), where \( \delta_{x_0} \) is the Dirac delta function. Then \( |m| = \alpha \delta_{x_0} \) and \( \frac{d\alpha}{|m|} = \sigma(x_0) \).

Let \( |B(x, r)| \) denote the Lebesgue measure of the ball of radius \( r \) centered at \( x \).

Then we have

\[
\alpha \frac{1_{B(x_0, r)}}{|B(x_0, r)|} \sigma(x_0) \, dx \rightharpoonup \alpha \sigma(x_0) \delta_{x_0} = m
\]

weakly in the sense of measures as \( r \to 0 \). Denote \( B = B(x_0, r) \). From claim 1 of Proposition 2.3 and Proposition 2.2, we have

\[
F^{**}(m) \leq \liminf_{r \to 0} F^{**} \left( \alpha \sigma(x_0) \frac{1_B}{|B|} \right) dx
\]

\[
\leq \liminf_{r \to 0} \int_{B(x_0, r)} F \left( \alpha \sigma(x_0) \frac{1_B}{|B|} \right) \, dx
\]

\[
= \liminf_{r \to 0} \int_B F \left( \alpha \sigma(x_0) \frac{1_B}{|B|} \right) \, dx
\]

\[
= \liminf_{r \to 0} F \left( \alpha \sigma(x_0) \frac{1}{|B|} \right) |B|
\]

\[
= F^{\infty}(\alpha \sigma(x_0)),
\]

by the definition of \( F^{\infty} \). Moreover, for \( \alpha > 0 \), we see that

\[
F^{\infty}(\alpha \sigma(x_0)) := \lim_{t \to \infty} \frac{1}{t} F(t \alpha \sigma(x_0))
\]

\[
= \lim_{t \to \infty} \frac{\alpha}{t \alpha} F(t \alpha \sigma(x_0))
\]

\[
= \alpha F^{\infty}(\sigma(x_0))
\]

\[
= \alpha \int_{\Omega} F^{\infty}(\sigma(x)) \, d\delta_{x_0}
\]

\[
= \int_{\Omega} F^{\infty}(\sigma(x)) \, d|m|.
\]

Thus we have

\[
F^{**}(m) \leq \int_{\Omega} F^{\infty}(\sigma(x)) \, d|m|.
\]

(2–11)
Now let $x_1, x_2, \ldots, x_k$ be a finite set of distinct points in $\Omega$ and $\alpha_1, \alpha_2, \ldots, \alpha_k > 0$. For each $i$, denote $\sigma(x_i) = \sigma_i \in S^{n-1}$. Denote the measure $\alpha_i \sigma_i \delta_{x_i} = m_i$. We will refer to a measure of the form

$$m = \sum_{i=1}^{k} \alpha_i \sigma_i(x_i) \delta_{x_i} = \sum_{i=1}^{k} m_i$$

as a simple measure. Observe that $m_i \perp m_j$ for $i \neq j$. Thus by Proposition 2.3 and (2–11), we have

$$F^{**}(m) = \sum_{i=1}^{k} F^{**}(m_i) \leq \sum_{i=1}^{k} \int_{\Omega} F^\infty(\sigma_i) \, d|m_i| \leq \int_{\Omega} F^\infty \left( \frac{dm}{d|m|} \right) \, d|m|.$$ 

For the last equality, we need the fact that $d|m_i| = \alpha_i \, d\delta_{x_i}$ is a weighted point mass, for distinct $x_i$. In such a case, we have the desired additivity. Hence for a simple measure $m$, we have shown

$$F^{**}(m) \leq \int_{\Omega} F^\infty \left( \frac{dm}{d|m|} \right) \, d|m|.$$ 

We now extend this to a general measure:

**Claim.** For any $m \in \mathcal{M}$, there is a sequence $\{m_i\}$ of simple measures such that $m_i \to m$ weakly and

$$\int_{\Omega} F^\infty \left( \frac{dm}{d|m|} \right) \, d|m| = \lim_{i \to \infty} \int_{\Omega} F^\infty \left( \frac{dm_i}{d|m_i|} \right) \, d|m_i|.$$

As a limit of continuous functions,

$$F^\infty(p) = \lim_{t \to \infty} \frac{F(tp)}{t}$$

is measurable. Let $\sigma : \Omega \to S^{n-1}$ be a measurable function such that $\sigma \, d|m| = dm$. Then $F^\infty(\sigma)$ is measurable as well. Since $F$ is convex and continuous and
$\frac{1}{t} F(tp)$ is increasing in $t$, convex and continuous, it follows that $F^\infty(p)$ is lower semi-continuous and convex. Since $F$ has linear growth, we have

$$0 \leq F^\infty(p) \leq \beta |p|.$$  

By a standard result on convex functions, we conclude that $F^\infty$ is indeed continuous. So by Dini’s Theorem, $\frac{1}{t} F(tp)$ converges uniformly on compact sets.

Let $\phi_1, \phi_2, \ldots$ be a sequence in the unit ball of the Banach space $C(\bar{\Omega}, \mathbb{R}^n)$ with the supremum. Then for any $\mu$ and any sequence $\mu_n \in M$, we define $\mu_n \rightharpoonup \mu$ weakly if and only if

$$\lim_{n \to \infty} \langle \phi_j, \mu_n \rangle = \langle \phi_j, \mu \rangle,$$

for all $j$.

By Lusin’s Theorem, for each $k$ we may choose a compact set $C_k$ such that $|m|(C^c_k) \leq \frac{1}{k}$, $\sigma$ and $F^\infty(\sigma)$ are continuous on $C_k$, and $C_{k-1} \subset C_k$. Since $\sigma, F^\infty(\sigma)$ and $\phi_i$ (1 $\leq$ $i$ $\leq$ $k$) are continuous and thus uniformly continuous on $C_k$, we may choose $\delta_k$ such that

$$|\sigma(x) - \sigma(y)| + |F^\infty(\sigma(x)) - F^\infty(\sigma(y))| + \sum_{i=1}^{k} |\phi_i(x) \cdot \sigma(x) - \phi_i(y) \cdot \sigma(y)| \leq \frac{1}{k},$$

whenever $x, y \in C_k$ with $|x - y| \leq \delta_k$. Split $C_k$ into disjoint subsets $A_{i,k}$ (1 $\leq$ $i$ $\leq$ $l_k$) such that $\text{diam}(A_{i,k}) \leq \delta_k$. Pick $x_{i,k} \in A_{i,k}$. Then the sequence of simple measures

$$m_k = \sum_{i=1}^{l_k} \sigma(x_{i,k}) |m| (A_{i,k}) \delta_{x_{i,k}}$$

satisfy the claim. That is to say, we have

$$\left| \int_{\Omega} F^\infty(\sigma) \, d|m| - \int_{\Omega} F^\infty \left( \frac{d m_k}{d |m_k|} \right) \, d|m_k| \right| \leq \frac{2 \|F^\infty\| \|m\|}{k},$$

where $\|F^\infty\| = \sup_{|p|=1} |F^\infty(p)|$. Thus the claim holds and the proposition is proved.

□
From (2–10) and the previous proposition, we have

$$F^{**}(m) \leq \int_{\Omega} F(m^a) \, dx + \int_{\Omega} F^{\infty} \left( \frac{d m^s}{d |m^s|} \right) \, d|m^s|.$$  

(2–12)

**Proposition 2.5.** Let \( f \in L^1 \). \( F^{**} \) is the largest convex functional on \( \mathcal{M} \) such that

$$F^{**}(f \, dx) \leq \int_{\Omega} F(f) \, dx.$$ 

**Proof.** We will show that \( F^{**} \) is the largest convex functional on \( \mathcal{M} \) in the sense that if \( G \) is also a lower semi-continuous and convex functional on \( \mathcal{M} \), and 

\( G(f \, dx) \leq \int_{\Omega} F(f) \, dx \), for all \( f \in L^1 \), then \( G(m) \leq F^{**}(m) \) for all \( m \in \mathcal{M} \).

Indeed, let \( m \in \mathcal{M} \) and suppose that \( G \) is lower semi-continuous and convex on \( \mathcal{M} \) such that \( G(f \, dx) \leq \int_{\Omega} F(f) \, dx \). Let \( \lambda \leq G(m) \). Then by the Hahn-Banach Theorem there exists \( \phi \in C_B \) and a number \( \kappa > 0 \) such that for all \( n \in \mathcal{M} \),

$$G(n) \geq \int_{\Omega} \phi \, dn - \kappa, \quad \text{and} \quad \lambda < \int_{\Omega} \phi \, dm - \kappa.$$ 

In particular, for any \( f \in L^1 \), we see that

$$G(f \, dx) \geq \int_{\Omega} \phi \cdot f \, dx - \kappa.$$ 

However, \( G(f \, dx) \leq \int_{\Omega} F(f) \, dx \) by assumption. Therefore,

$$\int_{\Omega} F(f) \, dx \geq \int_{\Omega} \phi \cdot f \, dx - \kappa.$$ 

Thus, for all \( f \in L^1 \), we have

$$\int_{\Omega} \phi \cdot f \, dx - \int_{\Omega} F(f) \, dx \leq \kappa.$$
Taking the supremum over $f \in L^1$, we conclude that $F^*(\phi) \leq \kappa$. Thus for $\phi \in C_B$ chosen above and all $n \in \mathcal{M}$ we have

$$F^{**}(n) = \sup_{|\psi| \leq 1} \int_{\Omega} \psi \, dn - F^*(\psi)$$

$$\geq \int_{\Omega} \phi \, dn - F^*(\phi)$$

$$\geq \int_{\Omega} \phi \, dn - \kappa.$$

In particular,

$$F^{**}(m) \geq \int_{\Omega} \phi \, dm - \kappa > \lambda.$$ 

Since $\lambda \leq G(m)$ was chosen arbitrarily, it follows that

$$G(m) \leq F^{**}(m).$$

In fact, if we suppose that $G(m) > F^{**}(m)$, we may choose $\lambda$ such that $G(m) > \lambda \geq F^{**}(m)$, a contradiction.

$\Box$

Finally, it remains to show that

**Proposition 2.6.** Suppose that $F$ is convex and $m \in \mathcal{M}$. Then

$$F^{**}(m) = \int_{\Omega} F(m^a) \, dx + \int_{\Omega} F^\infty \left( \frac{dm^s}{d|m^s|} \right) \, d|m^s|. \quad (2–13)$$

**Proof.** By Proposition 2.5, we need only show that the right hand side is convex. Since $F$ is convex, we know that $F^\infty$ is convex as well. For the convenience of the reader, we show the convexity of the map

$$m \mapsto \int_{\Omega} F(m^a) \, dx + \int_{\Omega} F^\infty \left( \frac{dm^s}{d|m^s|} \right) \, d|m^s|. \quad (2–14)$$

The absolutely continuous (singular) part of a sum of measures is the sum of their absolutely continuous (singular) parts. Thus we need only show the convexity
of
\[ m \mapsto \int_{\Omega} F^\infty \left( \frac{d m^s}{d |m|^s} \right) d |m^s| . \]  

(2–15)

To this end, let \( m, n \in \mathcal{M} \) be measures and let \( 0 \leq t \leq 1 \); denote \( s = 1 - t \). For brevity, denote
\[
A = \frac{d(t m + s n)}{d |tm + sn|} \quad \quad \quad B = \frac{d |tm + sn|}{t d |m| + s d |n|}
\]
\[
C = \frac{d |m|}{t d |m| + s d |n|} \quad \quad \quad D = \frac{d |n|}{t d |m| + s d |n|}
\]

Then
\[
\int_{\Omega} F^\infty \left( \frac{d(t m + s n)}{d |tm + sn|} \right) d |tm + sn| \\
= \int_{\Omega} F^\infty \left( \frac{d(t m + s n)}{d |tm + sn|} \right) B(t d |m| + s d |n|) \\
= \int_{\Omega} F^\infty \left( \frac{d(t m + s n)}{d |tm + sn|} B \right) (t d |m| + s d |n|)
\]

(as \( F^\infty(\alpha n) = \alpha F^\infty(n) \) for \( \alpha > 0 \))
\[
= \int_{\Omega} F^\infty \left( \frac{t d m + s d n}{t d |m| + s d |n|} \right) (t d |m| + s d |n|) \\
\leq \int_{\Omega} \left[ t F^\infty \left( \frac{d m}{t d |m| + s d |n|} \right) \\
+ s F^\infty \left( \frac{d n}{t d |m| + s d |n|} \right) \right] (t d |m| + s d |n|)
\]

(by the convexity of \( F^\infty \))
\[
= t \int_{\Omega} F^\infty \left( \frac{d m}{d |m|^C} \right) (t d |m| + s d |n|) \\
+ s \int_{\Omega} F^\infty \left( \frac{d n}{d |n|^D} \right) (t d |m| + s d |n|)
\]
\[ \begin{align*}
&= t \int_{\Omega} F^\infty \left( \frac{dm}{d|m|} \right) C(t \, d|m| + s \, d|n|) \\
&\quad + s \int_{\Omega} F^\infty \left( \frac{dn}{d|n|} \right) D(t \, d|m| + s \, d|n|) \\
&= t \int_{\Omega} F^\infty \left( \frac{dm}{d|m|} \right) d|m| + s \int_{\Omega} F^\infty \left( \frac{dn}{d|n|} \right) d|n|,
\end{align*} \]

proving the map \((2–15)\) is convex. \qed

The equality \((2–13)\) is precisely the result we have set out to establish, and thus Theorem \((2.1)\) is proved.
3.1 Introduction

Below, we establish a partial regularity result for the plasticity problem (1–6); for simplicity, we will take $\beta = 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, and functions $f \in L^\infty(\Omega)$ and $\varphi \in L^1(\partial\Omega)$ be given. For $u \in BV(\Omega)$, we decompose its gradient measure $Du$ into its absolutely continuous and singular parts with respect to Lebesgue measure: $Du = \nabla u \, dx + D^s u$. From our previous work, we may define

$$\int_{\Omega} F(Du) = \int_{\Omega} F(\nabla u) \, dx + \int_{\Omega} |D^s u|,$$

with the boundary condition $\varphi$ appropriately satisfied. The existence of a minimizer was discussed by Hardt and Kinderlehrer [18] for the Dirichlet problem. There exists a minimizer if $\|f\|_{L^\infty} \leq c_D(\Omega)^{-1}$, where $c_D$ is the smallest constant such that

$$\|\zeta\|_{L^1(\Omega)} \leq c_D(\Omega) \int_{\Omega} |D\zeta|,$$

for all $\zeta \in BV(\Omega)$ with $\zeta|_{\partial\Omega} = 0$. The condition (the so called safe-load condition) guarantees that the functional is bounded below. Thus a minimizer exists by the lower semi-continuity of the functional. Similarly, for the Neumann problem, the condition $\|f\|_{L^\infty} \leq c_N(\Omega)^{-1}$ guarantees the existence of a solution, where $c_N$ is the smallest constant such that

$$\|\zeta\|_{L^1(\Omega)} \leq c_N(\Omega) \int_{\Omega} |D\zeta|,$$
for all $\zeta \in BV(\Omega)$ with $\int_\Omega \zeta \, dx = 0$. Zhou [28] studied the parabolic problem associated with the above functional. Hardt, Tonegawa and Zhou [20, 21] study the related gradient flow, where $F$ is convex and has linear growth.

Below, we show the following regularity result at a point where the solution is close to a linear function with slope strictly smaller than one:

**Theorem 3.1.** Let $u$ be a minimizer of the functional $(1-6)$ with either Dirichlet or Neumann boundary condition. For any given $0 < \mu < 1$, there exist positive constants $\epsilon_0$ and $\kappa_0$, which depend only on $n$ and $\mu$ such that, if

$$\frac{1}{|B_r|} \int_{B_r(a)} |Du - l| \leq \epsilon_0$$

holds for some $B_r(a) \subset \subset \Omega$ and for some $l \in \mathbb{R}^n$, with

$$r \|f\|_{L^\infty} \leq \kappa_0 \quad \text{and} \quad |l| < 1 - 2\mu,$$

then

$$|D^s u|(B_{r/2}(a)) = 0, \quad |\nabla u| < 1 - \mu \quad \text{on } B_{r/2}(a),$$

and $u$ solves

$$-\Delta u = f \quad \text{on } B_{r/2}(a).$$

In particular, $u \in W^{2,p}(B_{r/2}(a))$ for any $p < \infty$.

Note that by the Sobolev embedding theorem, we have $u \in C^{1,\alpha}(B_{r/2}(a))$ for any $0 < \alpha < 1$. It follows by a standard result in measure theory that

**Theorem 3.2.** Let $u$ be a minimizer of $(1-6)$ with Dirichlet or Neumann boundary condition. If $\mathcal{L}^n(\{|\nabla u| < 1\}) > 0$, then there exists a nonempty open elastic region $E$ on which $u$ is in $C^{1,\alpha}$, $|\nabla u| < 1$ and $u$ satisfies

$$-\Delta u = f \quad \text{on } E.$$

Moreover, $|\nabla u(x)| \geq 1$ for $\mathcal{L}^n$-a.e. $x \in \Omega \setminus E$. 
We note that the previous two results depend only on the minimizer $u$ and not on the initial or boundary conditions. However, it may happen that the $L^n$-measure of the set $\{|\nabla u| < 1\}$ is zero for both the Dirichlet and Neumann problems. The following conditions on the boundary value $\varphi$ and the force term $f$, which are physically reasonable, assure that there exists a nonempty, open elastic region.

**Theorem 3.3.** There exists a constant $c_0 = c(\Omega)$ such that, if

$$\|\varphi\|_{L^1(\partial \Omega)} \leq c_0 \text{ and } \|f\|_{L^\infty} \leq c_0,$$

then any minimizer $u$ for the Dirichlet problem has a nonempty, open elastic region with the properties stated in Theorem 3.2.

For the Neumann problem, a restriction of the size of the force, $f$, guarantees the existence of a nonempty elastic region.

**Theorem 3.4.** Let $c_N(\Omega)$ be the smallest constant such that

$$\|\zeta\|_{L^1(\Omega)} \leq c_N \int_{\Omega} |D\zeta|,$$

for all $\zeta \in BV(\Omega)$ with $\int_{\Omega} \zeta \, dx = 0$. If $\|f\|_{L^\infty} < c_N/2$, then any minimizer for the Neumann problem has a nonempty, open elastic region with the properties stated in Theorem 3.2.

The key to the proof of Theorem 3.1 is the decay estimate given in Proposition 3.3 below. We show that the functional $\Phi$, defined below (3–7), which is an average of the gradient measure, decays for small balls with a linear correction. We achieve the decay by approximating a minimizer to (1–6) by a Lipschitz function and establishing an $L^\infty$ estimate for their difference. The linear correction arises by approximating $u$ with the solution of an appropriate PDE.
3.2 Decay Estimate

Below we fix $\mu > 0$ and denote constants depending only on $n$ and $\mu$ by $c_i$.

Our first lemma gives a lower bound for the functional (1-6) for some function $u$ of bounded variation in terms a function with gradient strictly smaller than one.

**Lemma 3.1.** Let $u \in BV(B_r(a))$ and $h \in C^1(\bar{B}_r(a)) \cap W^{1,2}(B_r(a))$ with $\sup_{B_r(a)} |\nabla h| \leq 1 - \mu$, then

$$
\int_{B_r} F(Du) - \int_{B_r} F(\nabla h) \, dx \geq \int_{B_r} \nabla (u - h) \cdot \nabla h \, dx + \frac{\mu^2}{2} \int_{B_r \cap \{|\nabla u| \geq 1\}} |\nabla u| \, dx \\
+ \frac{1}{2} \int_{B_r \cap \{|\nabla u| < 1\}} |\nabla (u - h)|^2 \, dx + \mu \int_{B_r} |D^s u| + \int_{B_r} D^s u \cdot \nabla h
$$

**Proof.** For the part where $|\nabla u| \geq 1$, we have

$$
F(\nabla u) - F(\nabla h) - \nabla (u - h) \cdot \nabla h \\
= |\nabla u| - \frac{1}{2} + \frac{1}{2} |\nabla h|^2 - \nabla u \cdot \nabla h \\
\geq \frac{1}{2} (2 |\nabla u| - 1 - |\nabla h|) (1 - |\nabla h|) \\
\geq \frac{1}{2} |\nabla u| \mu^2 = \frac{\mu^2}{2} |\nabla u|.
$$

For the part where $|\nabla u| < 1$, we have

$$
F(\nabla u) - F(\nabla h) - \nabla (u - h) \cdot \nabla h \\
= \frac{1}{2} |\nabla u|^2 - \frac{1}{2} |\nabla h|^2 - \nabla (u - h) \cdot \nabla h \\
= \frac{1}{2} |\nabla (u - h)|^2.
$$

For the singular part, we have

$$
\int_{B_r} |D^s u| \geq \int_{B_r} D^s u \cdot \nabla h + \int_{B_r} |D^s| (1 - |\nabla h|) \\
\geq \int_{B_r} D^s u \cdot \nabla h + \mu \int_{B_r} |D^s u|,
$$
since $|\nabla h| \leq 1 - \mu$. Hence, by combining the three inequalities above the desired estimate is deduced.

Let $f \in L^\infty(\Omega)$ be given, and fix $B_{2r}(a) \subseteq \Omega$. Also let $v \in C^{0,1}(B_{2r}(a))$ be given. Below, we use the smoothing of Lipschitz functions adopted from Schoen and Simon [24]. For $\delta > 0$ and $\beta > 0$, which we will choose later, assume that there exists $l \in \mathbb{R}^n$ such that

$$\sup_{B_{2r}(a)} |\nabla v - l| \leq \beta^{2\delta} \quad \text{and} \quad |l| \leq 1 - 2\mu.$$  

We denote $\bar{v}(x) = v(x) - l \cdot x$. Let $\psi \in C^\infty_0(\mathbb{R}^n)$ be the usual radially symmetric mollifier with compact support in $B_1(0)$ and with $\sup(|\psi| + |\nabla \psi|) \leq c_1$. Denote the scaled mollifier by $\psi_\alpha(x) = \alpha^{-n}\psi(x/\alpha)$ and let

$$\bar{v}_\alpha = \psi_{r\alpha} * \bar{v} \quad \text{and} \quad v_\alpha = \psi_{r\alpha} * v$$

be the usual convolution. We then have the following estimates [24].

**Lemma 3.2.** Using the notation above, we have

(i) $\sup_{B_r(a)} |\nabla v - l| = \sup_{B_r(a)} |\nabla \bar{v}_\beta \leq \beta^{2\delta}$,

(ii) $\sup_{B_r(a)} |v_\beta - v| = \sup_{B_r(a)} |\bar{v}_\beta - \bar{v}| \leq r^{\beta} \sup_{B_r(a)} |\nabla \bar{v}_\beta \leq r^{\beta + 2\delta}$,

(iii) $r^\delta \sup_{B_r(a)} |x - y|^{-\delta} |\nabla v_\beta(x) - \nabla v_\beta(y)|$

$$\leq c_2 r^\delta \sup_{B_r(a)} |\nabla v - l| \sup_{x,y \in B_r(a)} |x - y|^{-\delta} |\psi(x/(r\beta)) - \psi(y/(r\beta))|$$

$$\leq c_3 \beta^{2\delta} \beta^{-\delta} \leq c_3 \beta^\delta.$$
Proof. Let \( v \in C^{0,1}(B_{2r}(a)) \) and use the above notation. To verify the first estimate, for \( x \in B_r(a) \), we have

\[
|\nabla v_\beta(x) - l| = |\nabla(\psi_{r\beta} * v)(x) - l| = |\psi_{r\beta} \ast \nabla v(x) - l|
\]

\[
= \left| \int_{|x-y| \leq r\beta} \psi_{r\beta}(x-y) [\nabla v(y) - l] \, dy \right| \quad (\ast)
\]

\[
= |(\psi_{r\beta} \ast \nabla \bar{v})(x)| = |\nabla (\psi_{r\beta} * \bar{v})(x)|
\]

\[
= |\nabla \bar{v}_\beta(x)|.
\]

Hence \( \sup_{B_r(a)} |\nabla v_\beta - l| = \sup_{B_r(a)} |\nabla \bar{v}_\beta| \). From (\ast), it follows that

\[
|\nabla \bar{v}_\beta(x)| \leq \sup_{B_r(a)} |\nabla v - l| \int \psi_{r\beta}(x-y) \, dy \leq \sup_{B_r(a)} |\nabla v - l| \leq \beta^{2\delta},
\]

as desired.

For the second inequality, we observe that

\[
v_\beta(x) - v(x) = (\psi_{r\beta} * v)(x) - v(x)
\]

\[
= \int_{|x-y| \leq r\beta} \psi_{r\beta}(x-y) [v(y) - l \cdot x] \, dy - (v(x) - l \cdot x)
\]

\[
= \bar{v}_\beta(x) - \bar{v}(x).
\]

Thus \( \sup_{B_r(a)} |v_\beta - v| = \sup_{B_r(a)} |\bar{v}_\beta - \bar{v}|. \)

Furthermore, by the mean value theorem, there is a \( \xi \in B_{r\beta}(x) \) so that

\[
|\bar{v}_\beta(x) - \bar{v}(x)| = \left| \int_{|x-y| \leq r\beta} \psi_{r\beta}(y-x) [\bar{v}(y) - \bar{v}(x)] \, dy \right|
\]

\[
\leq \int_{|x-y| \leq r\beta} \psi_{r\beta}(y-x) |\nabla \bar{v}(\xi)| |y - x| \, dy
\]

\[
\leq (r\beta) \sup_{B_r(a)} |\nabla \bar{v}| \leq \beta^{2\delta} r\beta = r\beta^{1+2\delta},
\]

by the hypothesis on \( v \). Taking the supremum, the desired inequality follows.
Finally, for the last estimate, let \( x \neq y \in B_r(a) \). Then we have

\[
\frac{|\nabla v_\beta(x) - \nabla v_\beta(y)|}{|x - y|^{\delta}} = \frac{|\nabla \tilde{v}_\beta(x) - \nabla \tilde{v}_\beta(y)|}{|x - y|^{\delta}} = |x - y|^{-\delta} \left| \int_{\mathbb{R}^n} \left( \psi_{r \beta}(x - z) - \psi_{r \beta}(y - z) \right) (\nabla \tilde{v}(z)) \, dz \right|
\]

\[
\leq \sup_{B_r(a)} |\nabla \tilde{v}| \frac{|x - y|^{-\delta}}{|x - y|^{\delta}} \left| \int_{\mathbb{R}^n} |\psi_{r \beta}(x - z) - \psi_{r \beta}(y - z)| \, dz \right|
\]

\[
= \sup_{B_r(a)} |\nabla \tilde{v}| (r \beta)^{-n} \left| \int_{\mathbb{R}^n} \left| \psi \left( \frac{x - z}{r \beta} \right) - \psi \left( \frac{y - z}{r \beta} \right) \right| \, dz \right|
\]

\[
\leq \sup_{B_r(a)} |\nabla \tilde{v}| (r \beta)^{-n} c_2 (r \beta)^n \sup_{x' \neq y'} \left| \psi \left( \frac{x'}{r \beta} \right) - \psi \left( \frac{y'}{r \beta} \right) \right|
\]

\[
= c_2 \sup_{B_r(a)} |\nabla \tilde{v}| (r \beta)^{-\delta} \sup_{x' \neq y'} \left| \psi(x') - \psi(y') \right| \left| x' - y' \right|^{\delta}
\]

\[
\leq c_3 \beta^{2 \delta} r^{-\delta} \beta^{-\delta} = c_3 r^{-\delta} \beta^\delta.
\]

The desired estimate follows immediately. \( \square \)

From the usual theory of Poisson equations (see Gilbarg and Trudinger [15], for example), for any \( \bar{r} \in \left[ \frac{r}{2}, r \right] \), there exists a unique solution \( w \in W^{1,2}(B_{\bar{r}}(a)) \cap C^{1,\delta}(\overline{B_{\bar{r}}}(a)) \) for

\[
\begin{cases}
-\Delta w = f & \text{on } B_{\bar{r}}(a), \\
w = v_\beta & \text{on } \partial B_{\bar{r}}(a).
\end{cases}
\] (3–1)

Moreover, we have the following estimates.

**Proposition 3.1.** Fix \( \bar{r} \in \left[ \frac{r}{2}, r \right] \) and let \( w \) be a solution to (3–1). Then

(i) \( r^{\delta} \sup_{B_{\bar{r}}(a)} \frac{|\nabla w(x) - \nabla w(y)|}{|x - y|^{\delta}} \leq c_4 \left| r \| f \|_{L^\infty} + \beta^\delta \right| \),

(ii) \( \sup_{B_{\bar{r}/2}(a)} \frac{|\nabla w(x) - \nabla w(y)|}{|x - y|^{1/2}} \leq c_5 \left( \frac{1}{r^{n+1} \delta} \int_{\partial B_{\bar{r}}(a)} |v_\beta| \, dS + r^{1/2} \| f \|_{L^\infty} \right) \).
Proof. Let $w$ be a solution to (3–1) and write $w = w_1 + w_2$, where we have $w_1, w_2 \in W^{1,2}(B_{\bar{r}}(a)) \cap C^{1,\delta}(\bar{B}_{\bar{r}}(a))$ so that

\[
\begin{aligned}
\Delta w_1 &= f \quad \text{on } B_{\bar{r}}(a), \\
\Delta w_2 &= 0 \quad \text{on } B_{\bar{r}}(a), \\
w_1 &= 0 \quad \text{on } \partial B_{\bar{r}}(a); \\
w_2 &= v_\beta \quad \text{on } \partial B_{\bar{r}}(a).
\end{aligned}
\]

For $w_1$, we may use Green’s function to write

\[
w_1(x) = \int_{B_{2r}(a)} \frac{1}{|x - z|^{n-2}} f(z) \, dz.
\]

Differentiating, we get the estimate

\[
\frac{|\nabla w_1(x) - \nabla w_1(y)|}{|x - y|^\delta} \leq c \|f\|_{L^\infty} r^{1-\delta}.
\]

For $w_2$, we let $\tilde{w} = w_2 - v_\beta$ and consider the resulting problem

\[
\begin{aligned}
\Delta \tilde{w} &= -\Delta v_\beta \quad \text{on } B_{\bar{r}}, \\
\tilde{w} &= 0 \quad \text{on } \partial B_{\bar{r}}.
\end{aligned}
\]

We have (rescaling for $B_r$) that

\[
r^\delta [D\tilde{w}]_{\delta, B_r} \leq c \left[ \|\tilde{w}\|_{L^\infty(B_r)} + \|\nabla v_\beta\|_{L^\infty(B_r)} + r^\delta \|\nabla v_\beta\|_{\delta, B_r} \right];
\]

where $[\cdot]_{\delta, B_r}$ denotes the Hölder semi-norm on the ball of radius $r$ with exponent $\delta$ [15, Theorem 8.33].

It remains to estimate the right hand side only in terms of the Hölder norm. We have [15, Theorem 8.16],

\[
\|\tilde{w}\|_{L^\infty(B_r)} \leq \sup_{\partial B_r} \tilde{w} + c \|\nabla v_\beta\|_{L^\infty(B_r)} = c \|\nabla v_\beta\|_{L^\infty(B_r)}.
\]

Combining these last two inequalities, we have

\[
r^\delta [D\tilde{w}]_{\delta, B_r} \leq c \left[ \|\nabla v_\beta\|_{L^\infty(B_r)} + r^\delta \|\nabla v_\beta\|_{\delta, B_r} \right].
\]
Replace $v_\beta$ by $\tilde{v}_\beta = v_\beta - l \cdot (x - a)$, where $l = \nabla v_\beta(y)$, for some fixed $y \in B_r(a)$.

Note that $\Delta \tilde{v}_\beta = \Delta v_\beta$ and $[\nabla \tilde{v}_\beta]_{\delta, B_r} = [\nabla v_\beta]_{\delta, B_r}$. We claim that

$$\|\nabla v_\beta - l\|_{L^\infty(B_r)} \leq c r^\delta [\nabla v_\beta]_{\delta, B_r}.$$  

In fact, for $y \in B_r$ fixed above and any $x \in B_r$, we have $|x - y| \leq 2r$. Therefore, $|x - y|^\delta \leq (2r)^\delta$. Thus

$$|\nabla v_\beta(x) - \nabla v_\beta(y)| \leq \frac{(2r)^\delta |\nabla v_\beta(x) - \nabla v_\beta(y)|}{|x - y|^\delta}.$$  

Whence

$$\|\nabla v_\beta - l\|_{L^\infty(B_r)} \leq c r^\delta \sup_{x \neq y \in B_r} \frac{|\nabla v_\beta(x) - \nabla v_\beta(y)|}{|x - y|^\delta} = c r^\delta [\nabla v_\beta]_{\delta, B_r}.$$  

Thus we have

$$r^\delta [D\tilde{w}]_{\delta, B_r} \leq c \left( \|\nabla v_\beta - l\|_{L^\infty(B_r)} + r^\delta [\nabla v_\beta]_{\delta, B_r} \right) \leq c r^\delta [\nabla v_\beta]_{\delta, B_r} \leq c \beta^\delta,$$

where the last inequality follows from Lemma 3.2. It follows immediately that

$$[Dw_2]_{\delta, B_r} \leq c r^{-\delta} \beta^\delta.$$  

For $w = w_1 + w_2$ we assemble the above pieces to see that

$$[\nabla w]_{\delta, B_r} = \sup_{B_r(a)} \frac{|\nabla w(x) - \nabla w(y)|}{|x - y|^\delta} \leq \sup_{B_r(a)} \left[ \frac{|\nabla w_1(x) - \nabla w_1(y)|}{|x - y|^\delta} + \frac{|\nabla w_2(x) - \nabla w_2(y)|}{|x - y|^\delta} \right] \leq c_4 \left[ r^{1-\delta} \|f\|_{L^\infty} + r^{-\delta} \beta^\delta \right]$$

as desired.

For (ii), we note that for $\delta = 1/2$, the first part gives

$$[\nabla w_1]_{1/2, B_r/2} \leq [\nabla w_1]_{1/2, B_r} \leq c \|f\|_{L^\infty(B_r)} r^{1/2}.$$
For $w_2$, we use Green’s representation

$$w_2(x) = \int_{\partial B_r(0)} K(x,y) v_\beta(y) \, dS_y = \int_{\partial B_r(0)} \frac{r^2 - |x|^2}{n \omega_n r |x-y|^n} v_\beta(y) \, dS_y,$$

where $\omega_n$ is the size of the $n$-dimensional unit ball. For simplicity, we drop the subscript and write $w_2 = w$. Taking the derivative, we have

$$w_{x_i}(x) = \int_{\partial B_r} K_{x_i}(x,y) v_\beta(y) \, dS_y.$$

Therefore, for $x, \tilde{x} \in B_{r/2}(0)$ the mean value theorem gives

$$\frac{w_{x_i}(x) - w_{x_i}(\tilde{x})}{|x - \tilde{x}|^{1/2}} = \int_{\partial B_r} \frac{\nabla K_{x_i}(\xi,y)}{|x - \tilde{x}|^{1/2}} |x - \tilde{x}| v_\beta(y) \, dS_y,$$

for some $\xi \in B_{r/2}(0)$. Estimating $|\nabla K_{x_i}(\xi,y)|$, we have

$$\frac{|w_{x_i}(x) - w_{x_i}(\tilde{x})|}{|x - \tilde{x}|^{1/2}} \leq \int_{\partial B_r} |\nabla K_{x_i}(\xi,y)| |x - \tilde{x}|^{1/2} |v_\beta(y)| \, dS_y$$

$$\leq \frac{c}{\omega_n r^{n+1}} |x - \tilde{x}|^{1/2} \int_{\partial B_r} |v_\beta(y)| \, dS_y$$

$$\leq \frac{c}{r^{n+\frac{1}{2}}} \int_{\partial B_r} |v_\beta| \, dS_y.$$

Thus for $x, y \in B_{r/2}(a)$ it follows that

$$\frac{|\nabla w(x) - \nabla w(y)|}{|x - y|^{1/2}} \leq \frac{|\nabla w_1(x) - \nabla w_1(y)|}{|x - y|^{1/2}} + \frac{|\nabla w_2(x) - \nabla w_2(y)|}{|x - y|^{1/2}}$$

$$\leq c_5 \left( r^{1/2} \|f\|_{L^\infty} + \frac{1}{r^{n+\frac{1}{2}}} \int_{\partial B_r} |v_\beta| \, dS \right),$$

as desired.

\[ \square \]

**Remark.** It follows from Proposition 3.1 that

$$\sup_{B_r(a)} |\nabla w - l| \leq c \left( \beta^5 + r \|f\|_{L^\infty} \right).$$
Indeed, for $x \in B_{\tilde{r}}(a)$ and $y \in \partial B_{\tilde{r}}(a)$ we have

\[|\nabla w(x) - l| \leq |\nabla w(x) - \nabla v_\beta(y)| + |\nabla v_\beta(y) - l|\]

\[\leq (2r)^\delta \sup_{x,y \in B_{\tilde{r}}(a)} \frac{|\nabla w(x) - \nabla w(y)|}{|x - y|^\delta} + |\nabla v_\beta(y) - l|\]

\[\leq c \left( \beta^\delta + r \|f\|_{L^\infty} \right),\]

where we have used Lemma 3.2 to estimate $|\nabla v_\beta(y) - l| \leq \beta^\delta$.

**Lemma 3.3.** Suppose that $v \in C^{0,1}(B_{2\tilde{r}}(a))$ and that there exists $l \in \mathbb{R}^n$ such that $|l| \leq 1 - 2\mu$ and $\sup_{B_{2\tilde{r}}(a)} |\nabla v - l| \leq \beta^{2\delta}$. Let $v_\beta$, $\tilde{r}$, and $w$ be as described above.

Then there exists constants $c_6$ and $c_7$ such that, if $\beta^\delta \leq c_6$ and $r \|f\|_{L^\infty} \leq c_7$, then the following inequality holds:

\[
\int_{B_{\tilde{r}}} F(Du) - \int_{B_{\tilde{r}}} F(\nabla w) \, dx \geq \int_{\partial B_{\tilde{r}}} (u - v_\beta) \frac{\partial w}{\partial n} \, dS + \int_{B_{\tilde{r}}} (u - w) f \, dx
\]

\[+ \mu \int_{B_{\tilde{r}}} |D^2 u| + \mu \frac{2}{2} \int_{B_{\tilde{r}} \cap \{ |\nabla u| \geq 1 \}} |\nabla u| \, dx + \frac{1}{2} \int_{B_{\tilde{r}} \cap \{ |\nabla u| < 1 \}} |\nabla (u - w)|^2 \, dx.
\]

**Proof.** From the previous remark, it follows that

\[\sup_{B_{\tilde{r}}(a)} |\nabla w| \leq \sup_{B_{\tilde{r}}(a)} |\nabla w - l| + |l| \leq c \left( \beta^\delta + r \|f\|_{L^\infty} \right) + 1 - 2\mu.\]

Choose the constants $c_6$ and $c_7$ so that, if $\beta^\delta \leq c_6$ and $r \|f\|_{L^\infty} \leq c_7$, then

\[c \left( \beta^\delta + r \|f\|_{L^\infty} \right) \leq \mu.\]

Therefore,

\[\sup_{B_{\tilde{r}}(a)} |\nabla w| \leq 1 - \mu.\]
We may now apply Lemma 3.1 with \( w \) in place of \( h \) to get

\[
\int_{B_{\bar{r}}} F(Du) - \int_{B_{\bar{r}}} F(\nabla w) \, dx \\
\geq \int_{B_{\bar{r}}} \nabla (u - w) \cdot \nabla w \, dx + \int_{B_{\bar{r}}} D^s u \cdot \nabla w + \mu \int_{B_{\bar{r}}} |D^s u| \\
+ \frac{\mu^2}{2} \int_{B_{\bar{r}} \cap \{|\nabla u| \geq 1\}} |\nabla u| \, dx + \frac{1}{2} \int_{B_{\bar{r}} \cap \{|\nabla u| < 1\}} |\nabla (u - w)|^2 \, dx.
\]

To get the desired inequality, we integrate by parts as follows:

\[
\int_{B_{\bar{r}}} \nabla (u - w) \cdot \nabla w + \int_{B_{\bar{r}}} D^s u \cdot \nabla w \\
= \int_{B_{\bar{r}}} D(u - w) \cdot \nabla w = \int_{\partial B_{\bar{r}}} (u - v_\beta) \frac{\partial w}{\partial n} \, dS - \int_{B_{\bar{r}}} (u - w) f \, dx,
\]

as \( D^s w = 0 \). Substituting above, the lemma is proved.

A function \( u \in BV(\Omega) \) is said to be a \textit{local solution} in \( \Omega \) provided that

\[
\int_{\Omega} F(Du) - \int_{\Omega} fu \, dx \leq \int_{\Omega} F(D(u + \zeta)) - \int_{\Omega} f(u + \zeta) \, dx,
\]

for any \( \zeta \in BV_0(\Omega) \). Observe that a minimizer for the Dirichlet or the Neumann problem is a local solution. We note that the following argument requires only that \( u \) be a local solution.

Next, we establish a first variation formula by computing the Euler–Lagrange equation for the functional (1–6).

**First Variation.** Let \( u \in BV(\Omega) \) be a local solution and \( \Omega = \Omega_a \cup \Omega_s \) be the decomposition of \( \Omega \) into sets \( \Omega_a \), where \( Du = \nabla u \), and \( \Omega_s \), where \( Du = D^s u \). For any \( \zeta \in BV_0(\Omega) \) satisfying \( D^s \zeta \ll |D^s u| \), we have the first variation formula

\[
\int_{\Omega} \sigma \cdot \nabla \zeta \, dx + \int_{\Omega} \sigma \cdot \xi |D^s u| = \int_{\Omega} f \zeta \, dx,
\]
where \( \sigma = \sigma(u) \) is the stress tensor defined by

\[
\sigma(u) = \begin{cases} 
F_p(\nabla u) & \text{on } \Omega_a, \\
\frac{D^s u}{|D^s u|} & \text{on } \Omega_s,
\end{cases}
\]

and \( \xi \) is the Radon-Nikodym derivative of \( D^s \zeta \) with respect to \( |D^s u| \). For simplicity, we may denote the first variation as

\[
\int_\Omega \sigma \cdot D\zeta = \int_\Omega f\zeta \, dx.
\]

**Proof.** If \( u \in BV(\Omega) \) is a local solution, then for \( t > 0 \) we have

\[
\int_\Omega F(\nabla u) \, dx + \int_\Omega |D^s u| - \int_\Omega f u \, dx \\
\leq \int_\Omega F(\nabla (u + t\zeta)) \, dx + \int_\Omega |D^s(u + t\zeta)| - \int_\Omega f(u + t\zeta) \, dx.
\]

Moving everything to one side of the inequality and dividing by \( t > 0 \) yields

\[
0 \leq \frac{1}{t} \int_\Omega F(\nabla u + t\nabla\zeta) - F(\nabla u) \, dx + \frac{1}{t} \int_\Omega \left[ |D^s(u + t\zeta)| - |D^s u| \right] - \int_\Omega f\zeta \, dx.
\]

Since \( D^s \zeta \ll |D^s u| \) taking the limit as \( t \to 0^+ \) we get

\[
0 = \int_\Omega F_p(\nabla u) \cdot \nabla\zeta \, dx + \int_\Omega \frac{D^s u}{|D^s u|} \cdot D^s \zeta - \int_\Omega f\zeta \, dx.
\]

Since,

\[
\int_\Omega \frac{D^s u}{|D^s u|} \cdot D^s \zeta = \int_\Omega \frac{D^s u}{|D^s u|} \cdot \frac{D^s \zeta}{|D^s u|} |D^s u| = \int_\Omega \sigma \cdot \xi |D^s u|,
\]

the first variation formula follows immediately. \( \Box \)

Note that by Dirichlet’s principle (see Evans [10]), it now follows that \( u \) satisfies \( -\Delta u = f \), when \( |\nabla u| \leq 1 \) and \( u \) is free of singular part.
Lemma 3.4. Let $w$ be a solution to (3–1) and suppose that $u$ is a local solution. For $\tilde{r} \in [\frac{r}{2}, r]$, we have

$$\int_{B_{\tilde{r}}} F(Du) - \int_{B_{\tilde{r}}} F(\nabla w) - \int_{B_{\tilde{r}}} f(u-w) \, dx \leq \int_{\partial B_{\tilde{r}}} |u-v_\beta| \, dS.$$ 

Proof. Recall that $w \in W^{1,2}(B_{\tilde{r}}) \cap C^{1,\delta}(\bar{B}_{\tilde{r}})$. Let

$$\bar{w} = \begin{cases} 
  w & \text{in } B_{\tilde{r}}, \\
  u & \text{in } \Omega \setminus \bar{B}_{\tilde{r}}.
\end{cases}$$

Note that $\bar{w}$ has the same boundary data as $u$ and that from the usual theory of functions of bounded variation (see Evans and Gariepy [11], for example) we have $\bar{w} \in BV(\Omega)$. Thus from the definition of local solution, we see that

$$\int_{\Omega} F(Du) - \int_{\Omega} fu \, dx \leq \int_{\Omega} F(D\bar{w}) - \int_{\Omega} f\bar{w} \, dx$$

$$= \int_{B_{\tilde{r}}} F(\nabla w) + \int_{\Omega \setminus B_{\tilde{r}}} F(Du) + \int_{\partial B_{\tilde{r}}} F(D\bar{w})$$

$$- \int_{B_{\tilde{r}}} fw \, dx - \int_{\Omega \setminus \bar{B}_{\tilde{r}}} fu \, dx,$$ 

since $Dw = \nabla w$. Since $\int_{B_{\tilde{r}}} F(Du) + \int_{\partial B_{\tilde{r}}} |D^s u| \geq \int_{B_{\tilde{r}}} F(Du)$, (3–3) may be reduced to

$$\int_{B_{\tilde{r}}} F(Du) - \int_{B_{\tilde{r}}} F(\nabla w) - \int_{B_{\tilde{r}}} f(u-w) \, dx \leq \int_{\partial B_{\tilde{r}}} F(D\bar{w}).$$

It remains to check that

$$\int_{\partial B_{\tilde{r}}} F(D\bar{w}) \leq \int_{\partial B_{\tilde{r}}} |u-v_\beta| \, dS.$$ 

By the trace and extension theorems [11], we have that

$$D\bar{w} = (Tw-Tu)\eta \, d\mathcal{H}^{n-1} \text{ on } \partial B_{\tilde{r}},$$

where $\eta$ is the outward pointing normal vector to $\partial B_{\tilde{r}}$, and $Tw$ and $Tu$ are the traces of $w$ and $u$, respectively. By the definition of $F(p)$, we have $F(p) \leq |p|$ for all
$p$, so that
\[ \int_{\partial B_r} F(D\bar{w}) \leq \int_{\partial B_r} |(Tw - Tu)\eta| \, d\mathcal{H}^{n-1} \leq \int_{\partial B_r} |(Tu - Tw)| \, d\mathcal{H}^{n-1}. \]

Since $Tw = v_\beta$, we may (by slightly abusing the notation) write
\[ \int_{\partial B_r} |(Tu - Tw)| \, d\mathcal{H}^{n-1} = \int_{\partial B_r} |u - v_\beta| \, dS. \]

The desired inequality follows. \hfill \Box

**Lemma 3.5.** Let $\tilde{r} \in [\frac{r}{2}, r]$ and suppose that $v \in C^{0,1}(B_{2r}(a))$ is as in Lemma 3.2. If $u \in BV(\Omega)$ is a local solution and $w$ is a solution to (3–1), then
\[
\int_{B_{\tilde{r}}} |D^s u| + \int_{B_{\tilde{r}} \cap \{|\nabla u| \geq 1\}} |\nabla u| \, dx + \int_{B_{\tilde{r}} \cap \{|\nabla u| < 1\}} |\nabla (u - w)|^2 \, dx \\
\leq c_8 \int_{\partial B_{\tilde{r}}} |u - v| \, dS + c_9 r^{n \beta + 2\delta}.
\]

**Proof.** By Lemmas 3.4 and 3.3, we have
\[
\int_{\partial B_{\tilde{r}}} |u - v_\beta| \, dS + \int_{B_{\tilde{r}}} f(u - w) \, dx \geq \int_{B_{\tilde{r}}} F(Du) - \int_{B_{\tilde{r}}} F(\nabla w) \\
\geq \int_{\partial B_{\tilde{r}}} (u - v_\beta) \frac{\partial w}{\partial n} \, dS + \int_{B_{\tilde{r}}} f(u - w) \, dx \\
+ \mu \int_{B_{\tilde{r}}} |D^s u| + \frac{\mu^2}{2} \int_{B_{\tilde{r}} \cap \{|\nabla u| \geq 1\}} |\nabla u| \, dx \\
+ \frac{1}{2} \int_{B_{\tilde{r}} \cap \{|\nabla u| < 1\}} |\nabla (u - w)|^2 \, dx.
\]

Therefore
\[
\mu \int_{B_{\tilde{r}}} |D^s u| + \frac{\mu^2}{2} \int_{B_{\tilde{r}} \cap \{|\nabla u| \geq 1\}} |\nabla u| \, dx + \frac{1}{2} \int_{B_{\tilde{r}} \cap \{|\nabla u| < 1\}} |\nabla (u - w)|^2 \, dx \\
\leq \int_{\partial B_{\tilde{r}}} |u - v_\beta| \, dS + \int_{\partial B_{\tilde{r}}} \left| u - v_\beta \right| \left| \frac{\partial w}{\partial n} \right| \, dS \\
= \int_{\partial B_{\tilde{r}}} |u - v| \left( 1 + \left| \frac{\partial w}{\partial n} \right| \right) \, dS + \int_{\partial B_{\tilde{r}}} \left| v - v_\beta \right| \left( 1 + \left| \frac{\partial w}{\partial n} \right| \right) \, dS
\]
\[
\begin{align*}
\leq c \left( \int_{\partial B_r} |u - v| \, dS + \int_{\partial B_r} |v - v_\beta| \, dS \right) \\
\leq c \int_{\partial B_r} |u - v| \, dS + c'r^{n-1} (r^{1+2\delta}) ,
\end{align*}
\]
as the remark following Proposition 3.1 gives \( \frac{\partial w}{\partial n} \leq |l| + c_4 (\beta + r \| f \|_{L^\infty}) \leq c \) and we have \( |v - v_\beta| \leq r^{1+2\delta} \) by Lemma 3.2. The desired inequality follows immediately.

Below we show that if \( u \) is a local solution and \( v \) is a Lipschitz function with small gradient that coincide except for a set of small measure, then we can estimate \( \|u - v\|_{L^\infty} \). The result is a modification of Hardt and Kinderlehrer ([19, Theorem 2.2]), which we present here to illustrate the important changes.

**Proposition 3.2.** Suppose that \( u \in BV(\Omega) \) is a local solution in \( \Omega \) and that \( B_{2r}(a) \subset \subset \Omega \) with \( r \| f \|_{L^\infty} \leq c_7 \). Let \( v \in C^{0,1}(B_{2r}(a)) \) such that \( \sup_{B_{2r}(a)} |\nabla v| \leq 1 - \mu \) and

\[
\mathcal{L}^n \left( \{u \neq v\} \cap B_\rho(a) \right) \leq \frac{1}{2} |B_\rho| , \quad \text{for all} \quad r \leq \rho \leq 2r.
\]

Then there exists positive constants \( c_{10} \) and \( c_{11} \) so that if

\[
\mathcal{L}^n \left( \{u \neq v\} \cap B_{2r}(a) \right) \leq c_{10}r^n ,
\]

then

\[
\|u - v\|_{L^\infty} \leq c_{11} \left( \mathcal{L}^n \left( \{u \neq v\} \cap B_{2r}(a) \right) \right)^{1/n} .
\]

**Proof.** With out loss of generality, suppose that \( a = 0 \). We then center all balls in the discussion below at 0 and note that \( x/|x| \) denotes the outward pointing unit normal vector. Let \( u \) be a local solution and let \( \theta : \mathbb{R} \to \mathbb{R} \) be a bounded, increasing, piece-wise differentiable function with \( \theta'(t) \leq 1 \) for all \( t \), which we will
define later. Suppose that \( 0 < \rho < h \leq 2r \) and define
\[
\eta(x) = \begin{cases} 
1 & \text{in } B_\rho, \\
(h - \rho)^{-1}(h - |x|) & \text{in } B_h \setminus B_\rho, \\
0 & \text{in } \Omega \setminus B_h. 
\end{cases}
\]

Applying the variation formula \( \int_\Omega \sigma \cdot D\zeta = \int_\Omega f \zeta \, dx \) with \( \zeta = \eta \theta(u - v) \), the product rule gives
\[
\int_{B_h} \eta \sigma \cdot D[\theta(u - v)] - (h - \rho)^{-1} \int_{B_h \setminus B_\rho} \sigma \cdot \frac{x}{|x|} \theta(u - v) \, dx = \int_{B_h} \eta \theta(u - v) f \, dx. \tag{3-4}
\]

Observe that by convexity,
\[
F(Dv) = F(\nabla v) + |D^s v| \leq F_p(\nabla v) \cdot \nabla v + |D^s v| = \sigma \cdot Dv,
\]
for any \( v \in BV \) with \( D^s v \ll |D^s u| \). Also, by the linear growth constraint (2-5), there exists \( c \) so that \( |p| \leq c(F(p) + 1) \). Recall that \( |Dv| = |\nabla v| \leq 1 - \mu \). We have the following estimate:
\[
\int_{B_h} \eta |D[\theta(u - v)]| \leq \int_{B_h} \eta \theta'(u - v) |Du - Dv|
\leq \int_{B_h} \eta \theta'(u - v) |Du| + \int_{B_h} \eta \theta'(u - v) |Dv|
\leq c \int_{B_h} \eta \theta'(u - v) F(Du) + c' \int_{B_h} \eta \theta'(u - v)
\leq \int_{B_h} \eta \sigma \cdot D[\theta(u - v)] + \int_{B_h} |\eta \theta'(u - v)\sigma \cdot Dv|
+ c \int_{B_h} \eta \theta'(u - v)
\leq \int_{B_h} \eta \sigma \cdot D[\theta(u - v)] + c \int_{B_h} \theta'(u - v).
Substituting (3–4) into the above inequality, we see that

\[
\int_{B_h} \eta |D[\theta(u - v)]| \\
\leq (h - \rho)^{-1} \int_{B_h \setminus B_\rho} |\theta(u - v)| + \int_{B_h} |\theta(u - v)f| + c \int_{B_h} \theta'(u - v) \\
\leq (h - \rho)^{-1} \int_{B_h \setminus B_\rho} |\theta(u - v)| + \|f\|_{L^\infty} \int_{B_h} |\theta(u - v)| + c |\text{spt}(\eta\theta(u - v))|,
\]

(3–5)

as \text{spt}(\eta) = B_h, and \theta'(t) \leq 1.

Let 0 < k < s < \infty and define

\[
\theta(t) = \begin{cases} 
0 & \text{for } t \leq k, \\
t - k & \text{for } k < t < s, \\
s - k & \text{for } t \geq s.
\end{cases}
\]

Define

\[
A(k, h) := B_h \cap \{|u - v| > k\}
\]

and note that \( |\text{spt}(\eta\theta(u - v))| = |A(k, h)|. \) Since

\[
\int_{B_\rho} |D[\theta(u - v)]| \leq \int_{B_h} \eta |D[\theta(u - v)]|,
\]

we have from (3–5) that

\[
\int_{B_\rho} |D[\theta(u - v)]| \leq (\|f\|_{L^\infty} + (h - \rho)^{-1}) \int_{B_h} |\theta(u - v)| + c |A(k, h)|.
\]

Note that from our definition of \( A(k, h) \) and the conditions on \( u \) and \( v \) that

\[
|A(0, \rho)| = L^n(\{u \neq v\} \cap B_\rho) \leq \frac{1}{2} |B_\rho|,
\]

for all \( r \leq \rho \leq 2r. \)
From the definitions of $\theta$ and $A(k, h)$, and the isoperimetric inequality (see Evans and Gariepy [11, Theorem 5.6.1, part iii]) we have

$$ (s - k) |A(s, \rho)|^{\frac{n-1}{n}} \leq \left( \int_{B_\rho} |\theta(u - v)|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} $$

$$ \leq c \int_{B_\rho} |D[\theta(u - v)]| $$

$$ \leq c \left( \|f\|_{L^\infty} + (h - \rho)^{-1} \right) \int_{B_h} |\theta(u - v)| + c' |A(k, h)| $$

$$ \leq c \left[ (h - \rho)^{-1} (s - k) |A(k, h)| + |A(k, h)| \right]. $$

Whence

$$ |A(s, \rho)|^{\frac{n-1}{n}} \leq c \left[ (h - \rho)^{-1} + (s - k)^{-1} \right] |A(k, h)|, $$

(3–6)

for $r \leq \rho \leq h \leq 2r$ and $0 < k < s$. To deduce the desired result we use Lemma 2.1 from Hardt and Kinderlehrer [19], which we state here.

**Lemma 3.6.** Suppose

$$ \{A(k, h) : r \leq h \leq 2r, \, k > 0\} $$

is any collection of subsets of $B_{2r}$ that satisfy (3–6) and that $\gamma$ is a positive number with $\gamma \leq \frac{1}{2} |B_1|$ and $\gamma^{1/n} \leq c_{16}^{-1}$. If $|A(0, 2r)| \leq \gamma r^n$, then $|A(d, r)| = 0$, for $d = c_{16} |A(0, 2r)|^{1/n}$.

By the lemma, if

$$ |A(0, 2r)| = {\mathcal{L}}^n (\{u \neq v\} \cap B_{2r}) \leq c_{10} r^n, $$

then $|A(d, r)| = 0$, where

$$ d = c_{16} |A(0, 2r)|^{1/n} = c_{16} \left( {\mathcal{L}}^n (\{u \neq v\} \cap B_{2r}) \right)^{1/n}. $$
But \( A(d, r) = \{|u - v| > d\} \cap B_r \), so \( |A(d, r)| = 0 \) implies that \( |u - v| \leq d \) on \( B_r \).

That is to say that
\[
\|u - v\|_{L^\infty(B_r)} \leq d = c_{11} \left( \mathcal{L}^n(\{u \neq v\} \cap B_{2r}) \right)^{1/n}
\]
as desired. \( \square \)

**Remark.** To apply Lemma 3.6 to the proposition, we take \( c_{10} = \gamma \) so the condition
\[
\mathcal{L}^n(\{u \neq v\} \cap B_{2r}) = |A(0, 2r)| \leq \gamma r^n
\]
implies that
\[
\|u - v\|_{L^\infty} \leq c_{11} \left( \mathcal{L}^n(\{u \neq v\} \cap B_{2r}) \right)^{1/n} = c_{11} |A(0, 2r)| = d,
\]
if we take \( c_{11} = c_{16} \).

In the following proposition, we denote
\[
\Phi(r, l, x) := \frac{1}{|B_r|} \left[ \int_{B_r(x) \cap \{|\nabla u| \geq 1\}} |\nabla u| \, dx + \int_{B_r(x) \cap \{|\nabla u| < 1\}} |\nabla u - l|^2 \, dx + \int_{B_r(x)} |D^s u| \right], \tag{3–7}
\]
for \( r > 0, l \in \mathbb{R}^n \) and \( x \in \Omega \).

We show that one can find a Lipschitz function \( v \) that approximates a local solution \( u \). Choosing a “good slice” of ball and assuming that \( \Phi \) is small, we show that \( \Phi \) decays for a smaller ball, with the addition of a small linear correction.

For the proof of the standard Lipschitz approximation (steps 1 and 2, below), we reproduce the proof of Theorem 2 in section 6.6.2 [11] with some modification to suit our needs.

**Proposition 3.3.** Suppose that \( u \) is a local solution and that \( B_{4r}(a) \subset \subset \Omega \). Let \( l_1 \in \mathbb{R}^n \) be a vector such that \( |l_1| \leq 1 - 2\mu \). Then there exist positive constants \( \omega, \epsilon, \)
\( \kappa, c_{18}, c_{19}, \text{ and } c_{20}, \) depending only on \( n \) and \( \mu \), such that if

\[
\Phi(4r, l_1, a) \leq \epsilon,
\]

for \( r \) with \( r(\|f\|_{L^\infty} + 1) \leq \kappa \), then there exists \( l_2 \in \mathbb{R}^n \) for which

\[
\Phi(\omega r, l_2, a) \leq \frac{1}{2} \Phi(4r, l_1, a) + c_{18} r^2 \|f\|_{L^\infty}^2.
\]

Moreover,

\[
|l_1 - l_2| \leq c_{19} \Phi(4r, l_1, a)^{1/2} + c_{20} r \|f\|_{L^\infty}.
\]

**Remark.** The proof is long and technical; our approach will be to break it into several steps:

1. Define a set \( R^\lambda \) where \( \Phi(\rho, l_1, x) \) is small and estimate \( \mathcal{L}^n(B_{2r}(a) \setminus R^\lambda) \).
2. Let \( g(x) = u(x) - l_1 \cdot x \) and show that \( g \) is Lipschitz on \( R^\lambda \). Using a standard extension theorem, we establish the existence of a Lipschitz function \( v \) on \( B_{2r} \) such that \( v = u \mathcal{L}^n \)-a.e. on \( R^\lambda \).
3. Use step 1 to estimate the size of \( \{u \neq v\} \).
4. Estimate \( \int_{\partial B_r} |u - v| \, dS \) in terms of \( \Phi(4r, l_1, a) \) and apply Lemma 3.5.
5. Assemble the above pieces to get the desired estimate on \( \Phi(\omega r, l_2, a) \).
6. Note that in the process of the proof, we have what we need to obtain the desired estimate for \( |l_1 - l_2| \).

**Proof.** Step 1. For \( 0 < \lambda < 1 \), which we will fix later, define

\[
R^\lambda = \{x \in B_{2r}(a) : \Phi(\rho, l_1, x) \leq \lambda \text{ for all } 0 < \rho \leq 2r\}.
\]

By Vitali’s covering theorem, there exist disjoint balls \( \{B_{r_i}(x_i)\}_{i=1}^\infty \) so that

\[
B_{2r}(a) \setminus R^\lambda \subseteq \bigcup_{i=1}^\infty B_{5r_i}(x_i) \quad \text{and} \quad \Phi(r_i, l_1, x_i) \geq \lambda.
\]
Thus, we can estimate
\[
\mathcal{L}^n \left( B_{2r}(a) \setminus R^\lambda \right) \leq \sum_{i=1}^{\infty} |B_{5r_i}(x_i)| = 5^n \sum_{i=1}^{\infty} |B_{r_i}(x_i)| \\
\leq \frac{5^n}{\lambda} \sum_{i=1}^{\infty} |B_{r_i}(x_i)| \Phi(r_i, l_1, x_i) \\
\leq \frac{5^n}{\lambda} |B_{4r}| \Phi(4r, l_1, a).
\]

Step 2. Let \( g(x) = u(x) - l_1 \cdot x \). Below, we denote the average
\[
(g)_{x, \rho} = \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} g(y) \, dy.
\]

Using Hölder’s and Poincare’s inequalities, for any \( x \in R^\lambda \) and \( 0 < \rho \leq 2r \), we have
\[
\frac{1}{|B_\rho|} \int_{B_\rho(x)} \left| g(y) - (g)_{x, \rho} \right| \, dy \\
\leq \frac{c^\rho}{|B_\rho|} \left\| g - (g)_{x, \rho} \right\|_{L^\infty(B_\rho)} \\
\leq \frac{c'}{\rho^\beta} \int_{B_\rho(x)} |Dg| \\
= \frac{c'}{\rho^\beta} \left[ \int_{B_\rho(x)} |\nabla u - l_1| \, dx + \int_{B_\rho(x)} |D^su| \right] \\
\leq \frac{c'}{\rho^\beta} \left[ \int_{B_\rho(x) \cap \{|\nabla u| \geq 1\}} |\nabla u| + |l_1| \, dx + \int_{B_\rho(x)} |D^su| \right. \\
\left. + \| \nabla u - l_1 \|_{L^2(B_\rho(x) \cap \{|\nabla u| < 1\})} \| 1 \|_{L^2(B_\rho(x) \cap \{|\nabla u| < 1\})} \right] \\
\leq \frac{c''}{\rho^\beta} \left[ |B_\rho| \Phi(\rho, l_1, x) + |B_\rho|^{1/2} (|B_\rho| \Phi(\rho, l_1, x))^{1/2} \right] \\
\leq c_{22} \rho \Phi(\rho, l_1, x)^{1/2} \leq c_{22} \rho \lambda^{1/2},
\]

where we used the fact that all the terms of \( \Phi(\rho, l_1, x) \) are positive and \( 0 < \lambda < 1 \).

We note (for use later in step 5) that we have, in the process of establishing the above inequality, shown
\[
\frac{c}{\rho^n} \int_{\Omega} |Dg| = \frac{c}{\rho^n} \int_{\Omega} |Du - l_1| \leq c_{22} \Phi(\rho, l_1, x)^{1/2}.
\] (3–8)
It now follows that
\[
\left| (g)_{x,\rho/2^{k+1}} - (g)_{x,\rho/2^k} \right| = \frac{1}{|B_{\rho/2^{k+1}}(x)|} \int_{B_{\rho/2^{k+1}}(x)} \left| g(y) - (g)_{x,\rho/2^k} \right| dy \\
\leq \frac{2^n}{|B_{\rho/2^k}(x)|} \int_{B_{\rho/2^k}(x)} \left| g(y) - (g)_{x,\rho/2^k} \right| dy \\
\leq 2^n c_22 \lambda^{1/2} \left( \frac{\rho}{2^k} \right) = c_23 \lambda^{1/2} \rho \frac{\rho}{2^k}.
\]

Since \( g \in L^1(B_{2r}(a)) \), we conclude from the Lebesgue point theorem, that \( g(x) = \lim_{\rho \to 0} (g)_{x,\rho} \), for \( \mathcal{L}^n \)-a.e. \( x \in B_{2r}(a) \). Whence, for a.e. \( x \in R^\lambda \), we have
\[
\left| g(x) - (g)_{x,\rho} \right| \leq \sum_{k=0}^{\infty} \left| (g)_{x,\rho/2^{k+1}} - (g)_{x,\rho/2^k} \right| \leq c_23 \rho \lambda^{1/2}.
\]

For \( x, y \in R^\lambda \) with \( |x - y| \leq 2r \), set \( \rho = |x - y| \). Then for a.e. \( x, y \in R^\lambda \) we have
\[
\left| (g)_{x,\rho} - (g)_{y,\rho} \right| = \frac{1}{|B_{\rho}(x) \cap B_{\rho}(y)|} \int_{B_{\rho}(x) \cap B_{\rho}(y)} \left| (g)_{x,\rho} - (g)_{y,\rho} \right| dz \\
\leq \frac{|B_{\rho}|}{|B_{\rho}(x) \cap B_{\rho}(y)|} \left[ \frac{1}{|B_{\rho}|} \int_{B_{\rho}(x)} \left| (g)_{x,\rho} - g(z) \right| dz \right] \\
\quad + \frac{1}{|B_{\rho}|} \int_{B_{\rho}(y)} \left| g(z) - (g)_{y,\rho} \right| dz \\
\leq c \left[ \frac{1}{|B_{\rho}|} \int_{B_{\rho}(x)} \left| (g)_{x,\rho} - g(z) \right| dz + \frac{1}{|B_{\rho}|} \int_{B_{\rho}(y)} \left| g(z) - (g)_{y,\rho} \right| dz \right] \\
\leq c_25 \rho \lambda^{1/2}.
\]

Using the two previous inequalities, we have
\[
|g(x) - g(y)| \leq c_26 \lambda^{1/2} \rho = c_26 \lambda^{1/2} |x - y|,
\]
for $\mathcal{L}^n$-a.e. $x, y \in R^\lambda \subset B_{2r}(a)$. Define $\lambda = c_{2\theta}^{-2}\beta^{4\delta}$, so that for $\mathcal{L}^n$-a.e. $x, y \in R^\lambda$ we have

$$|u(x) - u(y)| = |(g(x) + l_1 \cdot x) - (g(y) + l_1 \cdot y)|$$

$$\leq |g(x) - g(y)| + |l_1 \cdot (x - y)|$$

$$\leq (\beta^{2\delta} + |l_1|) |x - y|.$$

Thus there exists a Lipschitz function $\tilde{u}: R^\lambda \to \mathbb{R}$ so that $u = \tilde{u}$ $\mathcal{L}^n$-a.e. on $R^\lambda$.

By the standard Lipschitz extension theorem (e.g., see section 3.1 of [11]), we may extend $\tilde{u}$ to a Lipschitz mapping $v: B_{2r}(a) \to \mathbb{R}$ such that

$$v = u \quad \mathcal{L}^n\text{-a.e. on } R^\lambda \quad \text{and} \quad \sup_{B_{2r}(a)} |\nabla v - l_1| = \sup_{B_{2r}(a)} |\nabla g| \leq \beta^{2\delta}.$$

Step 3. With the above choice of $\lambda$ and by setting

$$\beta = \Phi(4r, l_1, a) \quad \text{and} \quad \delta = \frac{1}{8(n + 1)},$$

we estimate the size of $\{u \neq v\}$ as follows: Since $u = v$ $\mathcal{L}^n$-a.e. on $R^\lambda \subset B_{2r}(a)$, we have from step 1 and definitions above that

$$\mathcal{L}^n(B_{2r}(a) \cap \{u \neq v\}) \leq \mathcal{L}^n(B_{2r}(a) \setminus R^\lambda)$$

$$\leq \frac{5^n}{\lambda} |B_{4r}| \Phi(4r, l_1, a) \leq c_{27}r^n \Phi(4r, l_1, a)^{1-4\delta}. \quad (3-9)$$

Step 4. We need the following

Claim. There exists an $\tilde{r} \in [\frac{r}{2}, r]$ so that

$$\int_{\partial B_{\tilde{r}}(a)} |u - v| \, dS \leq \frac{5}{r} \int_{B_{\tilde{r}}(a)} |u - v| \, dx$$

and

$$\int_{\partial B_{\tilde{r}}(a)} |u - (u)_{a,r} - l_1 \cdot (x - a)| \, dS \leq \frac{5}{r} \int_{B_{\tilde{r}}(a)} |u - (u)_{a,r} - l_1 \cdot (x - a)| \, dx$$
are satisfied simultaneously.

Indeed, for any function $f \in L^1(\Omega)$, Chebychev’s inequality and Fubini’s theorem gives

$$\left| \left\{ s \in \left[ \frac{r}{2}, r \right] : \int_{\partial B_s} |f| \, dS \geq \frac{5}{r} \int_{B_r} |f| \, dx \right\} \right| \leq \frac{2}{r} \int_{B_r} |f| \, dx \int_{r/2}^r |f| \, ds \leq \frac{r}{5} \int_{B_r} |f| \, dx \int_{B_r} |f| \, dx = \frac{r}{5}.$$  

Thus

$$\left| \left\{ s \in \left[ \frac{r}{2}, r \right] : \int_{\partial B_s} |f| \, dS \leq \frac{5}{r} \int_{B_r} |f| \, dx \right\} \right| \geq \frac{r}{2} - \frac{r}{5} = \frac{3}{10} r.$$  

Next we let

$$A = \left\{ s \in \left[ \frac{r}{2}, r \right] : \int_{\partial B_s} |u - v| \, dS \leq \frac{5}{r} \int_{B_r} |u - v| \, dx \right\}$$

and

$$B = \left\{ s \in \left[ \frac{r}{2}, r \right] : \int_{\partial B_s} |u - (u)_{a,r} - l_1 \cdot (x - a)| \, dS \leq \frac{5}{r} \int_{B_r} |u - (u)_{a,r} - l_1 \cdot (x - a)| \, dx \right\}.$$  

Since $u \in BV(\Omega)$ and $v \in C^{1,0}(\Omega)$ we note that the functions $u - v$ and $u - (u)_{a,r} - l_1 \cdot (x - a)$ are both in $L^1(\Omega)$, so that the above argument holds. Thus we have

$$|A| \geq \frac{3}{10} r \quad \text{and} \quad |B| \geq \frac{3}{10} r.$$  

However, $A, B \subset \left[ \frac{r}{2}, r \right]$, so their intersection must be non-empty. Hence, there is an $\tilde{r} \in \left[ \frac{r}{2}, r \right]$ which satisfies both the desired inequalities simultaneously as claimed.

For this choice of $\tilde{r}$, we have

$$\int_{\partial B_{\tilde{r}}(a)} |u - v| \, dS \leq \frac{5}{r} \int_{B_{\tilde{r}}(a)} |u - v| \, dx \leq \frac{5}{r} \|u - v\|_{L^\infty(B_{\tilde{r}}(a))} \mathcal{L}^n \left( B_{\tilde{r}}(a) \cap \{ u \neq v \} \right). \quad (3-10)$$
For \( r \| f \|_{L^\infty} \leq c_7 \) and \( \Phi(4r, l_1, a) \leq c_{28} \) (the latter being small), Proposition 3.2 and the estimate (3.9), we have

\[
\frac{5}{r} \| u - v \|_{L^\infty} \mathcal{L}^n (B_r(a) \cap \{u \neq v\}) \\
\leq \frac{5c}{r} \left( \mathcal{L}^n (B_{2r}(a) \cap \{u \neq v\}) \right)^{1/n} \mathcal{L}^n (B_r(a) \cap \{u \neq v\}) \\
\leq \frac{c'}{r} \left( \mathcal{L}^n (B_{2r}(a) \cap \{u \neq v\}) \right)^{n+1/n} \\
\leq \frac{c'}{r} \left[ c_{27} r^n \Phi(4r, l_1, a)^{1-4\delta} \right]^{n+1/n} \leq c_{29} r^n \Phi(4r, l_1, a)^{1+\frac{1}{2n}},
\]

as \((1-4\delta)^{n+1/n} = 1 + \frac{1}{2n}\) by our choice of \( \delta \) above. Thus, from (3–10), we conclude that

\[
\int_{\partial B_r(a)} |u - v| \, dS \leq c_{29} r^n \Phi(4r, l_1, a)^{1+\frac{1}{2n}}. \quad (3–11)
\]

By Lemma 3.5 and the preceding inequality, we have

\[
\int_{B_{\omega r} (a) \cap \{|\nabla u| \geq 1\}} |\nabla u| \, dx + \int_{B_{\omega r} (a) \cap \{|\nabla u| < 1\}} |\nabla (u - w)|^2 \, dx \\
\leq c_8 \int_{\partial B_r(a)} |u - v| \, dS + c_9 r^n \beta^{1+2\delta} \\
\leq c_8 c_{29} r^n \Phi(4r, l_1, a)^{1+\frac{1}{2n}} + c_9 r^n \Phi(4r, l_1, a)^{1+\frac{1}{4(n+1)}} \\
\leq c_{30} r^n \left[ \Phi(4r, l_1, a)^{1+\frac{1}{2n}} + \Phi(4r, l_1, a)^{1+\frac{1}{4(n+1)}} \right],
\]

for any \( 0 < \omega \leq 1/2 \).

Step 5. Note that \( |a + b|^2 \leq 2 \left[ |a|^2 + |b|^2 \right] \) for any \( a, b \in \mathbb{R} \). For any \( l_2 \in \mathbb{R} \), we can estimate

\[
\int_{B_{\omega r} (a) \cap \{|\nabla u| < 1\}} |\nabla u | - l_2|^2 \, dx \leq 2 \int_{B_{\omega r} (a) \cap \{|\nabla u| < 1\}} |\nabla (u - w)|^2 + |\nabla w - l_2|^2 \, dx,
\]

by taking \( a = \nabla (u - w) \) and \( b = \nabla w - l_2 \).
Using the last two estimates above, we have

\[
|B_{\omega r} \Phi(\omega r, l_2, a)| \leq \int_{B_{\omega r}(a)} |D^s u| + \int_{B_{\omega r}(a) \cap \{ |\nabla u| \geq 1 \}} |\nabla u| \, dx \\
+ 2 \int_{B_{\omega r}(a) \cap \{ |\nabla u| < 1 \}} |\nabla (u - w)|^2 + |\nabla w - l_2|^2 \, dx \\
\leq 2c_{30} r^n \left[ \Phi(4r, l_1, a)^{1 + \frac{1}{n}} + \Phi(4r, l_1, a)^{1 + \frac{1}{4(n+1)}} \right] \\
+ c_{36} \int_{B_{\omega r}(a)} |\nabla w - l_2|^2 \, dx \\
\leq c_{35} r^n \Phi(4r, l_1, a)^{1 + \frac{1}{4(n+1)}} + c_{36} \int_{B_{\omega r}(a)} |\nabla w - l_2|^2 \, dx.
\]

(3.12)

We now use Proposition 3.1 for the gradient of \( w \) to estimate the last term above. Let \( \tilde{w}(x) = w(x) - (u)_{a,r} - l_1 \cdot (x - a) \) and note the following:

\[
\begin{align*}
\Delta \tilde{w} &= \Delta w = f & \text{on } B_{\tilde{r}}(a), \\
\tilde{w} &= v_\beta - (u)_{a,r} - l_1 \cdot (x - a) & \text{on } \partial B_{\tilde{r}}(a),
\end{align*}
\]

for \( \tilde{r} \in [\frac{r}{2}, r] \) chosen above; and

\[
\nabla \tilde{w}(x) - \nabla \tilde{w}(y) = \nabla w(x) - \nabla w(y),
\]

for any \( x, y \in B_{\tilde{r}}(a) \). Hence, we may apply Proposition 3.1 to \( \tilde{w} \), yielding

\[
\sup_{x, y \in B_{\tilde{r}/4}(a)} \frac{|\nabla w(x) - \nabla w(y)|}{|x - y|^{1/2}} \leq \sup_{x, y \in B_{\tilde{r}}(a)} \frac{|\nabla \tilde{w}(x) - \nabla \tilde{w}(y)|}{|x - y|^{1/2}} \\
\leq c_5 \left[ \frac{1}{r^{n+1/2}} \int_{\partial B_{\tilde{r}}(a)} |v_\beta - (u)_{a,r} - l_1 \cdot (x - a)| \, dS + r^{1/2} ||f||_{L^\infty} \right].
\]

To complete this estimate, we verify the following claim, which will be used again later:

**Claim.** For \( \tilde{r} \), chosen above, we have

\[
\frac{1}{|B_{\tilde{r}}|} \int_{\partial B_{\tilde{r}}(a)} |v_\beta - (u)_{a,r} - l_1 \cdot (x - a)| \, dS \leq c_{34} \Phi(4r, l_1, a)^{1/2}.
\]
Indeed, we estimate as follows.

\[
\frac{1}{|B_r|} \int_{\partial B_r(a)} |v_\beta - (u)_{a,r} - l_1 \cdot (x - a)| \, dS
\]

\[
\leq \frac{1}{|B_r|} \left[ \int_{\partial B_r(a)} |v_\beta - v| \, dS + \int_{\partial B_r(a)} |v - u| \, dS + \int_{\partial B_r(a)} |u - (u)_{a,r} - l_1 \cdot (x - a)| \, dS \right]
\]

\[
:= \frac{1}{|B_r|} [I_1 + I_2 + I_3].
\]

From Lemma 3.2, we may estimate \(I_1\) by

\[
I_1 \leq \left( c\bar{r}^{n-1} \right) \left( \bar{r} \beta^{1+2\delta} \right) \leq c\bar{r}^n \Phi(4r, l_1, a)^{1+\frac{1}{n(n+1)}}.
\]

For \(I_2\), we have by (3–11) that

\[
I_2 := \int_{\partial B_r(a)} |u - v| \, dS \leq c_29 r^n \Phi(4r, l_1, a)^{1+\frac{1}{n}}.
\]

Finally, by our choice of \(\bar{r}\) and Poincare’s inequality, we have for \(I_3\)

\[
I_3 \leq \frac{5}{r} \int_{B_r(a)} |u - (u)_{a,r} - l_1 \cdot (x - a)| \, dx \leq 5c \int_{B_r(a)} |Du - l_1|.
\]

Putting these estimates together, we have

\[
\frac{1}{|B_r|} [I_1 + I_2 + I_3]
\]

\[
\leq \frac{c}{r^n} \left[ \bar{r}^n \Phi(4r, l_1, a)^{1+\frac{1}{n(n+1)}} + r^n \Phi(4r, l_1, a)^{1+\frac{1}{n}} + \int_{B_r(a)} |Du - l_1| \right]
\]

\[
\leq c\Phi(4r, l_1, a)^{1+\frac{1}{4(n+1)}} + 2^n c\Phi(4r, l_1, a)^{1+\frac{1}{n}} + \frac{2^n c}{r^n} \int_{B_r(a)} |Du - l_1|
\]

\[
\leq c_{32} \Phi(4r, l_1, a) + \frac{c}{r^n} \int_{B_r(a)} |Du - l_1|.
\]

Recall that from (3–8) above, we have

\[
\frac{c}{r^n} \int_{B_r(a)} |Du - l_1| \leq c' \Phi(r, l_1, a)^{1/2}.
\]
Whence,
\[
\frac{1}{|B_r|} \int_{\partial B_r(a)} |v_\beta - (u)_{a,r} - l_1 \cdot (x - a)| \, dS \leq c_{32} \Phi(4r, l_1, a) + \frac{c}{r^n} \int_{B_r(a)} |Du - l_1| \\
\leq c_{32} \Phi(4r, l_1, a) + c' \Phi(r, l_1, a)^{1/2} \leq c_{34} \Phi(4r, l_1, a)^{1/2}
\]
as desired, and the claim is proved.

Since \(|B_{r/2}| \leq |B_r|\), it follows that
\[
\sup_{x,y \in B_{r/4}(a)} \frac{|\nabla w(x) - \nabla w(y)|}{|x - y|^{1/2}} \leq c_5 \left\{ \frac{1}{r^{n+1/2}} \int_{\partial B_r(a)} |v_\beta - (u)_{a,r} - l_1 \cdot (x - a)| \, dS + r^{1/2} \|f\|_{L^\infty} \right\}
\leq c_{37} \left[ r^{-1/2} \Phi(4r, l_1, a)^{1/2} + r^{1/2} \|f\|_{L^\infty} \right].
\]

By defining \(l_2 = \nabla w(a)\) and taking \(\omega \leq \frac{1}{4}\), we may now establish a desirable estimate for
\[
\int_{B_{\omega r}(a)} |\nabla w - l_2|^2 \, dx \\
= \int_{B_{\omega r}(a)} |\nabla w(x) - \nabla w(a)|^2 \, dx \\
\leq c \int_{B_{\omega r}(a)} |x - a| \left( r^{-1} \Phi(4r, l_1, a) + r R \|f\|_{L^\infty}^2 + 2 \Phi(4r, l_1, a)^{1/2} \|f\|_{L^\infty} \right) \, dx \\
\leq c(r \omega)^n \left[ \omega \Phi(4r, l_1, a) + r^2 \omega \|f\|_{L^\infty}^2 + 2 r \omega \Phi(4r, l_1, a)^{1/2} \|f\|_{L^\infty} \right] \\
\leq c(r \omega)^n \left[ \omega \Phi(4r, l_1, a) + r^2 \omega \|f\|_{L^\infty}^2 + \omega \Phi(4r, l_1, a) + 4r^2 \omega \|f\|_{L^\infty}^2 \right] \\
\leq c_{39} (r \omega)^n \left[ \omega \Phi(4r, l_1, a) + r^2 \|f\|_{L^\infty}^2 \right],
\]
by Cauchy’s inequality. From (3–12), we now see that
\[
|B_{\omega r}| \Phi(\omega r, l_2, a) \leq c_{35} r^n \Phi(4r, l_1, a)^{1+\frac{1}{4(n+1)}} + c_{36} \int_{B_{\omega r}(a)} |\nabla w - l_2|^2 \, dx \\
\leq c_{35} r^n \Phi(4r, l_1, a)^{1+\frac{1}{4(n+1)}} + c(r \omega)^n \left[ \omega \Phi(4r, l_1, a) + r^2 \|f\|_{L^\infty}^2 \right].
\]
Dividing by $|B_{\omega r}| = c(r\omega)^n$, we conclude that
\[
\Phi(\omega r, l_2, a) \leq c_{40} \omega^{-n} \Phi(4r, l_1, a)^{1 + \frac{1}{4(n+1)}} + c_{41} \omega \Phi(4r, l_1, a) + c_{42} r^2 \|f\|^2_{L^\infty}.
\]

By choosing $\omega \leq 1/4$ small enough so that $c_{41} \omega < 1/4$ and restricting $\Phi(4r, l_1, a)$ so that $c_{40} \omega^{-n} \Phi(4r, l_1, a)^{1 + \frac{1}{4(n+1)}} < 1/4$, we have the desired decay estimate
\[
\Phi(\omega r, l_2, a) \leq \frac{1}{4} \Phi(4r, l_1, a) + \frac{1}{4} \Phi(4r, l_1, a) + c_{42} r^2 \|f\|^2_{L^\infty}.
\]

Step 6. It remains to verify that
\[
|l_1 - l_2| \leq c_{34} \Phi(4r, l_1, a)^{1/2} + c_{31} r \|f\|_{L^\infty}.
\]

To such ends, define
\[
h(x) = w(x) - (u)_{a,r} - l_1 \cdot (x - a)
\]
and note that
\[
\begin{cases}
\Delta h = \Delta w = f & \text{in } B_{\tilde{r}}(a), \\
h = v_\beta - (u)_{a,r} - l_1 \cdot (x - a) & \text{on } \partial B_{\tilde{r}}(a).
\end{cases}
\]

We may write $h = h_1 + h_2$ as the sum of a harmonic function and one with zero boundary data as follows:
\[
\begin{cases}
\Delta h_1 = 0, & \Delta h_2 = f & \text{in } B_{\tilde{r}}(a), \\
h_1 = v_\beta - (u)_{a,r} - l_1 \cdot (x - a), & h_2 = 0 & \text{on } \partial B_{\tilde{r}}(a).
\end{cases}
\]

From the gradient estimate for harmonic functions, we have
\[
\nabla h_1(x) = \frac{1}{|B_{\tilde{r}}|} \int_{\partial B_{\tilde{r}}(x)} h_1 \nu \, dS,
\]
where $\nu$ is the outward pointing unit normal vector [15]. Thus
\[
|\nabla h_1(x)| \leq \frac{1}{|B_{\tilde{r}}|} \int_{\partial B_{\tilde{r}}(x)} |h_1| \, dS.
\]
Moreover, for \( h_2 \), we have that
\[
|\nabla h_2| \leq c_{31} \tilde{r} \|f\|_{L^\infty}.
\]

Recalling that \( \nabla w(a) = l_2 \), we have
\[
|l_1 - l_2| = |\nabla h(a)|
\]
\[
\leq |\nabla h_1(a)| + |\nabla h_2(a)|
\]
\[
\leq \frac{1}{|B_\tilde{r}|} \int_{\partial B_\tilde{r}(a)} |h_1| \, dS + c_{31} \tilde{r} \|f\|_{L^\infty}
\]
\[
= \frac{1}{|B_\tilde{r}|} \int_{\partial B_\tilde{r}(a)} |v_\beta - (u)_{a,r} - l_1 \cdot (x - a)| \, dS + c_{31} \tilde{r} \|f\|_{L^\infty}
\]
\[
\leq c_{34} \Phi(4r, l_1, a)^{1/2} + c_{31} r \|f\|_{L^\infty},
\]
by the claim in step 5. Hence the proposition is proved.

**Theorem 3.1** is now proved by iterating the decay estimate.

**Proof.** (Theorem 3.1) We will use Proposition 3.3 iteratively. To initialize the inductive argument assume that
\[
\frac{1}{|B_r|} \int_{B_r(a)} |Du - l_1| \leq \epsilon_0,
\]
for some \( l_1 \in \mathbb{R}^n \) with \( |l_1| \leq 1 - 4\mu \) and for any \( r \) with \( r \|f\|_{L^\infty} \leq \kappa \). \( \epsilon_0 \) will be fixed later. For each \( x \in B_{r/2}(a) \), we have
\[
\Phi(r/2, l_1, x)
\leq \frac{2^n}{|B_r|} \left[ \int_{B_r(a) \cap \{ |\nabla u| < 1 \}} |\nabla u - l_1|^2 + \int_{B_r(a) \cap \{ |\nabla u| \geq 1 \}} |\nabla u| + \int_{B_r(a)} |D^s u| \right]
\leq 2^n \Phi(r, l_1, a) \leq c_{43} \frac{1}{|B_r|} \int_{B_r(a)} |Du - l_1| \leq c_{43} \epsilon_0,
\]
so Proposition 3.3 may be applied. For the last inequality, we check each term as follows: For the part with \( |\nabla u| < 1 \), we have
\[
|\nabla u - l_1|^2 = |\nabla u - l_1| |\nabla u - l_1| \leq |\nabla u - l_1| (|\nabla u| + |l_1|) \leq 2 |\nabla u - l_1|.
\]
For the part with \(|\nabla u| \geq 1\), it suffices to show that there exists a \(c > 0\) so that
\[
|\nabla u| \leq c |\nabla u - l_1|.
\]
In fact
\[
\frac{|\nabla u|}{|\nabla u - l_1|} \leq \frac{|\nabla u|}{|\nabla u - l_1|} \leq \frac{|\nabla u|}{|\nabla u - (1 - 4\mu)|},
\]
which is bounded, as the function \(\frac{t}{t - \text{constant}}\) is decreasing. Thus we have
\[
\int_{B_r(a) \cap \{|\nabla u| < 1\}} |\nabla u - l_1|^2 + \int_{B_r(a) \cap \{|\nabla u| \geq 1\}} |\nabla u| + \int_{B_r} |D^s u|
\]
\[
\leq 2 \int_{B_r(a) \cap \{|\nabla u| < 1\}} |\nabla u - l_1| + c \int_{B_r(a) \cap \{|\nabla u| \geq 1\}} |\nabla u - l_1| + \int_{B_r} |D^s u|
\]
\[
= c \int_{B_r} |Du - l_1|
\]
as claimed.

For the inductive step, choose \(\epsilon_0\) so that \(c_{43}\epsilon_0 \leq \epsilon\) and restrict \(r\) so that
\(c_{18}r^2 \|f\|_{L^\infty}^2 \leq \epsilon/2\). Furthermore, we assume that \(|l_{j-1}| \leq 1 - 2\mu\) and
\[
\Phi\left(\left(\frac{\omega}{4}\right)^{j-1} \frac{r}{2}, l_j, x\right) \leq \left(\frac{1}{2}\right)^{j-1} \Phi\left(\frac{r}{2}, l_1, x\right) + \sum_{i=1}^{j-1} \left(\frac{1}{2}\right)^{i-1} \omega^{j-i-1} c_{44}r^2 \|f\|_{L^\infty}^2,
\]
for \(j = 2, \ldots, k\). We need to show
\[
\Phi\left(\left(\frac{\omega}{4}\right)^{k-1} \frac{r}{2}, l_k, x\right) \leq \epsilon \quad \text{and} \quad |l_k| \leq 1 - 2\mu
\]
to apply Proposition 3.3 and continue the inductive step. Taking \(\omega \leq 1/4\), we have
for all \(k\)
\[
\sum_{i=1}^{k-1} \left(\frac{1}{2}\right)^{i-1} \omega^{k-i-1} \leq \left(\frac{1}{2}\right)^{k-1} \left(\frac{1}{2}\right)^{k-2} (k - 1) \leq c_{45} \left(\frac{1}{2}\right)^{k/2},
\]
where it can be shown by induction on \(k\) that \(c_{45} = \frac{2}{\sqrt{2} - 1}\). By further restricting \(r \|f\|_{L^\infty} \leq \epsilon/2\), we have from (3–13)
\[
\Phi\left(\left(\frac{\omega}{4}\right)^{k-1} \frac{r}{2}, l_k, x\right) \leq \left(\frac{1}{2}\right)^{k-1} \Phi\left(\frac{r}{2}, l_1, x\right) + c_{45} \left(\frac{1}{2}\right)^{k/2} c_{44}r^2 \|f\|_{L^\infty}^2 \leq \epsilon.
By Proposition 3.3 and the inductive assumption we see that

\[
|l_k| \leq \sum_{j=1}^{k-1} |l_{j+1} - l_j| + |l_1|
\]

\[
\leq \sum_{j=1}^{k-1} \left[ c_{19} \Phi \left( \left( \frac{\omega}{4} \right)^{j-1} \frac{r}{2}, l_j, x \right)^{1/2} + c_{20} \left( \frac{\omega}{4} \right)^{j-1} r \|f\|_{L^\infty} \right] + 1 - 4\mu
\]

\[
\leq c_{19} \sum_{j=1}^{k-1} \left[ \left( \frac{1}{2} \right)^{j-1} \Phi \left( \frac{r}{2}, l_1, x \right) + c_{45} \left( \frac{1}{2} \right) c_{44} r^2 \|f\|_{L^\infty} \right]^{1/2} + 1 - 4\mu
\]

\[
\leq c_{19} \sum_{j=1}^{k-1} \left[ \left( \frac{1}{2} \right)^{j-1} \Phi \left( \frac{r}{2}, l_1, x \right) + c_{45}^2 \left( \frac{1}{2} \right)^j c_{44} r \|f\|_{L^\infty} \right] + 1 - 4\mu
\]

\[
= c \left[ \Phi \left( \frac{r}{2}, l_1, x \right) + r \|f\|_{L^\infty} \right] + 1 - 4\mu,
\]

where the last inequality follows by bounding each term by the appropriate geometric series. Hence, by restricting \(\epsilon_0\) and \(r \|f\|_{L^\infty}\) again, we can estimate that \(|l_k| \leq 1 - 2\mu\). Therefore, we may apply Proposition 3.3 to show

\[
\Phi \left( \left( \frac{\omega}{4} \right)^k \frac{r}{2}, l_{k+1}, x \right) \leq \left( \frac{1}{2} \right)^k \Phi \left( \frac{r}{2}, l_1, x \right) + c \left( \frac{1}{2} \right)^{(k+1)/2} r^2 \|f\|_{L^\infty},
\]

where \(|l_{k+1}| \leq 1 - 2\mu\).

Whence

\[
\lim_{k \to \infty} \Phi \left( \left( \frac{\omega}{4} \right)^k \frac{r}{2}, l_{k+1}, x \right) = 0
\]

uniformly for all \(x \in B_{r/2}(a)\). Hence

\[
\lim_{\rho \to 0} \frac{1}{|B_{\rho}(x)|} \left[ \int_{B_{\rho}(x)} |D^s u| + \int_{B_{\rho} \cap \{|\nabla u| \geq 1\}} |\nabla u| \, dx \right] = 0. \quad (3-14)
\]
It follows from Lemma 1 in section 1.6 \cite{11} that
\[ |D^u| \left( B_{r/2}(a) \right) = 0. \]

Moreover, since
\[ \lim_{\rho \to 0} \frac{1}{|B_{\rho}(x)|} \int_{B_{\rho}(x) \cap \{ |\nabla u| \geq 1 \}} |\nabla u| \, dx = 0, \]
we conclude that
\[ \mathcal{L}^n \left( B_{r/2}(a) \cap \{ |\nabla u| \geq 1 \} \right) = 0. \]
Thus follows that $|\nabla u| < 1$, and by the first variation, $-\Delta u = f$ on $B_{r/2}(a)$ as desired.

### 3.3 Regularity of the Elastic Region

We prove Theorem 3.2. Assume that $u$ is a minimizer and that the set
\[ \tilde{E} = \{ |\nabla u| < 1 \} \]
has positive Lebesgue measure.

**Proof.** (Theorem 3.2) Since $u \in BV$, its derivative $Du$ is a Radon measure that can be decomposed into singular and absolutely continuous parts with respect to Lebesgue measure, $D^s u$ and $\nabla u$, respectively. It follows by Proposition 6.8 \cite{23} that $|D^s u|$ is singular with respect to Lebesgue measure as well. Thus, by Theorem 7.13 \cite{23}, we have from (3–14)
\[ \lim_{r \to 0} \frac{1}{|B_r|} \int_{B_r(x)} |D^s u| = 0, \]
for $\mathcal{L}^n$-a.e $x \in \tilde{E}$. Furthermore, since $\nabla u \in L^1(\Omega)$, we have
\[ \lim_{r \to 0} \frac{1}{|B_r|} \int_{B_r(x)} |\nabla u(y) - \nabla u(x)| \, dy = 0, \]
for $\mathcal{L}^n$-a.e. $x \in \tilde{E}$, by the Lebesgue point theorem.

From $\tilde{E}$, remove points at which either of the above fails to hold; call the resulting set $E$. Thus we have $\mathcal{L}^n(\tilde{E} \setminus E) = 0$, $|\nabla u| < 1$ on $E$ and both the above limits hold for each point in $E$. For each fixed $x \in E$, there exists some $\mu_x > 0$ such
that
\[ |\nabla u| < 1 - 2\mu_x. \]

Letting \( l = \nabla u(x) \), we see from above that
\[
\frac{1}{|B_r|} \int_{B_r(x)} |Du - l| \leq \frac{1}{|B_r|} \left[ \int_{B_r(x)} |\nabla u(y) - l| \, dy + \int_{B_r(x)} |D^* u| \right] \to 0.
\]

Thus, for \( \epsilon_0 \) given above, there exists \( r_x > 0 \) so that
\[
\frac{1}{|B_{rx}|} \int_{B_{rx}(x)} |Du - l| \leq \epsilon_0.
\]

Therefore, Theorem 3.1 gives that
\[
|D^* u|(B_{rx}(x)) = 0 \quad \text{and} \quad |\nabla u| < 1 - \mu_x \text{ on } B_{rx}(x),
\]

with \( u \in C^{1,\alpha}(B_{rx}(x)) \). Hence \( B_{rx}(x) \subset E \), showing that \( E \) is an open set with the desired properties.

\[ \square \]

Even with the safe-load condition (which guarantees the existence of a minimizer), it is not clear whether the set \( E \) has positive measure or not. We next prove Theorems 3.3 and 3.4, which are simple comparison arguments, discussed for example by Hardt and Kinderlehrer [18].

**Proof.** (Theorems 3.3 and 3.4) To get a contradiction, assume that
\[ \mathcal{L}^n(\{|\nabla u| < 1\}) = 0. \]

Under this assumption, \( F(Du) = |Du| - \frac{1}{2} \), a.e. For a given boundary data \( \varphi \) we can choose some \( v \in BV(\Omega) \) with \( v|_{\partial\Omega} = \varphi \) such that
\[ \|v\|_{BV(\Omega)} \leq c(\Omega) \|\varphi\|_{L^1(\partial\Omega)}. \]
by the trace theorem for $BV$ functions ([16]). By the minimality of $u$, we have
\[
\int_{\Omega} F(Du) \leq \int_{\Omega} F(Dv) - \int_{\Omega} f(v - u) \, dx \\
\leq c(\Omega) \| \varphi \|_{L^1(\partial \Omega)} + c_D(\Omega) \| f \|_{L^\infty} \int_{\Omega} (|Du| - |Dv|),
\]
where $c_D$ is the smallest constant such that $\| \zeta \|_{L^1(\Omega)} \leq c_D \int_{\Omega} |D\zeta|$, for all $\zeta \in BV(\Omega)$ such that $\zeta|_{\partial \Omega} = 0$. Assume that $\| f \|_{L^\infty} \leq \frac{1-\delta}{2c_D}$ for some $\delta > 0$. Then
\[
\frac{1}{2} \int_{\Omega} |Du| \leq \int_{\Omega} \left( |Du| - \frac{1}{2} \right) = \int_{\Omega} F(Du) \\
\leq c(\Omega) \| \varphi \|_{L^1(\partial \Omega)} + \frac{1-\delta}{2} \int_{\Omega} |Du| + c_D \| f \|_{L^\infty} \| \varphi \|_{L^1(\partial \Omega)}. 
\]
It follows that
\[
\frac{\delta}{2} |\Omega| \leq \frac{\delta}{2} \int_{\Omega} |Du| \leq [c_D \| f \|_{L^\infty} + c(\Omega)] \| \varphi \|_{L^1(\partial \Omega)}.
\]
If $\| \varphi \|_{L^1(\partial \Omega)}$ is sufficiently small, this is not possible. Hence, for small enough norms, we have a contradiction. Thus $\{ |\nabla u| < 1\}$ has positive measure and Theorem 3.2 applies. The proof of Theorem 3.4 follows similarly using $v = 0$ as a comparison.
CHAPTER 4
CONCLUSION

The main result of Chapter 2, Theorem 2.1, shows that the variational integral $\int F(m)$, for a vector-valued measure $m$, can be defined “naturally” by the Fenchel transform. An important consequence of the technique is that the lower semi-continuity of the functional follows immediately from well-known properties of the Fenchel transform from complex analysis. In our approach, we decomposed the measure with respect to Lebesgue measure: $m = m^a + m^s$, where $m^a$ is absolutely continuous to Lebesgue measure and $m^s$ is mutually singular to Lebesgue measure. One question of interest that was not explored is whether Lebesgue measure is the best choice for the base of this decomposition.

To establish the duality needed for the Fenchel transform, we considered the space of continuous bounded functions, $C_B$, which induces a topology (the weak topology) onto the space of bounded vector-valued measures, $M$. Is $C_B$ the best choice to establish the required duality? Another possible candidate is the Sobolev space $W^{1,\infty}(\Omega, \mathbb{R}^n)$; with this choice, the weak topology on $M$ is metrizable.

In Chapter 3, we established a partial regularity result for the plasticity problem. The basis of our technique is the decay estimate for the “excess,” given in Proposition 3.3. Hardt and Tonegawa [20] give a partial regularity result for a weak solution to the evolution problem

$$\frac{\partial u}{\partial t} = \text{div}_x F_p(\nabla u),$$

where $u = f$ on $\Omega \times \{0\}$ and $u = g$ on $\partial \Omega \times (0, T)$. Their result is similar to Theorem 3.1; however, the decay method they used is dependent upon the space

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variable having dimension \( n = 1 \) or 2 [20]. One extension of their result that to consider is to establish this result for the general \( n \)-dimensional space variable.

Another extension to explore is to obtain a partial regularity result for the problem
\[
\frac{\partial u}{\partial t} = \text{div}_x F_p(\nabla u) + \frac{1}{2}(u - I),
\]
where \( I \) is given and depends on the space variable \( x \). A weak solution to this problem minimizes
\[
\int_{\Omega} F(Du) + \frac{1}{2}(u - I(x))^2
\]
over \( BV(\Omega) \). An important application for this problem is image restoration. In this case \( F \) may be the function given in (1–3) and \( I \) is the observed image. The main distinction between this problem and the one considered by Hardt and Tonegawa[20] is the dependence of (4–1) on the function \( u \) and the space variable \( x \).
REFERENCES


BIOGRAPHICAL SKETCH

I received a Bachelor of Arts degree in 1994 from Ohio Wesleyan University. At OWU, I majored in mathematics and earned minors in economic management and philosophy. In 1997, I received a Master of Arts degree from Bowling Green State University in mathematics. I entered the doctoral program at the University of Florida in 1998; I began work with Dr. Chen in 2001.