MULTISCALE ANALYSIS
OF PARTIAL DIFFERENTIAL EQUATIONS
MODELING VOLTAGE POTENTIAL

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We study a nonlinear elliptic boundary value problem arising from electro-chemistry. The boundary condition is of an exponential type. We examine the questions of existence and uniqueness of solutions to this boundary value problem. We then treat the problem from the point of view of homogenization theory. The boundary condition has a periodic structure. We find a limiting or effective problem as the period approaches zero, along with a correction term and convergence estimates. This correction term satisfies a boundary value problem with Neumann boundary conditions. We do numerical experiments to investigate the behavior of galvanic currents near the boundary as the period approaches zero. We then consider a correction term which satisfies a boundary value problem with a Robin boundary condition. We do numerical experiments to investigate our approximation based on this corrector term. We then use asymptotics to analyze the behaviour of the steady state voltage potential of a conductor with a region of inhomogeneity. The boundary of the inhomogeneity shifts slightly, we do asymptotics utilizing the small shift.
CHAPTER 1
INTRODUCTION

Modern perturbation theory is primarily concerned with constructing approximations of solutions to mathematical models that have a parameter which is approaching zero. One such class of models are boundary value problems in which the domain has a periodic structure. In this case the period size is the small scale parameter. These types of boundary value problems often arise in the study of, for example, composite materials, macroscopic parameters of crystalline structures, fluid mechanics and aerodynamics. Perturbation theory is an example of an *analytical* approximation as opposed to a *numerical* approximation. There are many techniques to formulate these analytical approximations. One fundamental technique is through the use of a multiple scales asymptotic expansion. The foundational ideas for this approach appear in the early 1800s. In 1812 Laplace used asymptotic series to analyze some special functions. In 1823, Poisson constructed an expansion of a Bessel function. In 1886 Poincaré used asymptotic expansions to study solutions of differential equations. The idea behind the asymptotic approximations is that the solution can expressed as a sum of terms of different orders of magnitude with respect to the small scale. For example if $\phi_\epsilon(x)$ is the solution to some boundary value problem with small scale parameter $\epsilon$ then we begin by assuming the solution $\phi_\epsilon$ has an asymptotic expansion of the form

$$\phi_\epsilon \approx \phi_0 + \epsilon \phi^{(1)}_\epsilon + \epsilon^2 \phi^{(2)}_\epsilon + \cdots.$$ 

The general procedure is to then substitute the above expansion back into the original boundary value problem to determine associated boundary value problems for $\phi_0$, $\phi^{(1)}_\epsilon$, $\phi^{(2)}_\epsilon$, $\ldots$. We try to find simpler equations that describe the
behaviour of the solution on various orders of $\epsilon$. Here we wish to use multiple scales analysis to develop an asymptotic expansion of the solution to some phenomena related to conductivity. We first use perturbation theory to approximate the solution of a nonlinear boundary value problem which models galvanic corrosion. As a second application of asymptotics we wish to describe the conductivity properties of a material with a shifting dielectric boundary.

In the electrochemistry community there is much interest in the study of galvanic interactions on heterogeneous surfaces [12], [13]. When two different metals in electrical contact, referred to as anode and cathode, are immersed in an electrolytic solution, the difference in rest potential generates an electron flow. This electron flow is called a galvanic current and may lead to a deterioration (corrosion) of the anode.

In Figure 1–1 a strip of silver (Ag) and a strip of zinc (Zn) have been immersed in a saltwater solution. The zinc strip gives up electrons to the silver strip. The silver strip is said to be cathodic, and reduction takes place (Ag gains electrons.) Simultaneously oxidation takes place at the zinc strip, zinc loses electrons, and is said to be anodic. Zinc dissolves into the solution, the zinc electrode is being corroded and the electron flow is known as galvanic current. The driving force of the electron transport process is the difference in potential of the two metals involved. See Newman [15] for a complete introduction to the subject.

Here we study the electrostatic problem on a surface where anodes are arranged periodically in a cathodic matrix. Mathematically the potential is modeled as a function, $\phi$, over a Euclidean domain $\Omega$. Part of the boundary of $\Omega$ is electrochemically active while the rest of the boundary is inert. It is the active region of the boundary that is made up of anodic and cathodic portions. The potential over both of these regions satisfies an exponential boundary condition of Butler and Volmer, but with different material parameters on each portion.
In Morris and Smyrl [12] the authors study such a problem numerically, using finite elements. Additionally various interesting aspects of the two-dimensional, homogeneous model with the Butler–Volmer condition have been analyzed in Bryan and Vogelius [5], Turner and Hou [9], and Vogelius and Xu [19]. To the best of our knowledge, however, studies coming from the applied mathematics community have been restricted to two dimensions. The main reason for this is that one can bound exponentials of the two-dimensional weak solution on the boundary by using an Orlicz estimate [18], [19]. Such an estimate would require more than $H^1$ regularity in higher dimensions. In this paper, we attempt to treat a periodically heterogeneous problem, in two and three dimensions, from the point of view of homogenization theory.

Our second application of asymptotics is to a problem pertaining to electrostatic conductivity. We consider a PDE which models the steady state voltage potential of a metal with a small inhomogeneity. There is a jump in the conductivity across the boundary of the inhomogeneous region. The boundary of the inhomogeneous region shifts by some small amount due to some type of physical stress. We now wish to describe the voltage potential of the conductor with shifted boundary as a perturbation of the voltage potential of the original conductor. We do asymptotics to establish an estimate.
In Chapter 2 we formally present the electrolytic voltage potential model. We tackle the issue of existence and uniqueness of the solution to the model and then discuss the issue of regularity. In Chapter 3 we construct an asymptotic approximation of the solution to the original problem. Here the second term of the approximation satisfies a linear boundary value problem with Neumann data. We then establish some convergence estimates and do numerical implementation. Using a finite element method approach we implement and test our asymptotic approximation and convergence estimates. In Chapter 4 we propose an approximation in which the second term satisfies a boundary value problem with Robin boundary data. We then do numerical implementation and testing of this approximation. In Chapter 5 we perform asymptotics on the “shifting boundary” problem. We establish some convergence estimates and do some formal asymptotics. In Chapter 6 we discuss future work to be done.
CHAPTER 2
THE ELECTROLYTIC VOLTAGE POTENTIAL MODEL

2.1 Butler–Volmer Boundary Conditions

Now we formally present the three-dimensional model for electrolytic voltage potential on a heterogeneous surface. The domain $\Omega$ is of cylindrical shape with base some two-dimensional domain. The bottom base is assumed to contain a periodic arrangement of islands (anodes). We call this collection of islands $\partial \Omega_A$ and the remainder of the bottom of the base $\partial \Omega_C$ (cathodic plane). The electrolytic voltage potential, $\phi$, satisfies the following nonlinear elliptic boundary value problem,

$$\begin{align*}
\Delta \phi &= 0 \text{ in } \Omega \\
-\frac{\partial \phi}{\partial n} &= J_A[e^{\alpha_{aa}(\phi-V_A)} - e^{-\alpha_{ac}(\phi-V_A)}] \text{ on } \partial \Omega_A \\
-\frac{\partial \phi}{\partial n} &= J_C[e^{\alpha_{ca}(\phi-V_C)} - e^{-\alpha_{cc}(\phi-V_C)}] \text{ on } \partial \Omega_C \\
-\frac{\partial \phi}{\partial n} &= 0 \text{ on } \partial \Omega \setminus \{\partial \Omega_A \cup \partial \Omega_C\}
\end{align*} \tag{2.1}$$

where $\alpha_{aa}, \alpha_{ac}, \alpha_{ca}, \alpha_{cc}$ are the transfer coefficients and it is assumed that the sums $(\alpha_{aa} + \alpha_{ac})$ and $(\alpha_{ca} + \alpha_{cc})$ are equal to one. The positive constants $J_A, J_C$ are the anodic and cathodic polarization parameters and $V_A, V_C$ are the anodic and cathodic rest potentials respectively. Note that $\nabla \phi$ represents galvanic current.

These boundary conditions are the so-called the Butler–Volmer exponential boundary conditions.

In the numerical studies of [12], the authors observed that for fixed ratios of anodic to cathodic areas on the bottom base, the resulting current increased approximately linearly with the length of the perimeter between the two regions, and
Figure 2–1: The base is a heterogeneous surface

they hypothesized that it is the ratio of anodic area to perimeter that determines
the size of the resulting current.

Figure 2–2: Perimeter increases while anodic area fraction stays constant.

As a special case of increasing perimeter with approximately fixed area
fraction, we consider a periodic structure with period approaching zero. Our goal is
to expand the solution asymptotically with respect to the period size. Convergence
results involving these approximations could provide insight into the behavior of
the current for small period size; and possibly lead to techniques for computing
approximate solutions to (2.1).

We model the periodic structure by letting

\[
f(y,v) = \lambda(y)[e^{\alpha(y)(v-V(y))} - e^{-(1-\alpha(y))(v-V(y))}]
\]

for any \( v \in R \) and \( y \in Y \), the boundary period cell, which for simplicity we take
to be the unit square; \( Y = [0,1] \times [0,1] \). Here \( \lambda, \alpha, \) and \( V \) are all piecewise smooth
\( Y \)-periodic functions. We also assume there exist constants \( \lambda_0, \Lambda_0, \alpha_0, A_0 \) and \( V_0 \)
such that:

\[
0 < \lambda_0 \leq \lambda(y) \leq \Lambda_0, \quad (2.2)
\]

\[
0 < \alpha_0 \leq \alpha(y) \leq A_0 < 1, \quad (2.3)
\]
and

\[ |V(y)| \leq V_0. \]  \hspace{1cm} (2.4)

See [5] and [19] for an analysis of when \( \lambda < 0 \).

Consider the problem

\[
\begin{align*}
\Delta u_\epsilon &= 0 \text{ in } \Omega \\
- \frac{\partial u_\epsilon}{\partial n} &= f(x/\epsilon, u_\epsilon) \text{ on } \Gamma \\
- \frac{\partial u_\epsilon}{\partial n} &= 0 \text{ on } \partial \Omega \setminus \Gamma.
\end{align*}
\]  \hspace{1cm} (2.5)

As is typical in homogenization problems, one expects that as \( \epsilon \to 0 \), the solutions will converge in some sense to a solution of a problem with an averaged boundary condition. Define \( f_0(v) \) to be a cell average of \( f(y,v) \), that is,

\[ f_0(v) = \int_Y f(y,v) \, dy. \]

Consider the candidate for the homogenized problem

\[
\begin{align*}
\Delta u_0 &= 0 \text{ in } \Omega \\
- \frac{\partial u_0}{\partial n} &= f_0(u_0) \text{ on } \Gamma \\
- \frac{\partial u_0}{\partial n} &= 0 \text{ on } \partial \Omega \setminus \Gamma.
\end{align*}
\]  \hspace{1cm} (2.6)

**Remark** If, as is the case in [12], \( Y = Y_1 \cup Y_2 \) and the functions \( \lambda, \alpha, V \) are piecewise constant, each taking on the values \( \lambda_i, \alpha_i, V_i \) respectively in \( Y_i \), then

\[ f_0(v) = |Y_1| \lambda_1 \left[ e^{\alpha_1(v-V_1)} - e^{-(1-\alpha_1)(v-V_1)} \right] + |Y_2| \lambda_2 \left[ e^{\alpha_2(v-V_2)} - e^{-(1-\alpha_2)(v-V_2)} \right]. \]

That is, the above homogenized boundary condition would depend on the volume fraction of anodic to cathodic regions.
2.2 Existence and Uniqueness

In this section we show that the energy minimization forms of the nonlinear problem (2.5) and (2.6) have unique solutions in $H^1(Ω)$ in any dimension. Some elements of the proof are similar to those in [9] and [19]. For a given $ε$, define the following energy functional,

$$E_ε(v) = \frac{1}{2} \int_Ω |\nabla v|^2 \, dx + \int_Γ F(\frac{x}{ε}, v) dσ_x$$  \hspace{1cm} (2.7)

where,

$$F(y, v) = \frac{λ(y)}{α(y)}e^{α(y)(v - V(y))} + \frac{λ(y)}{1 - α(y)}e^{-(1-α(y))(v - V(y))}.$$  

We show the existence and uniqueness of a minimizer of (2.7). Formally, we show the existence of a function $u_ε \in H^1(Ω)$ such that

$$E_ε(u_ε) = \min_{u ∈ H^1(Ω)} E_ε(u).$$  \hspace{1cm} (2.8)

Note that $E_ε$ is not necessarily bounded on all of $H^1(Ω)$ (unless $n = 2$ for which we can use an Orlicz estimate). However this does not pose a problem. We set $E_ε$ equal to (2.7) where it is well defined and to $+∞$ where it is not, as in [7], p.444.

In the two-dimensional case of the model, due to the boundedness of $E_ε$ on $H^1(Ω)$, direct calculation shows $u_ε$ satisfies the variational form of (2.5),

$$\int_Ω \nabla u_ε \cdot \nabla v \, dx = -\int_Γ f(x/ε, u_ε)v \, dσ_x \text{ for any } v \in H^1(Ω).$$  \hspace{1cm} (2.9)

In the three-dimensional case, if $u_ε$ is an energy minimizer, we will have that

$$\int_Γ F(x/ε, u_ε) \, dσ_x < ∞,$$  \hspace{1cm} (2.10)

and hence by the positivity of each term of $F(x/ε, u_ε)$, we have that each term is separately in $L^1(Ω)$. Therefore,

$$E_ε(u_ε + tv) < ∞,$$  \hspace{1cm} (2.11)
for any \( t \in R \) and for any \( v \) which is smooth on \( \Gamma \). Standard arguments then show that \( u_\epsilon \) satisfies,

\[
\int_\Omega \nabla u_\epsilon \cdot \nabla v \, dx = -\int_\Gamma f(x/\epsilon, u_\epsilon)v \, d\sigma_x \quad \text{for any } v \in C^\infty(\Omega).
\]

Additionally, if we knew that \( u_\epsilon \in C^0(\overline{\Omega}) \) then \( f(x/\epsilon, u_\epsilon) \) is bounded and hence clearly in \( H^{-1/2}(\Gamma) \). So by the density of \( C^\infty(\overline{\Omega}) \) functions in \( H^1(\Omega) \), \( u_\epsilon \) in this case would satisfy

\[
\int_\Omega \nabla u_\epsilon \cdot \nabla v \, dx = -\int_\Gamma f(x/\epsilon, u_\epsilon)v \, d\sigma_x \quad \text{for any } v \in H^1(\Omega). \tag{2.12}
\]

Consider also the functional

\[
E_0(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx + \int_\Gamma F_0(v) \, d\sigma_x \tag{2.13}
\]

where,

\[
F_0(v) = \int_Y F(y, v) \, dy.
\]

Here again the energy \( E_0 \) is not necessarily bounded but as before, we set \( E_0 \) equal to (2.13) where it is well defined and to +\( \infty \) where it is not. Direct calculations show that a minimizer \( u_0 \) of (2.13) will satisfy,

\[
\int_\Omega \nabla u_0 \cdot \nabla v \, dx = -\int_\Gamma f(u_0)v \, d\sigma_x \quad \text{for any } v \in H^1(\Omega), \tag{2.14}
\]

assuming \( u_0 \) is continuous (actually we will see that \( u_0 \) is a constant.)

**Theorem 2.2.1** (Existence and Uniqueness of the Minimizer). Let \( E_\epsilon \) be defined by (2.7), where \( \lambda, \alpha, \) and \( V \) satisfy (2.2)-(2.4). Then there exists a unique function \( u_\epsilon \in H^1(\Omega) \) satisfying

\[
E_\epsilon(u_\epsilon) = \min_{u \in H^1(\Omega)} E_\epsilon(u).
\]

**Proof.** Note that

\[
\frac{\partial^2}{\partial v^2} F(y, v) = \lambda(y)\alpha(y)e^{\alpha(y)(v-V(y))} + \lambda(y)(1-\alpha(y))e^{-(1-\alpha(y))(v-V(y))},
\]
since \( \lambda > 0, \alpha > 0, \) and \( 1 - \alpha > 0 \) we have that \( \frac{\partial^2}{\partial v^2} F > 0. \) Clearly the partial derivative is bounded below. That is, there exists a constant \( c_0, \) independent of \( y \) and \( v \) such that,
\[
\frac{\partial^2}{\partial v^2} F(y, v) \geq c_0 > 0.
\]
Since \( F \) is smooth in the second variable, for any \( v, w \in H^1(\Omega) \) and for any \( y, \) there exists some \( \xi \) between \( v + w \) and \( v - w \) such that
\[
F(y, v + w) + F(y, v - w) - 2F(y, v) = \frac{\partial^2}{\partial v^2} F(y, \xi) w^2
\]
which from the lower bound yields
\[
F\left(\frac{x}{\epsilon}, v + w\right) + F\left(\frac{x}{\epsilon}, v - w\right) - 2F\left(\frac{x}{\epsilon}, v\right) \geq c_0 w^2
\]
whence
\[
E_\epsilon(v + w) + E_\epsilon(v - w) - 2E_\epsilon(v) \geq \int_{\Omega} |\nabla w|^2 \, dx + c_0 \int_{\Gamma} w^2 d\sigma_x
\geq c_0 \|w\|_{H^1(\Omega)}^2 \tag{2.15}
\]
where the last inequality follows by a variant of Poincaré. Now let \( \{u^n_\epsilon\}_{n=1}^\infty \) be a minimizing sequence, that is
\[
E_\epsilon(u^n_\epsilon) \to \inf_{u \in H^1(\Omega)} E_\epsilon(u) \quad \text{as} \quad n \to \infty.
\]
Since all the terms of (2.7) are nonnegative, clearly
\[
\inf_{u \in H^1(\Omega)} E_\epsilon(u) > - \infty.
\]
Note that without loss of generality we can choose the minimizing sequence so that all terms have finite energy (since \( \inf_{u \in H^1(\Omega)} E_\epsilon(u) \leq E(0) \) and \( E(0) \) is bounded independently of \( \epsilon. \) Let
\[
v = \frac{u^n_\epsilon + u^m_\epsilon}{2}
\]
and
\[ w = \frac{u^n - u^m}{2} . \]

Then \( v + w = u^n \) and \( v - w = u^m \), so (2.15) implies
\[
E_\epsilon(v + w) + E_\epsilon(v - w) - 2E_\epsilon(v) \geq \frac{\tilde{c}_0}{4} \| u^n - u^m \|_{H^1(\Omega)}^2 .
\]

which implies,
\[
E_\epsilon(u^n) + E_\epsilon(u^m) - 2 \inf_{v \in H^1(\Omega)} E_\epsilon(v) \geq \frac{\tilde{c}_0}{4} \| u^n - u^m \|_{H^1(\Omega)}^2 .
\]

Now if we let \( m, n \to \infty \), we see that \( \{u^n\}_n \) is a Cauchy sequence in the Hilbert Space \( H^1(\Omega) \). Define \( u_\epsilon \) to be its limit in \( H^1(\Omega) \). Then we have
\[
u^n \to u_\epsilon \quad \text{in} \quad H^1(\Omega)
\]
which by the Trace Theorem implies,
\[
u^n \to u_\epsilon \quad \text{in} \quad L^2(\Gamma)
\]
which implies ([17], p.68) there exists a subsequence \( \{u^{n_k}\}_k \), which we label \( \{u^k\}_k \), such that
\[
u^k \to u_\epsilon \quad \text{a.e. in} \quad \Gamma.
\]

Since \( F \) is smooth in the second variable, and \( u^k \to u_\epsilon \) a.e. in \( \Gamma \) we have that
\[
F(\frac{x}{\epsilon}, u_\epsilon) = \lim_{k \to \infty} F(\frac{x}{\epsilon}, u^k) \quad \text{a.e.}.
\]

Now note that clearly \( F(\frac{x}{\epsilon}, u^k) > 0 \) for any \( k \). So, by Fatou’s Lemma we have,
\[
\int_{\Gamma} F(\frac{x}{\epsilon}, u_\epsilon) \, d\sigma_x \leq \liminf_{k \to \infty} \int_{\Gamma} F(\frac{x}{\epsilon}, u^k) \, d\sigma_x .
\]

Thus from this and the fact that \( u^k \to u_\epsilon \) in \( H^1(\Omega) \), we can conclude that,
\[
E_\epsilon(u_\epsilon) \leq \liminf_{k \to \infty} E_\epsilon(u^k) .
\]
\[
\begin{align*}
&= \lim_{k \to \infty} E_\varepsilon(u^k) \\
&= \inf_{u \in H^1(\Omega)} E_\varepsilon(u).
\end{align*}
\]

Hence,

\[
E_\varepsilon(u_\varepsilon) = \inf_{u \in H^1(\Omega)} E_\varepsilon(u).
\]

So we have shown the existence of a minimizer.

Suppose \( u_\varepsilon \) and \( w_\varepsilon \) are both minimizers of the energy functional, i.e.

\[
E_\varepsilon(u_\varepsilon) = \inf_{u \in H^1(\Omega)} E_\varepsilon(u) = E_\varepsilon(w_\varepsilon).
\]

Now if we let

\[
v = (u_\varepsilon + w_\varepsilon)/2
\]

and

\[
w = (u_\varepsilon - w_\varepsilon)/2
\]

then substituting \( v \) and \( w \) into (2.15) yields,

\[
E_\varepsilon(u_\varepsilon) + E_\varepsilon(w_\varepsilon) - 2E_\varepsilon\left(\frac{u_\varepsilon + w_\varepsilon}{2}\right) \geq \frac{\tilde{c}_0}{4}\|u_\varepsilon - w_\varepsilon\|^2_{H^1(\Omega)}.
\]

However, this implies,

\[
\frac{\tilde{c}_0}{4}\|u_\varepsilon - w_\varepsilon\|^2_{H^1(\Omega)} \leq E_\varepsilon(u_\varepsilon) + E_\varepsilon(w_\varepsilon) - 2\inf_{u \in H^1(\Omega)} E_\varepsilon(u) = 0.
\]

Hence \( u_\varepsilon = w_\varepsilon \) in \( H^1(\Omega) \). Thus we have shown the uniqueness of the minimizer.

Note that this argument can be generalized to address the \( n \)-dimensional problem, i.e. the case in which we have \( \Omega \subset R^n \), \( \Gamma \subset R^{n-1} \) with boundary period cell \( Y = [0,1]^{n-1} \). The existence and uniqueness of a minimizer \( u_0 \) of \( E_0 \) follows from the same proof.
Corollary 2.2.2. There exists a constant $C$, depending on $\Lambda_0, a_0, A_0$ and $V_0$ but independent of $\epsilon$ such that,

$$\|u_\epsilon\|_{H^1(\Omega)} \leq C$$

where $u_\epsilon$ is a weak solution to (2.5).

Proof. Consider the function $v \equiv 0$. Then

$$E_\epsilon(v) = E_\epsilon(0) = \int_{\Gamma} F(\frac{x}{\epsilon}, 0) d\sigma_x \leq M$$

for $M$ independent of $\epsilon$ (but depending on $\Lambda_0, a_0, A_0$ and $V_0$). Then since $u_\epsilon$ is a minimizer,

$$E_\epsilon(u_\epsilon) \leq E_\epsilon(0) \leq M.$$

Since both terms in $E_\epsilon$ are positive,

$$\|\nabla u_\epsilon\|_{L^2(\Omega)}^2 \leq M.$$

We also have that

$$\int_{\Gamma} F(\frac{x}{\epsilon}, u_\epsilon) d\sigma_x \leq M.$$

By examining the form of $F(y, v)$, we see that there exists some constant $d$, depending on $\Lambda_0, a_0$, and $A_0$ but independent of $\epsilon$ and $x$ such that

$$d|u_\epsilon - V(\frac{x}{\epsilon})| \leq F(\frac{x}{\epsilon}, u_\epsilon).$$

Hence,

$$\int_{\Gamma} |u_\epsilon - V(\frac{x}{\epsilon})| d\sigma_x \leq M/d,$$

which by the boundedness of $V$ implies that

$$\int_{\Gamma} |u_\epsilon| d\sigma_x \leq \tilde{M}.$$
where \( \tilde{M} \) is independent of \( \epsilon \). One variant of the Poincaré inequality says that there exists \( \hat{C} \) such that

\[
\| u_\epsilon - \int_\Gamma u_\epsilon \, d\sigma_x \|_{L^2(\Omega)} \leq \hat{C} \| \nabla u_\epsilon \|_{L^2(\Omega)}.
\]

Finally the reverse triangle inequality yields,

\[
\| u_\epsilon \|_{L^2(\Omega)} \leq \hat{C} \| \nabla u_\epsilon \|_{L^2(\Omega)} + \tilde{M},
\]

which proves the corollary.

We conclude this section by establishing the fact that the solution to the homogenized problem (2.6) is in a fact a constant. This fact follows easily once we have established the following lemma.

**Lemma 2.2.3.** There exists a constant \( K \) such that \( f_0(K) = 0 \).

**Proof.** Recall that

\[
f_0(v) = \int_Y f(y, v) \, dy
\]

(2.16)

where

\[
f(y, v) = \lambda(y) \left[ e^{\alpha(y)(v-V(y))} - e^{-(1-\alpha(y))(v-V(y))} \right]
\]

for any \( v \in \mathbb{R} \) and \( y \in Y \), where \( Y = [0, 1] \times [0, 1] \). Also recall that \( \lambda, \alpha, \text{ and } V \) are all piecewise smooth \( Y \)-periodic functions for which there exist constants \( \lambda_0, \Lambda_0, \alpha_0, A_0 \) and \( V_0 \) such that:

\[
0 < \lambda_0 \leq \lambda(y) \leq \Lambda_0, \quad (2.17)
\]

\[
0 < \alpha_0 \leq \alpha(y) \leq A_0 < 1, \quad (2.18)
\]

and

\[
|V(y)| \leq V_0. \quad (2.19)
\]

Now if \( v > V_0 \) then (2.19) and (2.18) imply

\[
\alpha(y)(v-V(y)) > 0 \text{ and } -(1-\alpha(y))(v-V(y)) < 0 \text{ for all } y \in Y.
\]
So we have
\[ e^{-(1-\alpha(y))(v-V(y))} < e^{\alpha(y)(v-V(y))} \] for all \( y \in Y \).

Thus (2.17), (2.18), and (2.19) imply that if \( v > V_0 \) then \( f(y,v) > 0 \) for all \( y \in Y \).

Similarly if \( v < -V_0 \) then \( f(y,v) < 0 \) for all \( y \in Y \). Now note that (2.16) implies that if \( v > V_0 \) then \( f_0(v) > 0 \) and if \( v < -V_0 \) then \( f_0(v) < 0 \). Now since \( f_0(v) \) is continuous by the Intermediate Value Theorem there exists a constant \( K \in (-V_0, V_0) \) such that \( f_0(K) = 0 \). Thus the lemma is established.

**Theorem 2.2.4.** Let \( u_0 \) be a minimizer of (2.6) then \( u_0 \) is a constant.

**Proof.** Suppose \( K \) is a constant such that \( f_0(K) = 0 \). Such a constant exists by Lemma 2.2.3. Then clearly \( u_0 = K \) is a strong solution of (2.6). Note that this argument and Lemma 2.2.3 hold in any dimension.

### 2.3 Regularity

We conclude this chapter with a short discussion of the regularity of the solutions \( u_\epsilon \) and \( u_0 \) for the two and three-dimensional case. For the two-dimensional case of this problem, i.e. when the medium is layered as in Turner and Hou [9], and Vogelius and Xu [19] (see Figure 2–3), using embeddings of Sobolev Spaces \((W^{m,p})\) into Orlicz Spaces \((L^\Phi)\) we can show that \( f(x_2/\epsilon, u_\epsilon) \) and \( f_0(u_0) \) are bounded in \( L^2(\Gamma) \) independently of \( \epsilon \).

We first give a general definition of an Orlicz Space before presenting the imbedding result. See [2] or [16] for a thorough discussion of Orlicz Spaces. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Let \( \Phi \) be a Young function, i.e. a real-valued, convex function such that \( \Phi(x) = \Phi(-x) \), \( \Phi(0) = 0 \), and \( \Phi \nearrow \infty \) as \( x \nearrow \infty \). Then the Orlicz space \( L^\Phi(\Omega) \) is the set of all measurable functions \( f \) on \( \Omega \) such that
\[
\int_\Omega \Phi(\alpha f) \, dx < \infty \text{ for some } \alpha > 0.
\]
Now define the norm

$$\|f\|_{L^s(\Omega)} = \inf \left\{ k > 0 : \int_{\Omega} \Phi\left(\frac{f}{k}\right) \, dx < 1 \right\}$$

then $L^s(\Omega)$ becomes a Banach space. For example when $\Phi(x) = |x|^p$ then $L^s = L^p$.

We now state the imbedding result established by [18] and [2].

**Theorem 2.3.1 (Trudinger’s Theorem).** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ satisfying the cone condition. Let $mp = n$ and $p > 1$. Set

$$\Phi(x) = e^{p/(p-1)} - 1.$$

Then there exists the imbedding

$$W^{m,p}(\Omega) \to L^s(\Omega).$$

In the two-dimensional case of (2.5) we can conclude from Trudinger’s Theorem [18], [19] that there exists a constant $C$ such that for any $v \in H^1(\Omega)$ and any real $\beta$ we have,

$$\int_{\Gamma} e^{\beta |v|} \, dx_2 \leq e^{C\beta^2(p\|v\|_{H^1(\Omega)}+1)}(|\Gamma| + 1).$$

Then from standard elliptic regularity theory this implies that $u_\epsilon$ and $u_0$ are in $H^{3/2}(\Omega)$, with the norm bounded independently of $\epsilon$. By the trace theorem we
then obtain bounds for $u_\epsilon$ and $u_0$ in $H^1(\Gamma)$. Since $\Gamma$ is one-dimensional it follows that $u_\epsilon$ and $u_0$ are continuous on $\Gamma$ and bounded pointwise, and their tangential derivatives are bounded in $L^2(\Gamma)$. For the homogenized solution we have much more regularity, $u_0$ is in fact the constant that satisfies $f_0(u_0) = 0$. For nonzero boundary conditions on the inactive region, $u_0$ would still be a smooth bounded function. So for the two-dimensional version of this problem we have the following lemma:

**Lemma 2.3.2.** If $\Omega \subset \mathbb{R}^2$ is a rectangle and $\Gamma$ is an edge, then $u_\epsilon \in C(\bar{\Omega})$ where $u_\epsilon$ is a weak solution of (2.5). Furthermore, there exists a constant $D$, the value of which does not depend on $\epsilon$, such that,

$$\|u_\epsilon(x)\|_{C(\bar{\Omega})} \leq D.$$ 

In the three-dimensional case of (2.5) since $u_\epsilon \in H^1(\Omega)$ implies $m = 1$, so $mp \neq n$ and thus Trudinger’s Theorem does not apply. In fact there is a large class of Sobolev imbedding theorems pertaining to the case $mp = n$, however there are no imbeddings of $H^1(\Omega)$ for a cylindrically shaped domain $\Omega \subset \mathbb{R}^3$ that are applicable to (2.5). In general, there has been no rigorous analysis of the regularity of three-dimensional solutions of elliptic boundary value problems with $L^1$ Neumann boundary data. Now if we assume that (2.5) models electrolytic voltage potential then it is physically reasonable to make some assumptions about the regularity of $u_\epsilon$ in the three-dimensional case. In the next chapter we show that if these physically justifiable assumptions are made we can establish convergence estimates for our asymptotic approximation for the three-dimensional case.
CHAPTER 3
A CORRECTOR BASED ON NEUMANN BOUNDARY DATA

3.1 A Neumann Boundary Condition

To show $u_\epsilon$ converges to $u_0$, we will add a correction term and prove estimates in terms of powers of $\epsilon$. The convergence of $u_\epsilon$ to $u_0$ when $n = 2$ will then easily follow from this. The same estimate holds when $n = 3$ if we know that the solutions are continuous and uniformly bounded. When $n = 2$ we will see that the convergence is strong in $H^1(\Omega)$ and of the order of $\sqrt{\epsilon}$.

Let $u_0$ be a minimizer of (2.13) and define the correction $u_\epsilon^{(1)}$ to satisfy,

\[
\begin{align*}
\Delta u_\epsilon^{(1)} &= 0 \text{ in } \Omega \\
- \frac{\partial u_\epsilon^{(1)}}{\partial n} &= \frac{1}{\epsilon} \left( \frac{x}{\epsilon}, u_0 \right) - f_0(u_0) + \epsilon \text{ on } \Gamma \\
- \frac{\partial u_\epsilon^{(1)}}{\partial n} &= 0 \text{ on } \partial \Omega \setminus \Gamma \\
\int_{\Gamma} u_\epsilon^{(1)} \, d\sigma_x &= 0
\end{align*}
\]

(3.1)

where,

\[ e_\epsilon = \frac{1}{\epsilon} \int_{\Gamma} \left( f_0(u_0) - f(y/\epsilon, u_0) \right) d\sigma_x. \]

Hence $e_\epsilon$ is chosen such that the solution always exists and the condition (3.2) guarantees this solution is unique. We note that given $u_0$, this is a linear problem.

Now if $u_\epsilon$ and $u_0$ are in $L^\infty(\Gamma)$, let

\[ D_\epsilon = \max \left\{ \|u_\epsilon\|_{L^\infty(\Gamma)}, \|u_0\|_{L^\infty(\Gamma)} \right\} \]

(3.3)

and let,

\[ M_\epsilon = \sup_{(y,w) \in Y \times [-D_\epsilon, D_\epsilon]} \frac{\partial f}{\partial v}(y,w). \]

(3.4)
The next estimate holds for dimension $n = 2$ or $3$ but depends on the constant $M_\epsilon$. We do not know a priori that $D_\epsilon$ is finite in general when $n = 3$. However, such an assumption seems to be physically reasonable and known to be the case when the medium is layered.

**Proposition 3.1.1.** Let $n = 2$ or $3$ and let $u_\epsilon, u_0$ be minimizers of (2.7), (2.13) respectively, and let $u_\epsilon^{(1)}$ be the solution to (3.1). Assume also that $u_\epsilon \in C^0(\bar{\Omega})$. Then there exists constants $C$ and $D$ independent of $\epsilon$ such that

$$
\| u_\epsilon - u_0 - \epsilon u_\epsilon^{(1)} \|_{H^1(\Omega)} \leq C\epsilon(M_\epsilon + D),
$$

where $M_\epsilon$ is defined by (3.4). Furthermore, there exists constants $C_1$ and $C_2$ independent of $\epsilon$ such that,

$$
\| u_\epsilon^{(1)} \|_{L^2(\Gamma)} \leq C_1 \text{ and } |e_\epsilon| \leq C_2.
$$

**Proof.** Let

$$
z_\epsilon = u_\epsilon - u_0 - \epsilon u_\epsilon^{(1)},
$$

since $u_\epsilon$ is continuous, by (2.12), we have that for any $v \in H^1(\Omega)$,

$$
\int_\Omega \nabla z_\epsilon \cdot \nabla v \ dx = \int_\Omega \nabla u_\epsilon \cdot \nabla v \ dx - \int_\Omega \nabla u_0 \cdot \nabla v \ dx - \epsilon \int_\Omega \nabla u_\epsilon^{(1)} \cdot \nabla v \ dx
$$

$$
= - \int_\Gamma f(\frac{x}{\epsilon}, u_\epsilon)vd\sigma_x + \int_\Gamma f(\frac{x}{\epsilon}, u_0)vd\sigma_x + \epsilon \int_\Gamma e_\epsilon vd\sigma_x.
$$

So,

$$
\int_\Omega \nabla z_\epsilon \cdot \nabla v \ dx + \int_\Gamma [f(\frac{x}{\epsilon}, u_\epsilon) - f(\frac{x}{\epsilon}, u_0)]vd\sigma_x - \epsilon \int_\Gamma e_\epsilon vd\sigma_x = 0.
$$

Now note that $u_0$ and $u_\epsilon$ are defined pointwise on $\Gamma$. So, by the Mean Value Theorem, for each fixed $\epsilon$ and $x \in \Gamma$ there exists $\xi_\epsilon^x$ between $u_0(x)$ and $u_\epsilon(x)$ such that,

$$
f(\frac{x}{\epsilon}, u_\epsilon) - f(\frac{x}{\epsilon}, u_0) = (u_\epsilon - u_0)\frac{\partial f}{\partial v}(\frac{x}{\epsilon}, \xi_\epsilon^x).$$
By subtracting and adding $\epsilon u^{(1)}_\epsilon$ we have,

$$f\left(\frac{x}{\epsilon},u_\epsilon\right) - f\left(\frac{x}{\epsilon},u_0\right) = z_\epsilon \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon},\xi_x^x\right) + \epsilon u^{(1)}_\epsilon \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon},\xi_x^x\right)$$

which, if we pick $v = z_\epsilon$, yields,

$$\int_\Omega |\nabla z_\epsilon|^2 \, dx + \int_\Gamma z_\epsilon \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon},\xi_x^x\right) \, d\sigma_x = -\epsilon \int_\Gamma u^{(1)}_\epsilon \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon},\xi_x^x\right) z_\epsilon \, d\sigma_x + \epsilon e_\epsilon \int_\Gamma z_\epsilon \, d\sigma_x.$$

Since $\frac{\partial f}{\partial v} \geq c_0$, this implies

$$\tilde{c}_0 \left\| z_\epsilon \right\|^2_{H^1(\Omega)} \leq \int_\Omega |\nabla z_\epsilon|^2 \, dx + \int_\Gamma z_\epsilon \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon},\xi_x^x\right) \, d\sigma_x = -\epsilon \int_\Gamma u^{(1)}_\epsilon \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon},\xi_x^x\right) z_\epsilon \, d\sigma_x + \epsilon e_\epsilon \int_\Gamma z_\epsilon \, d\sigma_x.$$

So by applying Hölders Inequality and then the Trace Theorem we have,

$$\tilde{c}_0 \left\| z_\epsilon \right\|^2_{H^1(\Omega)} \leq \epsilon \left\| \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon},\xi_x^x\right) \right\|_{L^\infty(\Gamma)} \left\| u^{(1)}_\epsilon \right\|_{L^2(\Gamma)} \left\| z_\epsilon \right\|_{L^2(\Gamma)} + \epsilon |e_\epsilon| \left\| \Gamma \right\|^{1/2} \left\| z_\epsilon \right\|_{L^2(\Gamma)} \left\| u^{(1)}_\epsilon \right\|_{L^2(\Gamma)}$$

Thus, we can write,

$$\left\| z_\epsilon \right\|_{H^1(\Omega)} \leq C \epsilon \left( \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon},\xi_x^x\right) \right)_{L^\infty(\Gamma)} \left\| u^{(1)}_\epsilon \right\|_{L^2(\Gamma)} + |e_\epsilon| \left\| \Gamma \right\|^{1/2} \left\| z_\epsilon \right\|_{H^1(\Omega)}.$$

(3.5)

Now recall for any $v$ we have,

$$\int_Y (f(y,v) - f_0(v)) \, dy = 0$$

so there exists a continuous $Y$-periodic function $r(y,v)$ such that

$$\Delta_y r(y,v) = f(y,v) - f_0(v), \forall v \in \mathbb{R}. \quad (3.6)$$

So we have,

$$e_\epsilon = \frac{1}{\epsilon} \int_\Gamma (f_0(u_0) - f\left(\frac{x}{\epsilon},u_0\right)) \, d\sigma_x$$

$$= \frac{-1}{\epsilon} \int_\Gamma \Delta_y r\left(\frac{x}{\epsilon},u_0\right) \, d\sigma_x.$$
\[= - \int_{\partial \Omega} \nabla_y r \left( \frac{x}{\epsilon}, u_0 \right) \cdot \nu \, ds_x \]

where the last equality is arrived at using integration by parts and the fact that the chain rule implies \( \frac{\partial r}{\partial y}(x/\epsilon, u_0) = \epsilon \frac{\partial r}{\partial x}(x/\epsilon, u_0) \). Note that the differential operators \( \nabla_y \) and \( \Delta_y \) are with respect to \( y \in Y \), that is, they are surface operators. Now since \( u_0 \) is bounded pointwise on \( \Gamma \) and since \( r(y, v) \) is a continuously differentiable \( Y \)-periodic function we have,

\[ e_\epsilon \leq C \quad (3.7) \]

where \( C \) is bounded independent of \( \epsilon \). Now we show that \( \|u_{\epsilon}^{(1)}\|_{L^2(\Gamma)} \) is similarly bounded. Let \( w_\epsilon \in H^1(\Omega) \) satisfy,

\[
\begin{align*}
\Delta w_\epsilon &= 0 \text{ in } \Omega \\
\frac{\partial w_\epsilon}{\partial n} &= u_{\epsilon}^{(1)} \text{ on } \Gamma \\
\frac{\partial w_\epsilon}{\partial n} &= 0 \text{ on } \partial \Omega \setminus \Gamma \\
\int_{\Gamma} w_\epsilon \, d\sigma_x &= 0
\end{align*}
\]

(3.8)

then,

\[
\int_{\Gamma} (u_{\epsilon}^{(1)})^2 d\sigma_x = \int_{\Gamma} u_{\epsilon}^{(1)} \frac{\partial w_\epsilon}{\partial n} d\sigma_x = \int_{\Omega} \nabla u_{\epsilon}^{(1)} \nabla w_\epsilon \, dx = \int_{\partial \Omega} \frac{\partial u_{\epsilon}^{(1)}}{\partial n} w_\epsilon d\sigma_x
\]

where the last two equalities follow from integration by parts. Now, since \( u_{\epsilon}^{(1)} \) satisfies (3.1), we have

\[
\int_{\partial \Omega} \frac{\partial u_{\epsilon}^{(1)}}{\partial n} w_\epsilon d\sigma_x = - \int_{\Gamma} \left[ \frac{f(x/\epsilon, u_0) - f_0(u_0)}{\epsilon} + e_\epsilon \right] w_\epsilon d\sigma_x
\]

\[
= - \frac{1}{\epsilon} \int_{\Gamma} \Delta_y r(x/\epsilon, u_0) w_\epsilon d\sigma_x - e_\epsilon \int_{\Gamma} w_\epsilon d\sigma_x
\]

\[
= - \frac{1}{\epsilon} \int_{\Gamma} \Delta_y r(x/\epsilon, u_0) w_\epsilon d\sigma_x
\]
where the second equality follows from (3.6) and the last equality holds since \( \int_{\Gamma} w_{\epsilon} d\sigma = 0 \). Now using the chain rule we can write

\[
\Delta_y r(x/\epsilon, u_0) = \epsilon^2 \Delta_x r(x/\epsilon, u_0),
\]

where \( \Delta_x \) is a surface Laplacian on \( \Gamma \). Thus, we have,

\[
\int_{\Gamma} (u^{(1)}_{\epsilon})^2 d\sigma_x = -\epsilon \int_{\Gamma} \Delta_x r(x/\epsilon, u_0) w_{\epsilon} d\sigma_x
\]

\[
= \epsilon \int_{\Gamma} \nabla_x r(x/\epsilon, u_0) \nabla w_{\epsilon} d\sigma_x - \epsilon \int_{\partial \Gamma} \frac{\partial_r}{\partial \nu} w_{\epsilon} ds_x, \tag{3.9}
\]

where \( \nu \) is the outward unit normal to \( \partial \Gamma \). Note that when \( n = 2 \), we use the last integral to represent endpoint evaluation. So, by H"older's Inequality,

\[
\epsilon \int_{\Gamma} \nabla_x r(x/\epsilon, u_0) \nabla w_{\epsilon} d\sigma_x - \epsilon \int_{\partial \Gamma} \frac{\partial_r}{\partial \nu} w_{\epsilon} ds_x \leq \epsilon \| \nabla_x r \|_{L^2(\Gamma)} \| \nabla w_{\epsilon} \|_{L^2(\Gamma)}
\]

\[
+ \epsilon \| \frac{\partial_r}{\partial \nu} \|_{L^2(\partial \Gamma)} \| w_{\epsilon} \|_{L^2(\partial \Gamma)}. \tag{3.10}
\]

Then by the Trace Theorem we have,

\[
\| w_{\epsilon} \|_{L^2(\partial \Gamma)} \leq C_1 \| w_{\epsilon} \|_{H^1(\Gamma)} \leq C_2 \| w_{\epsilon} \|_{H^{3/2}(\Omega)}. \tag{3.11}
\]

Similarly,

\[
\| \nabla w_{\epsilon} \|_{L^2(\Gamma)} \leq C_3 \| w_{\epsilon} \|_{H^1(\Gamma)} \leq C_4 \| w_{\epsilon} \|_{H^{3/2}(\Omega)}. \tag{3.12}
\]

Then (3.9), (3.10), (3.11) and (3.12) imply,

\[
\| u^{(1)}_{\epsilon} \|_{L^2(\Gamma)}^2 \leq \epsilon \left( \| \nabla_x r \|_{L^2(\Gamma)} + \| \frac{\partial_r}{\partial \nu} \|_{L^2(\partial \Gamma)} \right) \| w_{\epsilon} \|_{H^{3/2}(\Omega)}.
\]

Now since \( w_{\epsilon} \) satisfies (3.8) we have from standard elliptic regularity theory [10],

\[
\| w_{\epsilon} \|_{H^{3/2}(\Omega)} \leq C \| u^{(1)}_{\epsilon} \|_{L^2(\Gamma)}
\]

where \( C \) is independent of \( \epsilon \) and so we can write,

\[
\| u^{(1)}_{\epsilon} \|_{L^2(\Gamma)} \leq C \epsilon \left( \| \nabla_x r(x/\epsilon, u_0) \|_{L^2(\Gamma)} + \| \frac{\partial_r(x/\epsilon, u_0)}{\partial \nu} \|_{L^2(\partial \Gamma)} \right)
\]
\[ C \left( \| \nabla_y r(x/\epsilon, u_0) \|_{L^2(\Gamma)} + \| \frac{\partial_y r(x/\epsilon, u_0)}{\partial \nu} \|_{L^2(\partial \Gamma)} \right), \]

where the last equality follows from the chain rule. Consequently, since we have that \( u_0 \) is continuous on \( \Gamma \) and bounded pointwise and since \( r(y, v) \) is a continuously differentiable \( Y \)-periodic function we can conclude that

\[ \| u^{(1)}_\epsilon \|_{L^2(\Gamma)} \leq D \]  

where \( D \) is bounded independently of \( \epsilon \). Then (3.5), (3.7) and (3.13) imply the main result of the proposition:

\[ \| z_\epsilon \|_{H^1(\Omega)} \leq C\epsilon \left( \| \frac{\partial f}{\partial v} (x, \xi_\epsilon^n) \|_{L^\infty(\Gamma)} \| u^{(1)}_\epsilon \|_{L^2(\Gamma)} + |e_\epsilon| \right) \leq \tilde{C}\epsilon (M_\epsilon + \hat{D}), \]

where \( M_\epsilon \) is defined by (3.4).

Note that in light of Lemma 2.3.2, we can easily establish the following corollaries:

**Corollary 3.1.2.** When \( n = 2 \), i.e. for the case in which \( \Omega \subset \mathbb{R}^2, \Gamma \subset \mathbb{R} \) with boundary period cell \( Y = [0,1] \) there exists a constant \( C \) independent of \( \epsilon \) such that,

\[ \| u_\epsilon - u_0 - \epsilon u^{(1)}_\epsilon \|_{H^1(\Omega)} \leq C\epsilon. \]  

**Corollary 3.1.3.** When \( n = 2 \), for \( u_\epsilon \) the weak solution of (2.5), and \( u_0 \) the weak solution of (2.6), there exists a constant \( C \) independent of \( \epsilon \) such that,

\[ \| u_\epsilon - u_0 \|_{H^1(\Omega)} \leq C\sqrt{\epsilon}. \]  

Estimate (3.15) follows from the fact that,

\[ \| u^{(1)}_\epsilon \|_{H^2(\Omega)} \leq C \| \frac{\partial u^{(1)}_\epsilon}{\partial n} \|_{H^{-1/2}(\Gamma)} \leq C\epsilon^{-1/2}, \]

where the last inequality follows by interpolating between \( L^2(\Gamma) \) and \( H^1(\Gamma) \) (see [11], Section 11.5) and then using duality (as in [14]). Finally note that
estimate (3.14) also holds for $n = 3$ if we know that $D_\epsilon$ defined by (3.3) is uniformly bounded.

### 3.2 Finite Element Method Implementation

We wish to numerically observe the behaviour of the homogenized boundary value problems as a way to describe the behaviour of the current near the boundary. We use a finite element method approach to the 2-D problem. For the 2-D problem the domain $\Omega$ is a unit square and the boundary $\Gamma$ is the left side of the unit square, that is

$$\Gamma = \{(x_1, x_2) \in \Omega : x_1 = 1\}$$

(see Figure 2–3). In this case we impose a grid of points (called nodes) on the unit square and triangulate the domain, then introduce a finite set of piecewise continuous basis functions. Here we use standard finite element method “tent” functions. We impose a grid of node points on the unit square which are evenly spaced both on the $x$ and $y$-axes. We label the nodes starting at the origin and moving to the right. We label the nodes $P_i$, $i = 1, \ldots, m$, where $P_1 = (0, 0)$ and $P_m = (1, 1)$. Furthermore, if there are $N$ nodes on the axis (i.e. if the axis is divided into $N - 1$ pieces) then $m = N^2$ and $P_{N-1} = (1 - 1/N, 0)$, $P_N = (1, 0)$, $P_{N+1} = (0, 1/N)$, $P_{N+2} = (1/N, 1/N)$ etc. Once the node points have been established we triangulate the domain in a predetermined fashion. Note that the node points $P_1$, $P_2$, $P_{N+1}$ and $P_{N+2}$ form a square. We form one triangle by using as vertices the node points $P_1$, $P_2$ and $P_{N+1}$ and another triangle by the node points $P_2$, $P_{N+1}$ and $P_{N+2}$. Proceeding in this fashion we can form yet one more triangle by using as vertices the node points $P_2$, $P_3$ and $P_{N+2}$ and yet another by using $P_3$, $P_{N+2}$ and $P_{N+3}$. The entire domain can be triangulated in this fashion using the node points as vertices. Once the domain has been triangulated we introduce basis functions. Note that in the original problem (2.5) we attempt to
minimize the energy functional

\[ E_\varepsilon(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx + \int_\Gamma F(\frac{x}{\varepsilon}, v) d\sigma_x \]

over the space \( H^1(\Omega) \). However, numerically we minimize the energy functional over the space \( V_h \) where,

\[ V_h = \{ v \in C(\Omega) : v \text{ is a linear function when restricted to each triangle in } \Omega \} \]

Now we introduce a set of basis functions \( b_i(x_1, x_2), i = 1, \ldots, m \). We use the standard finite element method “tent” function, that is for each \( i \) define \( b_i \in V_h \) by,

\[ b_i(P_j) = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases} \quad \text{for } j = 1, \ldots, m. \]

So any function \( v \in V_h \) can be written as \( v(x) = \sum_{j=1}^{m} \eta_j b_j(x) \), where \( \eta_j = v(P_j) \).

Furthermore, we can reformulate the energy minimization problem in the following way, if we denote the minimizer of the energy functional as \( u_h \) then

\[ E_\varepsilon(u_h) = \min_{v \in V_h} E_\varepsilon(v) = \min_{\eta_1, \ldots, \eta_m} E_\varepsilon(\sum_{j=1}^{m} \eta_j b_j(x)). \]

So if \( \xi_1, \ldots, \xi_m \) minimizes the energy functional then we can write \( u_h = \sum_{j=1}^{m} \xi_j b_j(x) \).

Thus, by triangulating the domain and introducing basis functions of \( V_h \) we are able to discretize the problem. Now we wish to solve the energy minimization problem. Note that if we let \( v(x) = \sum_{j=1}^{m} \eta_j b_j(x) \) then the energy functional has the form,

\[ E_\varepsilon(\eta) = \frac{1}{2} \eta^T A \eta + \int_\Gamma F(\frac{x}{\varepsilon}, \sum_{j=1}^{m} \eta_j b_j(x)) d\sigma_x. \]

where \( A = (a_{ij}), 1 \leq i, j \leq m \) such that \( a_{ij} = \int_\Omega \nabla b_i \cdot \nabla b_j \, d\tilde{x} \) and \( \eta \) is the \( m \times 1 \) vector \( \eta = (\eta_1, \ldots, \eta_m)^T \). Hence we wish to minimize the energy \( E_\varepsilon(\eta) \) over all
$\eta \in \mathbb{R}^m$. Since we plan to use a gradient descent based optimization method to minimize the energy we are required to calculate the gradient of the energy as well.

For the homogenized problem (2.6) we wish to minimize the functional

$$E_0(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx + \int_{\Gamma} F_0(v) d\sigma_x$$

over the space $H^1(\Omega)$. Thus numerically we wish to minimize

$$E_0(\eta) = \frac{1}{2} \eta^T A \eta + \int_{\Gamma} F_0(\sum_{j=1}^{m} \eta_j b_j(x)) d\sigma_x.$$

over all $\eta \in \mathbb{R}^m$. To conclude this section, as an example, we include some detailed calculations to determine the boundary integral $\int_{\Gamma} F_0(u_h) dx_2$ explicitly as a function of $\xi$. Then we find the gradient of the boundary integral with respect to $\xi$.

Note that

$$\int_{\Gamma_i} F_0(u_h) dx_2 = \sum_{i=1}^{N-1} \int_{\Gamma_i} F_0(u_h) dx_2$$

where $\Gamma_i = \left\{ x_2 \in \Gamma \middle| \frac{i-1}{N-1} \leq x_2 \leq \frac{i}{N-1} \right\}$

and,

$$\int_{\Gamma_i} F_0(u_h) dx_2 = \int_{\Gamma_i} \int_Y F(y, u_h) dy dx_2 = \int_{\Gamma_i} \left\{ \int_Y \frac{\lambda}{\alpha} e^{\alpha(u_h-V)} + \frac{\lambda}{1-\alpha} e^{-(1-\alpha)(u_h-V)} dy \right\} dx_2.$$

In the two-dimensional case $Y = Y_1 \cup Y_2 = [0, 1/k] \cup (1/k, 1]$ and furthermore,

$$\lambda(y) = \begin{cases} 
\lambda_1, & \text{if } y \in Y_1 \\
\lambda_2, & \text{if } y \in Y_2 
\end{cases}, \quad
\alpha(y) = \begin{cases} 
\alpha_1, & \text{if } y \in Y_1 \\
\alpha_2, & \text{if } y \in Y_2 
\end{cases}, \quad \text{and } V(y) = \begin{cases} 
V_1, & \text{if } y \in Y_1 \\
V_2, & \text{if } y \in Y_2 
\end{cases}.$$

Hence,

$$\int_{\Gamma_i} F_0(u_h) dx_2 = \int_{\Gamma_i} \left\{ \frac{\lambda_1}{k\alpha_1} e^{\alpha_1(u_h-V_1)} + \frac{\lambda_1}{k(1-\alpha_1)} e^{-(1-\alpha_1)(u_h-V_1)} \\
+ \frac{(k-1)\lambda_2}{k\alpha_2} e^{\alpha_2(u_h-V_2)} + \frac{(k-1)\lambda_2}{k(1-\alpha_2)} e^{-(1-\alpha_2)(u_h-V_2)} \right\} dx_2.$$
Now, note that \( u_h = \sum_{j=1}^{N^2} \xi_j \varphi_j \) and \( u_h|_\Gamma = \sum_{i=1}^{N} \xi_i \varphi_{iN} \). Also note that

\[
\varphi_N|_\Gamma = \psi_N(x_2) = \begin{cases} 
- (N - 1)x_2 + 1, & \text{on } \Gamma_1 \\
0, & \text{otherwise}
\end{cases},
\]

and for \( i = 2, \ldots, N - 1 \) we have

\[
\varphi_{iN}|_\Gamma = \psi_{iN}(x_2) = \begin{cases} 
(N - 1)x_2 - (i - 2), & \text{on } \Gamma_{i-1} \\
-(N - 1)x_2 + i, & \text{on } \Gamma_i \\
0, & \text{otherwise}
\end{cases}
\]

and finally

\[
\varphi_{N^2}|_\Gamma = \psi_{N^2}(x_2) = \begin{cases} 
(N - 1)x_2 - (N - 2), & \text{on } \Gamma_{N-1} \\
0, & \text{otherwise}
\end{cases}.
\]

This implies \( u_h|_{\Gamma_i} = \xi_i \psi_{iN} + \xi_{(i+1)N} \psi_{(i+1)N} \) for \( i = 1, \ldots, N - 1 \). Whence,

\[
u_h|_{\Gamma_i} = \xi_i \psi_{iN} + \xi_{(i+1)N} \psi_{(i+1)N} = \xi_i (- (N - 1)x_2 + i) + \xi_{(i+1)N} ((N - 1)x_2 - (i - 1))
\]

\[
= (\xi_{(i+1)N} - \xi_i) ((N - 1)x_2 - i) + \xi_{(i+1)N} = G_i(x_2).
\]

So,

\[
u_h|_{\Gamma_i} = \begin{cases} 
(\xi_{(i+1)N} - \xi_i) ((N - 1)x_2 - i) + \xi_{(i+1)N}, & \text{if } \xi_i \neq \xi_{(i+1)N} \\
\xi_{(i+1)N}, & \text{if } \xi_i = \xi_{(i+1)N}
\end{cases}.
\]

Thus,

\[
\int_{\Gamma_i} F_0(u_h) dx_2 = \int_{\Gamma_i} \left\{ \frac{\lambda_1}{k \alpha_1} e^{\alpha_1(G_i(x_2) - V_1)} + \frac{\lambda_1}{k(1 - \alpha_1)} e^{-(1 - \alpha_1)(G_i(x_2) - V_1)} + \frac{(k - 1) \lambda_2}{k \alpha_2} e^{\alpha_2(G_i(x_2) - V_2)} + \frac{(k - 1) \lambda_2}{k(1 - \alpha_2)} e^{-(1 - \alpha_2)(G_i(x_2) - V_2)} \right\} dx_2.
\]
So, if \( \xi_{iN} \neq \xi_{(i+1)N} \) then,

\[
\int_{\Gamma_i} F_0(u_h) dx_2 = \int_{\Gamma_i} \left\{ \frac{\lambda_1}{k\alpha_1} e^{\alpha_1(\xi_{(i+1)N} - \xi_iN)((N-1)x_2-i)+\xi_{(i+1)N}-V_1)} \right. \\
+ \frac{\lambda_1}{k(1-\alpha_1)} e^{-(1-\alpha_1)(\xi_{(i+1)N} - \xi_iN)((N-1)x_2-i)+\xi_{(i+1)N}-V_1)} \\
+ \int_{\Gamma_i} \left\{ \frac{(k-1)\lambda_2}{k\alpha_2} e^{\alpha_2(\xi_{(i+1)N} - \xi_iN)((N-1)x_2-i)+\xi_{(i+1)N}-V_2)} \\
+ \frac{(k-1)\lambda_2}{k(1-\alpha_2)} e^{-(1-\alpha_2)(\xi_{(i+1)N} - \xi_iN)((N-1)x_2-i)+\xi_{(i+1)N}-V_2)} \right\} dx_2,
\]

and, if \( \xi_{iN} = \xi_{(i+1)N} \) then,

\[
\int_{\Gamma_i} F_0(u_h) dx_2 = \int_{\Gamma_i} \left\{ \frac{\lambda_1}{k\alpha_1} e^{\alpha_1(\xi_{(i+1)N} - V_1)} + \frac{\lambda_1}{k(1-\alpha_1)} e^{-(1-\alpha_1)(\xi_{(i+1)N} - V_1)} \right\} dx_2 \\
+ \int_{\Gamma_i} \left\{ \frac{(k-1)\lambda_2}{k\alpha_2} e^{\alpha_2(\xi_{(i+1)N} - V_2)} + \frac{(k-1)\lambda_2}{k(1-\alpha_2)} e^{-(1-\alpha_2)(\xi_{(i+1)N} - V_2)} \right\} dx_2.
\]

Recall \( \Gamma_i = \{ x_2 \in \Gamma \mid \frac{i-1}{N-1} \leq x_2 \leq \frac{i}{N-1} \} \), so if \( \xi_{iN} \neq \xi_{(i+1)N} \) then for \( i = 1, \ldots, N - 1 \) we have,

\[
\int_{\Gamma_i} F_0(u_h) dx_2 = \frac{\lambda_1}{k\alpha_1^2(N-1)} \left( e^{\alpha_1(\xi_{(i+1)N} - V_1)} - e^{\alpha_1(\xi_iN - V_1)} \right) \\
- \frac{\lambda_1}{k(1-\alpha_1)^2(N-1)} \left( e^{-(1-\alpha_1)(\xi_{(i+1)N} - V_1)} - e^{-(1-\alpha_1)(\xi_iN - V_1)} \right) \\
+ \frac{(k-1)\lambda_2}{k\alpha_2^2(N-1)} \left( e^{\alpha_2(\xi_{(i+1)N} - V_2)} - e^{\alpha_2(\xi_iN - V_2)} \right) \\
- \frac{(k-1)\lambda_2}{k(1-\alpha_2)^2(N-1)} \left( e^{-(1-\alpha_2)(\xi_{(i+1)N} - V_2)} - e^{-(1-\alpha_2)(\xi_iN - V_2)} \right),
\]

and, if \( \xi_{iN} = \xi_{(i+1)N} \) then for \( i = 1, \ldots, N - 1 \) we have,

\[
\int_{\Gamma_i} F_0(u_h) dx_2 = \frac{\lambda_1}{k\alpha_1(N-1)} e^{\alpha_1(\xi_{(i+1)N} - V_1)} + \frac{\lambda_1}{k(1-\alpha_1)(N-1)} e^{-(1-\alpha_1)(\xi_{(i+1)N} - V_1)} \\
+ \frac{(k-1)\lambda_2}{k\alpha_2(N-1)} e^{\alpha_2(\xi_{(i+1)N} - V_2)} + \frac{(k-1)\lambda_2}{k(1-\alpha_2)(N-1)} e^{-(1-\alpha_2)(\xi_{(i+1)N} - V_2)}.\]
Recall that \( u_h = \sum_{j=1}^{N^2} \xi_j \varphi_j \) so that \( E_0(u_h) = E_0(\xi_1, \ldots, \xi_{N^2}) \). But \( u_h|_{\Gamma_i} = \xi_i \psi_iN + \xi_{(i+1)N} \psi_{(i+1)N} \) for \( i = 1, \ldots, N - 1 \) implies that the boundary integral is a function of \( \xi_N, \xi_{2N}, \ldots, \xi_{(N-1)N}, \xi_{N^2} \) only, i.e. we can write \( \int_{\Gamma} F_0(u_h) = g(\xi_N, \xi_{2N}, \ldots, \xi_{(N-1)N}, \xi_{N^2}) \). So we see that the gradient of the boundary integral with respect to \( \xi \) is

\[
\nabla_\xi \int_{\Gamma} F_0(u_h) = \langle 0, \ldots, 0, \frac{\partial g}{\partial \xi_N}, 0, \ldots, 0, \frac{\partial g}{\partial \xi_{2N}}, 0, \ldots, 0, \frac{\partial g}{\partial \xi_{N^2}} \rangle.
\]

Now note that,
\[
\frac{\partial g}{\partial \xi_N} = \frac{\partial}{\partial \xi_N} \left\{ \int_{\Gamma_1} F_0(\xi_N, \xi_{2N}) dx_2 \right\},
\]
\[
\frac{\partial g}{\partial \xi_{iN}} = \frac{\partial}{\partial \xi_{iN}} \left\{ \int_{\Gamma_{i-1}} F_0(\xi_{(i-1)N}, \xi_{iN}) dx_2 + \int_{\Gamma_i} F_0(\xi_{iN}, \xi_{(i+1)N}) dx_2 \right\} \text{ for } i = 2, \ldots, N - 1,
\]
and
\[
\frac{\partial g}{\partial \xi_{N^2}} = \frac{\partial}{\partial \xi_{N^2}} \left\{ \int_{\Gamma_{N-1}} F_0(\xi_{(N-1)N}, \xi_{N^2}) dx_2 \right\}.
\]
Thus if \( \xi_N \neq \xi_{2N} \) then,
\[
\frac{\partial g}{\partial \xi_N} = \frac{\partial}{\partial \xi_N} \left\{ \int_{\Gamma_1} F_0(\xi_N, \xi_{2N}) dx_2 \right\} =
\frac{\lambda_1}{k \alpha_1^2 (N - 1)} \left( e^{\alpha_1 (\xi_{2N} - \xi_N)} - e^{\alpha_1 (\xi_N - \xi_N)} - \alpha_1 (\xi_{2N} - \xi_N) e^{\alpha_1 (\xi_N - \xi_N)} \right)
\frac{\lambda_1}{k (1 - \alpha_1)^2 (N - 1)} \left( e^{-(1 - \alpha_1) (\xi_N - \xi_N)} - e^{-(1 - \alpha_1) (\xi_{2N} - \xi_N)} + (1 - \alpha_1) (\xi_{2N} - \xi_N) e^{-(1 - \alpha_1) (\xi_{2N} - \xi_N)} \right)
\frac{(k - 1) \lambda_2}{k \alpha_2^2 (N - 1)} \left( e^{\alpha_2 (\xi_{2N} - \xi_N)} - e^{\alpha_2 (\xi_{2N} - \xi_N)} - \alpha_2 (\xi_{2N} - \xi_N) e^{\alpha_2 (\xi_{2N} - \xi_N)} \right)
\frac{(k - 1) \lambda_2}{k (1 - \alpha_2)^2 (N - 1)} \left( e^{-(1 - \alpha_2) (\xi_{2N} - \xi_N)} - e^{-(1 - \alpha_2) (\xi_{2N} - \xi_N)} + (1 - \alpha_2) (\xi_{2N} - \xi_N) e^{-(1 - \alpha_2) (\xi_{2N} - \xi_N)} \right),
\]

and if \( \xi_{(N-1)N} \neq \xi_{N^2} \) then,
\[
\frac{\partial g}{\partial \xi_{N^2}} = \frac{\partial}{\partial \xi_{N^2}} \left\{ \int_{\Gamma_{N-1}} F_0(\xi_{(N-1)N}, \xi_{N^2}) dx_2 \right\} =
\frac{\lambda_1}{k \alpha_1^2 (N - 1)} \left( e^{\alpha_1 (\xi_{(N-1)N} - \xi_{N^2})} - e^{\alpha_1 (\xi_{N^2} - \xi_{N^2})} + \alpha_1 (\xi_{(N-1)N} - \xi_{N^2}) e^{\alpha_1 (\xi_{N^2} - \xi_{N^2})} \right)
\frac{\lambda_1}{k (1 - \alpha_1)^2 (N - 1)} \left( e^{-(1 - \alpha_1) (\xi_{(N-1)N} - \xi_{N^2})} - e^{-(1 - \alpha_1) (\xi_{N^2} - \xi_{N^2})} + (1 - \alpha_1) (\xi_{N^2} - \xi_{N^2}) e^{-(1 - \alpha_1) (\xi_{N^2} - \xi_{N^2})} \right)
\frac{(k - 1) \lambda_2}{k \alpha_2^2 (N - 1)} \left( e^{\alpha_2 (\xi_{N^2} - \xi_{N^2})} - e^{\alpha_2 (\xi_{N^2} - \xi_{N^2})} - \alpha_2 (\xi_{N^2} - \xi_{N^2}) e^{\alpha_2 (\xi_{N^2} - \xi_{N^2})} \right)
\frac{(k - 1) \lambda_2}{k (1 - \alpha_2)^2 (N - 1)} \left( e^{-(1 - \alpha_2) (\xi_{N^2} - \xi_{N^2})} - e^{-(1 - \alpha_2) (\xi_{N^2} - \xi_{N^2})} + (1 - \alpha_2) (\xi_{N^2} - \xi_{N^2}) e^{-(1 - \alpha_2) (\xi_{N^2} - \xi_{N^2})} \right).
So if $\lambda$ and if $\lambda i \partial \xi$ and, if $\partial g k k k k\alpha k N\alpha N\alpha N\alpha N\alpha N\alpha N$ then for $i = 2, \ldots, N - 1$ we have,

\[
\frac{\partial g}{\partial \xi N} = \frac{\lambda_1}{2 k (N - 1)} (e^{\alpha_1 (\xi N - V_1)} - e^{-(1 - \alpha_1) (\xi N - V_1)}) + \frac{(k - 1) \lambda_2}{2 k (N - 1)} (e^{\alpha_2 (\xi N - V_2)} - e^{-(1 - \alpha_2) (\xi N - V_2)}),
\]

and if $\xi_{(N-1)N} = \xi_{N2}$ then,

\[
\frac{\partial g}{\partial \xi_{N2}} = \frac{\lambda_1}{2 k (N - 1)} (e^{\alpha_1 (\xi_{(N-1)N} - V_1)} - e^{-(1 - \alpha_1) (\xi_{(N-1)N} - V_1)}) + \frac{(k - 1) \lambda_2}{2 k (N - 1)} (e^{\alpha_2 (\xi_{(N-1)N} - V_2)} - e^{-(1 - \alpha_2) (\xi_{(N-1)N} - V_2)}).
\]

Now recall that for $i = 2, \ldots, N - 1$ we have,

\[
\frac{\partial g}{\partial \xi_{iN}} = \frac{\partial}{\partial \xi_{iN}} \left\{ \int_{\Gamma_{i-1}} F_0 (\xi_{(i-1)N}, \xi_{iN}) dx_2 + \int_{\Gamma_i} F_0 (\xi_{iN}, \xi_{(i+1)N}) dx_2 \right\}.
\]

So if $\xi_{(i-1)N} \neq \xi_{iN}$ then for $i = 2, \ldots, N - 1$ we have,

\[
\frac{\partial}{\partial \xi_{iN}} \left\{ \int_{\Gamma_{i-1}} F_0 (\xi_{(i-1)N}, \xi_{iN}) dx_2 \right\} =
\frac{k \lambda_1}{k (1 - \alpha_1)^2 (N - 1)} (\xi_{iN} - \xi_{(i-1)N})^2 (\xi_{(i-1)N} - \xi_{iN})^2
\]

and, if $\xi_{iN} \neq \xi_{(i+1)N}$ then for $i = 2, \ldots, N - 1$ we have,

\[
\frac{\partial}{\partial \xi_{iN}} \left\{ \int_{\Gamma_i} F_0 (\xi_{iN}, \xi_{(i+1)N}) dx_2 \right\} =
\frac{k \lambda_1}{k (1 - \alpha_1)^2 (N - 1)} (\xi_{(i+1)N} - \xi_{iN})^2 (\xi_{iN} - \xi_{(i+1)N})^2
\]
\[
\begin{align*}
\frac{(k - 1)\lambda_2}{k\alpha_2^2(N - 1)} (\xi_{i+1}N - \xi_iN)^2 & - \frac{(k - 1)\lambda_2}{k(1 - \alpha)^2(N - 1)} (\xi_{i+1}N - \xi_iN)^2 - 2e^{\alpha_1(\xi_{i+1}N - V_1)} - e^{(1 - \alpha_1)(\xi_{i+1}N - V_1)} \\
& + (k - 1)\lambda_2 (\xi_{i+1}N - \xi_iN)^2 - e^{\alpha_2(\xi_{i+1}N - V_2)} - e^{(1 - \alpha_2)(\xi_{i+1}N - V_2)}
\end{align*}
\]

Finally, if \(\xi_{(i-1)N} = \xi_{iN}\) then for \(i = 2, \ldots, N - 1\) we have

\[
\frac{\partial}{\partial \xi_{iN}} \left\{ \int_{\Gamma} \mathcal{F}_0(\xi_{i-1}N, \xi_{iN}) d\Gamma \right\} = \frac{\lambda_1}{2k(N - 1)} (e^{\alpha_1(\xi_{i-1}N - V_1)} - e^{(1 - \alpha_1)(\xi_{i-1}N - V_1)})
\]

\[+ \frac{(k - 1)\lambda_2}{2k(N - 1)} (e^{\alpha_2(\xi_{i-1}N - V_2)} - e^{(1 - \alpha_2)(\xi_{i-1}N - V_2)}),\]

and if \(\xi_{iN} = \xi_{(i+1)N}\) then for \(i = 2, \ldots, N - 1\) we have,

\[
\frac{\partial}{\partial \xi_{iN}} \left\{ \int_{\Gamma} \mathcal{F}_0(\xi_{iN}, \xi_{(i+1)N}) d\Gamma \right\} = \frac{\lambda_1}{2k(N - 1)} (e^{\alpha_1(\xi_{i+1}N - V_1)} - e^{(1 - \alpha_1)(\xi_{i+1}N - V_1)})
\]

\[+ \frac{(k - 1)\lambda_2}{2k(N - 1)} (e^{\alpha_2(\xi_{i+1}N - V_2)} - e^{(1 - \alpha_2)(\xi_{i+1}N - V_2)}).
\]

Once we are able to determine the energy and the gradient of the energy we can implement the optimization strategy developed in [8]. Coding was done in FORTRAN and the results of our implementation are discussed in the next section.

### 3.3 Numerical Results

Here we will both test the accuracy of our asymptotic expansion and observe the behavior of the current by performing numerical experiments in two dimensions. Note that for the two-dimensional problem the domain \(\Omega\) is a unit square and the boundary \(\Gamma\) is the right side of the unit square, that is

\[\Gamma = \{(x_1, x_2) \in \Omega : x_1 = 1\},\]

(see Figure 2–3). To compute solutions \(u_\epsilon, u_0, \) and \(u_\epsilon^{(1)}\), we use piecewise linear finite elements on a regular mesh. To avoid singularities within elements, we chose a grid which conforms to the medium. To perform the nonlinear minimization
(when solving for \( u_\varepsilon \)), we use a conjugate gradient descent based algorithm developed by Hager and Zhang, [8]. Note that the homogenized solution \( u_0 \) is simply a constant value here, which we can find by Newton’s Method. The correction, \( u_\varepsilon^{(1)} \), we compute using standard finite elements for a linear problem, again conforming to the media.

We perform these computations for \( \varepsilon = 1/5, \varepsilon = 1/11, \varepsilon = 1/25 \) and \( \varepsilon = 1/40 \). We use the following parameter values for our simulation: \( J_A = 1, J_C = 10, V_A = 0.5, V_C = 1.0, \alpha_{aa} = 0.5, \alpha_{ca} = 0.85 \), and \( Y = Y_A \cup Y_C \) where \( Y_A = [0, 1/3] \) and \( Y_C = [1/3, 1] \). Note that for the parameter values used in this implementation, we have \( u_0 = 0.9758 \). We have analytically shown that the estimates below hold for the case of layered media and wish to numerically verify these estimates:

\[
\|u_\varepsilon - u_0 - \varepsilon u_\varepsilon^{(1)}\|_{H^1(\Omega)} \leq C_1 \varepsilon \\
\|u_\varepsilon - u_0\|_{H^1(\Omega)} \leq C_2 \sqrt{\varepsilon}
\]

The results are summarized in Table 3–1. The estimates above are all bounded by a term of the form \( C\varepsilon^\alpha \). We estimate this exponent \( \alpha \) in the table below. Note that the numerical results in Table 3–1 are in compliance with the given estimates.

Table 3–1: Table of estimates over \( \Omega \) and convergence rates

| \( \varepsilon \) | 1/5 | 1/11 | 1/25 | 1/40 | \( \alpha \)
<table>
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<tr>
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<tbody>
<tr>
<td>( |u_\varepsilon - (u_0 + \varepsilon u_\varepsilon^{(1)})|_{H^1(\Omega)} )</td>
<td>.0189</td>
<td>.0090</td>
<td>.0040</td>
<td>.0025</td>
<td>.9699</td>
</tr>
<tr>
<td>( |u_\varepsilon - u_0|_{H^1(\Omega)} )</td>
<td>.0537</td>
<td>.0360</td>
<td>.0238</td>
<td>.0188</td>
<td>.5057</td>
</tr>
<tr>
<td>( |u_\varepsilon - u_0|_{L^2(\Omega)} )</td>
<td>.0063</td>
<td>.0027</td>
<td>.0011</td>
<td>.0007</td>
<td>1.0808</td>
</tr>
</tbody>
</table>

Table 3–2: Table of estimates over \( \Gamma \) and estimates of the gradient over \( \Gamma \)

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>1/5</th>
<th>1/11</th>
<th>1/25</th>
<th>1/40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( |u_\varepsilon - (u_0 + \varepsilon u_\varepsilon^{(1)})|_{L^2(\Gamma)} )</td>
<td>.0108</td>
<td>.0050</td>
<td>.0027</td>
<td>.0014</td>
</tr>
<tr>
<td>( |u_\varepsilon - u_0|_{L^2(\Gamma)} )</td>
<td>.0128</td>
<td>.0057</td>
<td>.0025</td>
<td>.0015</td>
</tr>
<tr>
<td>( |\nabla u_\varepsilon - \nabla (u_0 + \varepsilon u_\varepsilon^{(1)})|_{L^2(\Gamma)} )</td>
<td>.1027</td>
<td>.0710</td>
<td>.0475</td>
<td>.0377</td>
</tr>
<tr>
<td>( |\nabla u_\varepsilon - \nabla u_0|_{L^2(\Gamma)} )</td>
<td>.1235</td>
<td>.0817</td>
<td>.0536</td>
<td>.0422</td>
</tr>
</tbody>
</table>

In Figure 3–2 and Figure 3–3 we plot the “correct” and asymptotic approximation of the potential on \( \Omega \) when \( \varepsilon = 1/5 \). We see that the macroscopic behavior
is captured by the expansion. Figure 3–4 and Figure 3–5 show the same for \( \epsilon = 1/11 \). In Figures 3–1(a)-3–1(d) we can view the limiting behavior of \( u_\epsilon \) on \( \Gamma \) as \( \epsilon \) approaches 0. To examine the influence of the corrector term more closely, in Figures 3–6–3–9 we graph both the “correct” solution and the asymptotic expansion over \( \Gamma \) with material regions indicated. Note that the asymptotic approximation is not exact and in fact is slightly skewed. This is probably due to the linearization of the corrector term. In Figure 3–10 we graph the \( L^\infty \)-norm of \( \nabla u_\epsilon \) on the boundary for various values of \( \epsilon \). We see that according to our simulations of the layered media case, the current remains bounded as the perimeter becomes arbitrarily large, suggesting that the linear relation between current and perimeter observed in [12] may not hold for all geometries. Our results, however, do not directly contradict the observations made in [12], where the computations were done for a fixed number of anodes with a varying geometry. Furthermore, since the estimates here are merely in \( H^1(\Omega) \), pointwise estimates for the gradient (current) on the boundary do not follow.
Figure 3–1: Limiting behaviour of $u_\epsilon$ on $\Gamma$ as $\epsilon$ approaches zero for: (a) $\epsilon = 1/5$, (b) $\epsilon = 1/11$, (c) $\epsilon = 1/25$, (d) $\epsilon = 1/40$
Figure 3–2: $u_v, \epsilon = 1/5$

Figure 3–3: $u_0 + \epsilon u_v^{(1)}, \epsilon = 1/5$
Figure 3–4: $u_\epsilon, \epsilon = 1/11$

Figure 3–5: $u_0 + \epsilon u_\epsilon^{(1)}, \epsilon = 1/11$
Figure 3–6: The potential on the boundary $\Gamma$, $\epsilon = 1/5$
Figure 3-7: The potential on the boundary $\Gamma$, $\epsilon = 1/11$
Figure 3-8: The potential on the boundary $\Gamma$, $\epsilon = 1/25$
Figure 3–9: The potential on the boundary $\Gamma$, $\epsilon = 1/40$
Figure 3–10: $L^\infty$ norm of $\nabla u_\epsilon$ on $\Gamma$ as $\epsilon$ approaches 0.
CHAPTER 4
A CORRECTOR BASED ON ROBIN BOUNDARY DATA

4.1 A Robin Boundary Condition

Note that from Figures 3–6 through 3–9 we see that the corrector developed in the Chapter 3 is slightly shifted away from the original. While the shape of this approximation is good, the overall location of the approximation is poor. In this section we attempt to improve the model used to determine the corrector term $u^{(1)}_\epsilon$.

Thus instead of using (3.1), suppose the correction $u^{(1)}_\epsilon$ satisfies the Robin boundary condition problem

$$
\begin{align*}
\Delta u^{(1)}_\epsilon &= 0 \text{ in } \Omega \\
\frac{\partial u^{(1)}_\epsilon}{\partial n} &= \frac{1}{\epsilon} (f(x/\epsilon, u_0) - f_0(u_0)) + u^{(1)}_\epsilon \frac{\partial f}{\partial v}(x/\epsilon, u_0) \text{ on } \Gamma \\
-\frac{\partial u^{(1)}_\epsilon}{\partial n} &= 0 \text{ on } \partial \Omega \setminus \Gamma
\end{align*}
$$

(4.1)

Before we develop rigorous estimates or provide numerical data to verify that this approximation will be more accurate let us first intuitively motivate our reason for proposing (4.1). The original model utilized in Chapter 3 to determine $u^{(1)}_\epsilon$ requires only Neumann boundary data. By adding Dirichlet boundary data to the Neumann boundary data we hope that the resulting approximation will have good shape as well as location. Note that it is not a priori obvious that the term $u^{(1)}_\epsilon \frac{\partial f}{\partial v}(x/\epsilon, u_0)$ should be added to the boundary condition.

Assuming $u_\epsilon \approx u_0 + \epsilon u^{(1)}_\epsilon$ implies $u^{(1)}_\epsilon \approx (u_\epsilon - u_0)/\epsilon$ which motivates the boundary condition $(f(x/\epsilon, u_\epsilon) - f_0(u_0))/\epsilon$. Using $f(x/\epsilon, u_\epsilon) \approx f(x/\epsilon, u_0)$ motivates the Neumann boundary conditioned used in Chapter 3. Here using the Taylor
Series expansion of $f(y, v)$ in the variable $v$ about $u_0$, i.e. using

$$f(x/\epsilon, u_0) \approx f(x/\epsilon, u_0) + \frac{\partial f}{\partial v}(x/\epsilon, u_0)(u - u_0)$$

with $\epsilon u^{(1)}_\epsilon \approx (u - u_0)$ then yields the Robin boundary condition (4.1).

**Proposition 4.1.1.** Let $n = 2$ and let $u_\epsilon$, $u_0$ be minimizers of (2.7), (2.13) respectively, and let $u^{(1)}_\epsilon$ be the solution to (4.1). Then there exists a constant $C$ independent of $\epsilon$ such that

$$\|u_\epsilon - u_0 - \epsilon u^{(1)}_\epsilon\|_{H^1(\Omega)} \leq C\epsilon^2.$$  

**Proof.** Let

$$z_\epsilon = u_\epsilon - u_0 - \epsilon u^{(1)}_\epsilon,$$

since $u_\epsilon$ is continuous, by (2.12), we have that for any $v \in H^1(\Omega)$,

$$\int_{\Omega} \nabla z_\epsilon \cdot \nabla v \, dx = \int_{\Omega} \nabla u_\epsilon \cdot \nabla v \, dx - \int_{\Omega} \nabla u_0 \cdot \nabla v \, dx - \epsilon \int_{\Omega} \nabla u^{(1)}_\epsilon \cdot \nabla v \, dx$$

$$= -\int_{\Gamma} f(x/\epsilon, u_\epsilon)vd\sigma_x + \int_{\Gamma} f(x/\epsilon, u_0)vd\sigma_x + \epsilon \int_{\Gamma} \frac{\partial f}{\partial v}(x/\epsilon, u_0)u^{(1)}_\epsilon vd\sigma_x.$$

So,

$$\int_{\Omega} \nabla z_\epsilon \cdot \nabla v \, dx + \int_{\Gamma}[f(x/\epsilon, u_\epsilon) - f(x/\epsilon, u_0)]vd\sigma_x - \epsilon \int_{\Gamma} \frac{\partial f}{\partial v}(x/\epsilon, u_0)u^{(1)}_\epsilon vd\sigma_x = 0.$$

Now note that $u_0$ and $u_\epsilon$ are defined pointwise on $\Gamma$. So, by Taylors Theorem, using Lagrange's form of the remainder term we have that for each fixed $\epsilon$ and $x \in \Gamma$ there exists $\xi^\epsilon_x$ between $u_0(x)$ and $u_\epsilon(x)$ such that,

$$f(x/\epsilon, u_\epsilon) - f(x/\epsilon, u_0) = \frac{\partial f}{\partial v}(x/\epsilon, u_0)(u - u_0) + \frac{1}{2} \frac{\partial^2 f}{\partial v^2}(x/\epsilon, \xi^\epsilon_x)(u - u_0)^2.$$

By subtracting and adding $\epsilon u^{(1)}_\epsilon$ within the parentheses of the first term on the right hand side we have,

$$f(x/\epsilon, u_\epsilon) - f(x/\epsilon, u_0) = \frac{\partial f}{\partial v}(\epsilon, u_0)z_\epsilon + \frac{\partial f}{\partial v}(\epsilon, u_0)u^{(1)}_\epsilon + \frac{1}{2} \frac{\partial^2 f}{\partial v^2}(\epsilon, \xi^\epsilon_x)(u - u_0)^2.$$


Thus making the above substitution yields
\[
\int_{\Omega} \nabla z_{\epsilon} \cdot \nabla v \, dx + \int_{\Gamma} \frac{\partial f}{\partial v}(x/\epsilon, u_0) z_{\epsilon} \, d\sigma_x = -\frac{1}{2} \int_{\Gamma} \frac{\partial^2 f}{\partial v^2}(x/\epsilon, \xi^z_{\epsilon}) (u_\epsilon - u_0)^2 \, d\sigma_x.
\]
Now if we pick \( v = z_{\epsilon} \), this yields
\[
\int_{\Omega} |\nabla z_{\epsilon}|^2 \, dx + \int_{\Gamma} \frac{\partial f}{\partial v}(x/\epsilon, u_0) z_{\epsilon}^2 \, d\sigma_x = -\frac{1}{2} \int_{\Gamma} \frac{\partial^2 f}{\partial v^2}(x/\epsilon, \xi^z_{\epsilon}) (u_\epsilon - u_0)^2 z_{\epsilon} \, d\sigma_x.
\]
Since \( \frac{\partial f}{\partial v} \geq c_0 \), using a variant of Poincaré yields
\[
\tilde{c}_0 \| z_{\epsilon} \|^2_{H^1(\Omega)} \leq \int_{\Omega} |\nabla z_{\epsilon}|^2 \, dx + \int_{\Gamma} \frac{\partial^2 f}{\partial v^2}(x/\epsilon, \xi^z_{\epsilon}) (u_\epsilon - u_0)^2 z_{\epsilon} \, d\sigma_x.
\]
Now note that
\[
\left| \frac{\partial^2 f}{\partial v^2}(y, v) \right| = \left| \lambda(y)\alpha(y)^2 e^{\alpha(y)(v-V(y))} - (1-\alpha(y))^2 e^{-(1-\alpha(y))(v-V(y))} \right|
\leq \lambda(y)\alpha(y)^2 e^{\alpha(y)(v-V(y))} + (1-\alpha(y))^2 e^{-(1-\alpha(y))(v-V(y))}
\leq \frac{1}{2} \int_{\Gamma} \frac{\partial f}{\partial v}(x/\epsilon, \xi^z_{\epsilon}) (u_\epsilon - u_0)^2 z_{\epsilon} \, d\sigma_x.
\]
where the last inequality follows from the fact that \( 0 < \alpha(y) < 1 \) for all \( y \in Y \). So
\[
\tilde{c}_0 \| z_{\epsilon} \|^2_{H^1(\Omega)} \leq \frac{1}{2} \left\| \frac{\partial f}{\partial v}(x/\epsilon, \xi^z_{\epsilon}) \right\|_{L^\infty(\Gamma)} \int_{\Gamma} (u_\epsilon - u_0)^2 |z_{\epsilon}| \, d\sigma_x. \quad (4.2)
\]
Now note that when \( n = 2 \)
\[
\| u_\epsilon - u_0 \|_{L^\infty(\Gamma)} \leq C_1 \| u_\epsilon - u_0 \|_{H^1(\Gamma)} \leq C_2 \| u_\epsilon - u_0 \|_{H^{3/2}(\Omega)} \quad (4.3)
\]
where the first inequality follows from the Sobolev Imbedding Theorem [2] and the second inequality follows from the Trace Theorem. Now from standard elliptic regularity theory [10] we have
\[
\| u_\epsilon - u_0 \|_{H^{3/2}(\Omega)} \leq C \left\| \frac{\partial}{\partial n}(u_\epsilon - u_0) \right\|_{L^2(\Gamma)}. \quad (4.4)
\]
By the Mean Value Theorem, for each fixed $\epsilon$ and $x \in \Gamma$ there exists $\eta^x_\epsilon$ between $u_0(x)$ and $u_\epsilon(x)$ such that,

$$f\left(\frac{x}{\epsilon}, u_\epsilon\right) - f\left(\frac{x}{\epsilon}, u_0\right) = (u_\epsilon - u_0) \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \eta^x_\epsilon\right).$$

Now recall for any $v$ we have,

$$\int_Y (f(y, v) - f_0(v)) dy = 0$$

so there exists a continuous $Y$-periodic function $g(y, v)$ such that

$$\frac{\partial g}{\partial v}(y, v) = f(y, v) - f_0(v), \forall v \in R.$$ 

This implies

$$f(x/\epsilon, u_\epsilon) - f_0(u_0) = f(x/\epsilon, u_\epsilon) - f(x/\epsilon, u_0) + f(x/\epsilon, u_0) - f_0(u_0)$$

$$= (u_\epsilon - u_0) \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \eta^x_\epsilon\right) + \epsilon \frac{\partial g}{\partial x}(x/\epsilon, u_0).$$

So,

$$\left\| \frac{\partial}{\partial n}(u_\epsilon - u_0) \right\|_{L^2(\Gamma)} = \left\| f(x/\epsilon, u_\epsilon) - f_0(u_0) \right\|_{L^2(\Gamma)}$$

$$= \left\| (u_\epsilon - u_0) \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \eta^x_\epsilon\right) + \epsilon \frac{\partial g}{\partial x}(x/\epsilon, u_0) \right\|_{L^2(\Gamma)}$$

$$\leq \left\| (u_\epsilon - u_0) \frac{\partial f}{\partial v}\left(\frac{x}{\epsilon}, \eta^x_\epsilon\right) \right\|_{L^2(\Gamma)} + \epsilon \left\| \frac{\partial g}{\partial x}(x/\epsilon, u_0) \right\|_{L^2(\Gamma)}$$

$$\leq M_\epsilon \| u_\epsilon - u_0 \|_{L^2(\Gamma)} + \epsilon \left\| \frac{\partial g}{\partial x}(x/\epsilon, u_0) \right\|_{L^2(\Gamma)},$$

where $M_\epsilon$ is defined by (3.4). Note that if $\tilde{u}_\epsilon$ is a weak solution to (3.1) then

$$\| u_\epsilon - u_0 \|_{L^2(\Gamma)} = \| u_\epsilon - u_0 - \epsilon \tilde{u}_\epsilon + \epsilon \tilde{u}_\epsilon \|_{L^2(\Gamma)}$$

$$\leq \| u_\epsilon - u_0 - \epsilon \tilde{u}_\epsilon \|_{L^2(\Gamma)} + \epsilon \| \tilde{u}_\epsilon \|_{L^2(\Gamma)}$$

$$\leq C\epsilon(M_\epsilon + D) + \epsilon C_1$$
where the last inequality follows from Proposition 3.1.1. So in the two-dimensional case we have

$$\|u_\epsilon - u_0\|_{L^2(\Gamma)} \leq \tilde{C}\epsilon$$  \hspace{1cm} (4.6)

for some constant \(\tilde{C}\) independent of \(\epsilon\). Thus in the two-dimensional case (4.3), (4.4), (4.5) and (4.6) imply that

$$\|u_\epsilon - u_0\|_{L^\infty(\Gamma)} \leq C\epsilon$$  \hspace{1cm} (4.7)

for some constant \(C\) independent of \(\epsilon\). Then clearly

$$(u_\epsilon - u_0)^2 \leq \|u_\epsilon - u_0\|_{L^\infty(\Gamma)}^2 \leq (C\epsilon)^2.$$  

So applying the above estimate to (4.2) yields,

$$\tilde{c}_0\|\tilde{z}_\epsilon\|_{H^1(\Omega)}^2 \leq \frac{(C\epsilon)^2}{2}\|\frac{\partial f}{\partial v}(\frac{x}{\epsilon}, \xi_\epsilon)\|_{L^\infty(\Gamma)}\|\tilde{z}_\epsilon\|_{L^2(\Gamma)} \leq \tilde{C}\epsilon^2\|\tilde{z}_\epsilon\|_{L^2(\Gamma)} \leq \tilde{C}\epsilon^2\|\tilde{z}_\epsilon\|_{H^1(\Omega)}.$$  

where the last inequality follows from the Trace Theorem. Thus, we can write,

$$\|\tilde{z}_\epsilon\|_{H^1(\Omega)} \leq C\epsilon^2$$  

where \(C\) is independent of \(\epsilon\) and so the proposition is proved.

4.2 Numerical Results

We use the same parameter values as before and utilize the same finite element based approach as outline in Chapter 3 to discretize the model. The linear problem (4.1) can be solved directly in MATLAB. We provide graphs of the approximation as a three-dimensional function in Figures 4–2 and 4–4. In Figures 4–5, 4–7, 4–10 and 4–12 we graph the new approximation on the active boundary. In Figures 4–6, and 4–8 we graph both the approximation and the
“correct” solution on the boundary. As we see from the graphs and Table 4–1 this approximation is much more accurate than the approximation used in Chapter 3. Using a corrector based on Robin boundary data yields a substantially more accurate approximation without becoming numerically cumbersome. The new corrector is a substantial improvement over the corrector used in Chapter 3.

Table 4–1: Table of estimates

<table>
<thead>
<tr>
<th>s</th>
<th>1/2</th>
<th>1/3</th>
<th>1/4</th>
<th>1/5</th>
<th>1/6</th>
</tr>
</thead>
<tbody>
<tr>
<td>|u_ε - (u_0 + \varepsilon u^{(1)})|_{L^\infty(\Gamma)}</td>
<td>0.0119</td>
<td>0.0113</td>
<td>0.0009</td>
<td>0.0007</td>
<td>0.0004</td>
</tr>
<tr>
<td>|u_ε - (u_0 + \varepsilon u^{(1)})|_{L^2(\Gamma)}</td>
<td>0.5729e-4</td>
<td>0.2908e-4</td>
<td>0.1791e-4</td>
<td>0.1212e-4</td>
<td>0.0736e-4</td>
</tr>
<tr>
<td>|u_ε - (u_0 + \varepsilon u^{(1)})|_{L^2(\Omega)}</td>
<td>0.5254e-4</td>
<td>0.2427e-4</td>
<td>0.1290e-4</td>
<td>0.0720e-4</td>
<td>0.0417e-4</td>
</tr>
<tr>
<td>|u_ε - (u_0 + \varepsilon u^{(1)})|_{\mathcal{H}^1(\Omega)}</td>
<td>0.8406e-4</td>
<td>0.4420e-4</td>
<td>0.2877e-4</td>
<td>0.2124e-4</td>
<td>0.2104e-4</td>
</tr>
</tbody>
</table>
Figure 4–1: $u_\varepsilon, \varepsilon = 1/5$

Figure 4–2: $u_0 + \varepsilon u_\varepsilon^{(1)}, \varepsilon = 1/5$
Figure 4–3: \( u_\varepsilon, \varepsilon = 1/11 \)

Figure 4–4: \( u_0 + \varepsilon u^{(1)}_\varepsilon, \varepsilon = 1/11 \)
Figure 4–5: Graph of $u_\epsilon$ (above) and $u_0 + u_\epsilon^{(1)}$ (below) on the boundary $\Gamma$, $\epsilon = 1/5$
Figure 4–6: The approximation and the original, $\epsilon = 1/5$
Figure 4–7: Graph of $u_\epsilon$ (above) and $u_0 + u_\epsilon^{(1)}$ (below) on the boundary $\Gamma$, $\epsilon = 1/11$
Figure 4–8: The approximation and the original, $\epsilon = 1/11$
Figure 4–9: Graph of $u_\varepsilon$ on the boundary $\Gamma$, $\varepsilon = 1/25$
Figure 4–10: Graph of $u_0 + u^{(1)}_\epsilon$ on the boundary $\Gamma$, $\epsilon = 1/25$
Figure 4–11: Graph of $u_\epsilon$ on the boundary $\Gamma$, $\epsilon = 1/40$
Figure 4–12: Graph of $u_0 + u^{(1)}_\epsilon$ on the boundary $\Gamma$, $\epsilon = 1/40$
CHAPTER 5
SHifting材料 boundaries

5.1 The Electrostatic Conductivity Model

We wish to utilize asymptotic expansions to approximate the solution of a linear elliptic boundary value problem over a two-dimensional domain with shifting material boundaries. This is a problem that pertains to electrostatic conductivity and has applications to photonic bandgap (PBG) optical materials. We consider a PDE which models the steady state voltage potential of a conductor with a small inhomogeneity in which there is a discontinuity in the conductivity across the boundary of the inhomogeneity. The boundary of the inhomogeneity shifts by some small amount \( h \) (Figure 5–1). The shift results in a new steady state voltage potential for the conductor. Let \( u_0 \) be the solution to the boundary value problem

\[
\nabla \cdot (\sigma_0 \nabla u_0) = 0 \text{ on } \Omega \\
\sigma_0 \frac{\partial u_0}{\partial n} = g \text{ on } \partial \Omega
\]

(5.1)

where \( \Omega \subset \mathbb{R}^2 \), and \( g \in L^2(\Omega) \) and

\[
\sigma_0(x) = \begin{cases} 
\sigma_1(x), & \text{if } x \in D \\
\sigma_2(x), & \text{if } x \in \Omega \setminus D
\end{cases}
\]

where \( D \) is the region of the small inhomogeneity and \( \sigma_1, \sigma_2 > 0 \). Let \( u_h \) be the solution to the problem with shifted boundaries, that is \( u_h \) is a solution to

\[
\nabla \cdot (\sigma_h \nabla u_h) = 0 \text{ on } \Omega \\
\sigma_h \frac{\partial u_h}{\partial n} = g \text{ on } \partial \Omega.
\]

(5.2)

Here,
\[ \sigma_h(x) = \begin{cases} 
\sigma_1(x), & \text{if } x \in D_h \\
\sigma_2(x), & \text{if } x \in \Omega \setminus D_h 
\end{cases} \]

where \( D_h \) is the region of the small inhomogeneity but with shifted boundary.

Here we assume \( D \subset D_h \). The steady state voltage potential of the conductor with shifted inhomogeneity is viewed as a perturbation of the original steady state voltage potential. We wish to establish an estimate and do formal asymptotics.

![Figure 5–1: Perturbation due to shifting between two dielectrics \( \sigma_1 \) and \( \sigma_2 \).](image)

### 5.2 Estimating the \( H^1(\Omega) \) norm of \( u_0 - u_h \)

We conclude this chapter with an estimate. We use energy methods to rigorously develop an estimate characterizing the limiting behaviour of \( u_h \). First we must establish a lemma.

**Lemma 5.2.1.** Let \( C = 1 / \min\{\sigma_1, \sigma_2\} \) then for \( \epsilon > C/4 \) we have

\[
\| \nabla (u_0 - u_h) \|_{L^2(\Omega)} \leq \left( \frac{4C\epsilon^2(\sigma_2 - \sigma_1)^2}{4\epsilon - C} \right) \| \nabla u_0 \|_{L^2(D_h \setminus D)}.
\]

**Proof.** For \( v \in H^1(\Omega) \), let \( E_h(v) \) be the energy defined by

\[
E_h(v) = \frac{1}{2} \int_{\Omega} \sigma_h |\nabla v|^2 \, dx - \int_{\partial\Omega} gv \, dx,
\]
then

\[ E_h(u_0) = \frac{1}{2} \int_\Omega \sigma_h |\nabla u_0|^2 \, dx - \int_{\partial \Omega} g u_0 \, dx. \]

Thus we have

\[
\int_\Omega \sigma_h |\nabla (u_0 - u_h)|^2 \, dx = \int_\Omega \sigma_h |\nabla u_0|^2 \, dx - 2 \int_\Omega \sigma_h (\nabla u_h \cdot \nabla u_0) \, dx + \int_\Omega \sigma_h |\nabla u_h|^2 \, dx
\]

\[
= \int_\Omega \sigma_h |\nabla u_0|^2 \, dx - 2 \int_{\partial \Omega} g u_0 \, dx + \int_\Omega \sigma_h |\nabla u_h|^2 \, dx
\]

\[
= 2E_h(u_0) + \int_\Omega \sigma_h |\nabla u_h|^2 \, dx \tag{5.3}
\]

where we used the fact that the variational form of (5.2) implies

\[
\int_\Omega \sigma_h (\nabla u_h \cdot \nabla u_0) \, dx = \int_{\partial \Omega} g u_0 \, dx.
\]

Now note that the variational form of (5.1) implies that

\[
\int_{\partial \Omega} g u_0 \, d\sigma = \int_\Omega \sigma_0 |\nabla u_0|^2 \, dx
\]

and the definition of \( \sigma_h \) implies

\[
\int_\Omega \sigma_h |\nabla u_0|^2 \, dx = \int_{\Omega \setminus D_h} \sigma_2 |\nabla u_0|^2 \, dx + \int_{D \setminus D_h} \sigma_1 |\nabla u_0|^2 \, dx + \int_{D_h \setminus D} \sigma_1 |\nabla u_0|^2 \, dx
\]

and similarly the definition of \( \sigma_0 \) implies

\[
\int_\Omega \sigma_0 |\nabla u_0|^2 \, dx = \int_{\Omega \setminus D_h} \sigma_2 |\nabla u_0|^2 \, dx + \int_{D \setminus D_h} \sigma_1 |\nabla u_0|^2 \, dx + \int_{D_h \setminus D} \sigma_2 |\nabla u_0|^2 \, dx. \tag{5.4}
\]

So

\[
E_h(u_0) = \frac{1}{2} \int_\Omega \sigma_h |\nabla u_0|^2 \, dx - \int_\Omega \sigma_0 |\nabla u_0|^2 \, dx
\]

\[
= \frac{1}{2} \left( \int_{\Omega \setminus D_h} \sigma_2 |\nabla u_0|^2 \, dx + \int_{D \setminus D_h} \sigma_1 |\nabla u_0|^2 \, dx + \int_{D_h \setminus D} \sigma_1 |\nabla u_0|^2 \, dx \right)
\]

\[
- \left( \int_{\Omega \setminus D_h} \sigma_2 |\nabla u_0|^2 \, dx + \int_{D \setminus D_h} \sigma_1 |\nabla u_0|^2 \, dx + \int_{D_h \setminus D} \sigma_2 |\nabla u_0|^2 \, dx \right)
\]

\[
= -\frac{1}{2} \int_{\Omega \setminus D_h} \sigma_2 |\nabla u_0|^2 \, dx - \frac{1}{2} \int_{D \setminus D_h} \sigma_1 |\nabla u_0|^2 \, dx + \int_{D_h \setminus D} (\frac{1}{2} \sigma_1 - \sigma_2) |\nabla u_0|^2 \, dx.
\]
Equation (5.4) then implies

\[
E_h(u_0) = -\frac{1}{2} \int_\Omega |\nabla u_0|^2 \, dx + \frac{1}{2} \int_{D_h \setminus D} \sigma_0 |\nabla u_0|^2 \, dx \quad \frac{1}{2} \int_{D_h \setminus D} (\sigma_1 - \sigma_2) |\nabla u_0|^2 \, dx
\]

\[
= -\frac{1}{2} \int_\Omega |\nabla u_0|^2 \, dx + \int_{D_h \setminus D} \frac{1}{2}(\sigma_1 - \sigma_2) |\nabla u_0|^2 \, dx.
\]

So

\[
2E_h(u_0) + \int_\Omega \sigma_h |\nabla u_h|^2 \, dx = \int_\Omega \sigma_h |\nabla u_h|^2 \, dx - \int_\Omega \sigma_0 |\nabla u_0|^2 \, dx + \int_{D_h \setminus D} (\sigma_1 - \sigma_2) |\nabla u_0|^2 \, dx. \tag{5.5}
\]

Now the variational form of (5.1) and (5.2) imply

\[
\int_\Omega \sigma_0 |\nabla u_0|^2 \, dx = \int_\Omega \sigma_0 (\nabla u_0 \cdot \nabla u_0) \, dx,
\]

and

\[
\int_\Omega \sigma_h |\nabla u_h|^2 \, dx = \int_\Omega \sigma_0 (\nabla u_0 \cdot \nabla u_h) \, dx
\]

so we have

\[
\int_\Omega (\sigma_h |\nabla u_h|^2 - \sigma_0 |\nabla u_0|^2) \, dx = \int_{D_h \setminus D} (\sigma_2 - \sigma_1) \nabla u_0 \cdot \nabla u_h \, dx. \tag{5.6}
\]

So by (5.3), (5.5) and (5.6) we have that

\[
\int_\Omega \sigma_h |\nabla (u_0 - u_h)|^2 \, dx = 2E_h(u_0) + \int_\Omega \sigma_h |\nabla u_h|^2 \, dx
\]

\[
= \int_{D_h \setminus D} (\sigma_2 - \sigma_1) \nabla u_0 \cdot \nabla u_h \, dx + \int_{D_h \setminus D} (\sigma_1 - \sigma_2) |\nabla u_0|^2 \, dx.
\]

Thus

\[
\int_\Omega \sigma_h |\nabla (u_0 - u_h)|^2 \, dx = \int_{D_h \setminus D} (\sigma_2 - \sigma_1) \nabla u_0 \cdot \nabla u_h \, dx - \int_{D_h \setminus D} (\sigma_2 - \sigma_1) |\nabla u_0|^2 \, dx
\]

\[
= \int_{D_h \setminus D} (\sigma_2 - \sigma_1) (\nabla u_h - \nabla u_0) \cdot \nabla u_0 \, dx.
\]
So, we have
\[
\int_{\Omega} \sigma |\nabla (u_0 - u_h)|^2 \, dx = \int_{\mathcal{D}} (\sigma_2 - \sigma_1) \nabla (u_h - u_0) \cdot \nabla u_0 \, dx \\
\leq \int_{\mathcal{D}} \frac{1}{4\epsilon} |\nabla (u_h - u_0)|^2 \, dx + \int_{\mathcal{D}} \epsilon (\sigma_2 - \sigma_1)^2 |\nabla u_0|^2 \, dx \\
\leq \int_{\Omega} \frac{1}{4\epsilon} |\nabla (u_h - u_0)|^2 \, dx + \int_{\mathcal{D}} \epsilon (\sigma_2 - \sigma_1)^2 |\nabla u_0|^2 \, dx
\]
thus \(C = 1 / \min \{\sigma_1, \sigma_2\}\) implies
\[
\int_{\Omega} |\nabla (u_0 - u_h)|^2 \, dx \leq C \int_{\Omega} \sigma |\nabla (u_0 - u_h)|^2 \, dx \\
\leq \int_{\Omega} \frac{C}{4\epsilon} |\nabla (u_h - u_0)|^2 \, dx + \int_{\mathcal{D}} \epsilon (\sigma_2 - \sigma_1)^2 |\nabla u_0|^2 \, dx.
\]
Hence
\[
(1 - \frac{C}{4\epsilon}) \int_{\Omega} |\nabla (u_0 - u_h)|^2 \, dx \leq \int_{\mathcal{D}} \epsilon (\sigma_2 - \sigma_1)^2 |\nabla u_0|^2 \, dx
\]
which implies
\[
\int_{\Omega} |\nabla (u_0 - u_h)|^2 \, dx \leq \left(\frac{4\epsilon}{4\epsilon - C}\right) C \epsilon (\sigma_2 - \sigma_1)^2 \int_{\mathcal{D}} |\nabla u_0|^2 \, dx
\]
and so
\[
\|\nabla (u_0 - u_h)\|_{L^2(\Omega)} \leq \sqrt{\frac{4C\epsilon^2(\sigma_2 - \sigma_1)^2}{4\epsilon - C}} \|\nabla u_0\|_{L^2(\mathcal{D})}
\]
and thus the lemma is proved.

We now establish an estimate for the \(H^1(\Omega)\) norm of \(u_0 - u_h\).

**Proposition 5.2.2.** Let \(C = 1 / \min \{\sigma_1, \sigma_2\}\) and suppose \(\epsilon > C/4\) then there exists a constant \(K(\sigma_1, \sigma_2)\), independent of \(h\), such that
\[
\int_{\Omega} \left(|u_0 - u_h|^2 + |\nabla (u_0 - u_h)|^2\right) \, dx \leq K(\sigma_1, \sigma_2) h.
\]

**Proof.** By a variant of Poincaré we have
\[
\int_{\Omega} |u_0 - u_h|^2 \, dx \leq \tilde{C}_1 \left(\int_{\Omega} |\nabla (u_0 - u_h)|^2 \, dx + \int_{\partial \Omega} |u_0 - u_h|^2 \, d\sigma\right)
\]
where the last equality follows from the fact that $u_0 = u_h$ on $\partial \Omega$. Thus it suffices to show

$$\int_{\Omega} |\nabla (u_0 - u_h)|^2 \, dx \leq \tilde{C}_2 h.$$  

Now by Lemma 5.2.1 we have that

$$\|\nabla (u_0 - u_h)\|_{L^2(\Omega)} \leq \sqrt{\frac{4C\varepsilon^2(\sigma_2 - \sigma_1)^2}{4\varepsilon - C}} \|\nabla u_0\|_{L^2(D_h \setminus D)}.$$  

Note that due to elliptic regularity $|\nabla u_0|$ is uniformly bounded on $D_h \setminus D$ thus

$$\|\nabla u_0\|_{L^2(D_h \setminus D)} \leq \tilde{C}_3 |D_h \setminus D|^{1/2}$$

and note that there exists some positive real number $\alpha$, independent of $h$, such that $|D_h \setminus D| \leq \alpha h$ for all $h$ as $h \to 0$. So it follows that

$$\|\nabla u_0\|_{L^2(D_h \setminus D)} \leq \tilde{C}_3 \sqrt{\alpha h}. \tag{5.9}$$

Define

$$K(\sigma_1, \sigma_2) = (\tilde{C}_1 + 1)\tilde{C}_3 \alpha \frac{4C\varepsilon^2(\sigma_2 - \sigma_1)^2}{4\varepsilon - C}$$

then (5.7), (5.8), and (5.9) imply that

$$\int_{\Omega} (|u_0 - u_h|^2 + |\nabla (u_0 - u_h)|^2) \, dx \leq K(\sigma_1, \sigma_2)h$$

and thus the proposition is proved. \hfill \square

### 5.3 Formal Asymptotics

We conclude this chapter with some formal asymptotics. We attempt to characterize the new steady state voltage potential by developing an asymptotic expansion in terms of the shift $h$ using integral equation asymptotics and Green’s function. Note that the shifting boundary also causes a perturbation in the conductivity of the conductor. The asymptotic expansion must be developed in
such a way as to address this discontinuity. Let $\delta_x(x)$ be the Dirac delta function centered at $z$, and let $N(x, z)$ be the solution to

$$
-\nabla_x \cdot \sigma_0(x) \nabla_x N(x, z) = \delta_x(x) \text{ in } \Omega
$$

$$
\sigma_2 \frac{\partial N(x, z)}{\partial n_x} = -\frac{1}{|\partial \Omega|} \text{ on } \partial \Omega.
$$

Then

$$
u_0(z) = \int_\Omega u_0(x) \delta_x(x) \, dx
$$

$$
= -\int_\Omega u_0(x) (\nabla_x \cdot \sigma_0(x) \nabla_x N(x, z)) \, dx
$$

$$
= \int_\Omega \nabla_x u_0(x) \cdot \sigma_0(x) \nabla_x N(x, z) \, dx - \int_{\partial \Omega} \sigma_2 u_0 \frac{\partial N}{\partial n_x} \, d\sigma_x
$$

$$
= \int_\Omega \nabla_x u_0(x) \cdot \sigma_0(x) \nabla_x N(x, z) \, dx
$$

where the last inequality follows from the fact that $u_0 \in H^1(\Omega)$ and we assume $\int_{\partial \Omega} u_0 \, d\sigma_x = 0$. Now using integration by parts yields

$$
u_0(z) = -\int_\Omega (\nabla_x \cdot \sigma_0(x) \nabla_x u_0(x)) N(x, z) \, dx + \int_{\partial \Omega} \sigma_2 \frac{\partial u_0}{\partial n_x} N(x, z) \, d\sigma_x.
$$

So (5.1) implies

$$
u_0(z) = \int_{\partial \Omega} \sigma_2 \frac{\partial u_0}{\partial n_x} N(x, z) \, d\sigma_x
$$

$$
= \int_{\partial \Omega} \sigma_2 g(x) N(x, z) \, d\sigma_x.
$$

(5.10)

Similarly we have

$$
u_h(z) = \int_\Omega u_h(x) \delta_x(x) \, dx
$$

$$
= -\int_\Omega u_h(x) (\nabla_x \cdot \sigma_0(x) \nabla_x N(x, z)) \, dx
$$

$$
= \int_\Omega \nabla_x u_h(x) \cdot \sigma_0(x) \nabla_x N(x, z) \, dx - \int_{\partial \Omega} \sigma_2 u_h \frac{\partial N}{\partial n_x} \, d\sigma_x$$
where the last inequality follows by integration by parts. Now assuming
\[ \int_{\partial \Omega} u_h \, d\sigma = 0 \]
yields
\[
\begin{align*}
  u_h(z) &= \int_{\Omega} \sigma_0(x) \nabla u_h(x) \cdot \nabla N(x, z) \, dx \\
  &= \int_{\Omega} \sigma_0(x) \nabla u_h(x) \cdot \nabla N(x, z) \, dx + \int_{D_h} \sigma_0(x) \nabla u_h(x) \cdot \nabla N(x, z) \, dx.
\end{align*}
\]

Then \( D \subset D_h \) and integration by parts in the first integral implies
\[
\begin{align*}
  u_h(z) &= \int_{\partial(\Omega \setminus D_h)} \sigma_0(x) \frac{\partial u_h}{\partial n_x} N(x, z) \, d\sigma_x + \int_{D} \sigma_1(x) \nabla u_h(x) \cdot \nabla N(x, z) \, dx \\
  &\quad + \int_{D_h \setminus D} \sigma_2(x) \nabla u_h(x) \cdot \nabla N(x, z) \, dx.
\end{align*}
\]
Note that \( D_h \subset \Omega \) implies
\[
\int_{\partial(\Omega \setminus D_h)} \sigma_0(x) \frac{\partial u_h}{\partial n_x} N(x, z) \, d\sigma_x = \int_{\partial \Omega} \sigma_2(x) \frac{\partial u_h}{\partial n_x} N(x, z) \, d\sigma_x - \int_{\partial D_h} \sigma_2(x) \frac{\partial u_h}{\partial n_x} N(x, z) \, d\sigma_x
\]
and using integration by parts on the last two integrals appearing in the right hand side of (5.11) yields
\[
\begin{align*}
  u_h(z) &= \int_{\partial \Omega} \sigma_2(x) \frac{\partial u_h}{\partial n_x} N(x, z) \, d\sigma_x - \int_{\partial D_h} \sigma_2(x) \frac{\partial u_h}{\partial n_x} N(x, z) \, d\sigma_x - \int \sigma_2(x) \frac{\partial u_h}{\partial n_x} N(x, z) \, d\sigma_x \\
  &\quad + \int_{\partial D} \sigma_1(x) \frac{\partial u_h}{\partial n_x} N(x, z) \, d\sigma_x + \int_{\partial(D_h \setminus D)} \sigma_2(x) \frac{\partial u_h}{\partial n_x} N(x, z) \, d\sigma_x.
\end{align*}
\]

Now recall that \( D \subset D_h \) and let \( \partial D = \Gamma_1 \) and let \( \Gamma_2 \) and \( \Gamma_3 \) be such that \( \Gamma_3 \subset \Gamma_1 \) and \( \partial D_h = (\Gamma_1 \setminus \Gamma_3) \cup \Gamma_2 \) (Figure 5–1). Equation (5.10) implies
\[
\begin{align*}
  u_h(z) &= u_0(z) \int_{(\Gamma_1 \setminus \Gamma_3) \cup \Gamma_2} \sigma_2(x) \left( \frac{\partial u_h}{\partial n_x} \right)^+ N(x, z) \, d\sigma_x \\
  &\quad + \int_{\Gamma_1} \sigma_1(x) \frac{\partial u_h}{\partial n_x} N(x, z) \, d\sigma_x + \int_{\Gamma_3 \cup \Gamma_2} \sigma_2(x) \frac{\partial u_h}{\partial n_x} N(x, z) \, d\sigma_x.
\end{align*}
\]
Note we have that
\[ \sigma_1 \left( \frac{\partial u_h}{\partial n_x} \right)^- = \sigma_2 \left( \frac{\partial u_h}{\partial n_x} \right)^+ \text{ on } \Gamma_2 \]
and since \( \partial u_h / \partial n_x \) is continuous across \( \Gamma_3 \) we have
\[ \left( \frac{\partial u_h}{\partial n_x} \right)^- = \left( \frac{\partial u_h}{\partial n_x} \right)^+ \text{ on } \Gamma_3. \]
Thus
\[
\begin{align*}
u_h(z) &= u_0(z) - \int_{\Gamma_2} (\sigma_1(x) - \sigma_2(x)) \left( \frac{\partial u_h}{\partial n_x} \right)^- N(x, z) \, d\sigma_x \\
&\quad + \int_{\Gamma_3} (\sigma_1(x) - \sigma_2(x)) \left( \frac{\partial u_h}{\partial n_x} \right)^- N(x, z) \, d\sigma_x,
\end{align*}
\]
and so
\[
u_h(z) = u_0(z) - (\sigma_1(x) - \sigma_2(x)) \left[ \int_{\Gamma_2} \left( \frac{\partial u_h}{\partial n_x} \right)^- N(x, z) \, d\sigma_x - \int_{\Gamma_3} \left( \frac{\partial u_h}{\partial n_x} \right)^- N(x, z) \, d\sigma_x \right].
\]
Let
\[ \vec{p}(s) = \langle x_1(s), x_2(s) \rangle, \ s \in I \]
be a parametric equation for the curve \( \partial D \) and let
\[ \vec{q}(s) = \langle \tilde{x}_1(s), \tilde{x}_2(s) \rangle, \ s \in I \]
be a parametric equation for the perturbed boundary \( \partial D_h \) where
\[ \langle \tilde{x}_1(s), \tilde{x}_2(s) \rangle = \langle x_1(s), x_2(s) \rangle + h\phi(s)\nu(s) \]
and where \( \phi(s) \) is a positive real, function and \( \nu \) is the outward pointing normal vector field on \( D \). In particular let \( \vec{p}(s) = \langle x_1(s), x_2(s) \rangle, \ s \in [a, b] \) be a parametrization of \( \Gamma_3 \) and let \( \vec{q}(s) = \langle \tilde{x}_1(s), \tilde{x}_2(s) \rangle, \ s \in [a, b] \) be a parametrization of \( \Gamma_2 \).
Then
\[
\int_{\Gamma_2} \left( \frac{\partial u_h}{\partial n_x} \right)^{-} N(x, z) \, d\sigma_x = \int_a^b \frac{\partial u_h}{\partial n_x}^{-} (\bar{x}_1(s), \bar{x}_2(s)) N((\bar{x}_1, \bar{x}_2), (z_1, z_2)) |\vec{q}'(s)| \, ds
\]
and
\[
\int_{\Gamma_3} \left( \frac{\partial u_h}{\partial n_x} \right)^{-} N(x, z) \, d\sigma_x = \int_a^b \frac{\partial u_h}{\partial n_x}^{-} (x_1(s), x_2(s)) N((x_1, x_2), (z_1, z_2)) |\vec{p}'(s)| \, ds.
\]
Now we use a first order Taylor Series expansion of \( \frac{\partial u_h}{\partial n_x}^{-} (\bar{x}_1, \bar{x}_2) \) and \( N((\bar{x}_1, \bar{x}_2), (z_1, z_2)) \) about the point \((x_1, x_2)\) in \( \Gamma_3 \). Recall that the first order Taylor Series expansion of \( f(\bar{x}_1, \bar{x}_2) \) about the point \((x_1, x_2)\) is given by
\[
f(\bar{x}_1, \bar{x}_2) = f(x_1, x_2) + \frac{\partial f(x_1, x_2)}{\partial \bar{x}_1} (\bar{x}_1 - x_1) + \frac{\partial f(x_1, x_2)}{\partial \bar{x}_2} (\bar{x}_2 - x_2).
\]
To approximate \(|\vec{q}'(s)|\) we use a first order Taylor series expansion of \( g(y_1, y_2) = \sqrt{y_1^2 + y_2^2} \) about the point \((x_1, x_2)\) since \(|\vec{q}'(s)| = g(\bar{x}', \bar{x}'_2)\). Note that for any \( s \in [a, b] \) we have \( \bar{x}(s) - x(s) = h\phi(s)\nu(s) \) and thus
\[
u_h(z) = u_0(z) + h(\sigma_2 - \sigma_1) I_1
\]
where
\[
I_1 = \int_a^b \left[ \frac{\partial u_h}{\partial n_x}^{-} \phi(s)(\nabla N(x_1(s), x_2(s), z) N(x_1(s), x_2(s), z) |\vec{p}'(s)| \right.
+ \left. \frac{\partial u_h}{\partial n_x}^{-} \phi(s)(\nabla N(x_1(s), x_2(s), z) \cdot \nu(s)) |\vec{p}'(s)| \right.
+ \left. \frac{\partial u_h}{\partial n_x}^{-} N(x_1(s), x_2(s), z) \frac{\vec{p}'(s) \cdot (\phi'\nu + \phi\nu')}{\sqrt{(x_1')^2 + (x_2')^2}} \right] \, ds.
\]
Now if we assume \( \partial D \) and \( \partial D_h \) are parametrized by arclength then \(|\vec{p}'(s)| = 1 \) and thus
\[
u_h(z) = u_0(z) + h(\sigma_2 - \sigma_1) I_2
\]
where

\[
I_2 = \int_a^b \left[ \phi(s) \left( \nabla \left( \frac{\partial u_h}{\partial n_x} \right) \cdot \nu(s) \right) N(x_1(s), x_2(s), z) \\
+ \left( \frac{\partial u_h}{\partial n_x} \right)^- \phi(s) \left( \nabla N(x_1(s), x_2(s), z) \cdot \nu(s) \right) \\
+ \left( \frac{\partial u_h}{\partial n_x} \right)^- N(x_1(s), x_2(s), z) (\vec{p}'(s) \cdot (\phi' \nu + \phi \nu')) \right] ds.
\]

Note that to complete the formal asymptotics we need to show

\[
\frac{\partial u_h^-}{\partial n} \to \frac{\partial u_0^-}{\partial n} \text{ as } h \to 0.
\]

We leave this as the topic of future work.
CHAPTER 6
CONCLUSION

We have analyzed a Butler–Volmer type model which describes the potential distribution in a system of anodic islands in a coplanar cathodic matrix with a periodic structure. By using a multi-scale approach we have determined the limiting problem for the boundary value problem (2.5) as the period approaches zero. Furthermore, by introducing a linear correction, we have developed an asymptotic expansion which closely estimates the solution of the original boundary value problem. Essentially, we have taken a nonlinear heterogeneous problem and decomposed it, in a sense, into a nonlinear homogeneous problem and a linear heterogeneous problem.

Hence the homogenization approach to this problem gives insight into the behaviour of the solution while also providing an efficient computational technique. The corrector term, although inhomogeneous, solves a linear problem, and was therefore not difficult to compute in our experiments. However, in higher dimensions or for very small scale problems, one may want to homogenize the corrector term itself. This could perhaps be done by solving a cell problem or looking at the tail behaviour, as in Achdou et al. [1] or Allaire and Amar [3]. In this paper we have used the language and terminology of galvanic corrosion but this analysis could also carry over to a more general class of elliptic problems with nonlinear boundary conditions having periodic structure (assuming the appropriate convexity conditions.) Future work must address the continuity and boundedness issues of the three-dimensional problem, i.e. the lack of an applicable Orlicz estimate must be resolved. We wish to do three-dimensional numerics and we also wish to consider the model for the case $\lambda < 0$. 

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With respect to the corrector introduced in Chapter 4 we wish to develop a convergence estimate for the three-dimensional case. In addition to implementing three-dimensional numerical simulations we wish to use the multiscale analysis developed in Chapter 2 and Chapter 4 to construct a first order approximation of solutions to other nonlinear PDE.

With regards to the electrostatic voltage potential model of Chapter 5, we wish to complete the asymptotic analysis presented there. We also wish to do numerics simulating electrostatic voltage potential. The end goal is to work up to a three-dimensional time harmonic Maxwell’s equation so that we may model propagation phenomena and apply this research to PBG structures.
REFERENCES


BIOGRAPHICAL SKETCH

Sujeet Bhat was born in Bangalore, India in 1972. He lived in Malaysia, Indonesia and the Philippines from 1973 to 1990. In 1990 he graduated from the International School, Manila. Sujeet obtained a bachelors degree in mathematics from the University of Florida in 1995, and a master’s degree in applied mathematics from the University of Texas at Dallas in May, 1998. He began his study towards a doctoral degree under the supervision of Professor Shari Moskow in 2003 and received his doctorate in mathematics from the University of Florida in May, 2006. He accepted a two year Industrial Postdoctoral Fellowship from the Institute for Mathematics and its Applications (IMA) at the University of Minnesota in April, 2006.