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EFFICIENT FOURIER TRANSFORMS ON HEXAGONAL ARRAYS

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A main concern of my research is the discrete Fourier transform (DFT) on two sequences of arrays, each of which consists of a finite number of lattice points (pixels) on a hexagonal grid. There are efficient addressing schemes for these arrays that allow for zooming in and out on an image in a hexagonal grid to view fine image details or global image features. We consider the formulation and the efficient computation of the DFT on those arrays. Some related problems such as the arithmetic for the labels of those lattice points are studied as well.

Each array in the first sequence consists of all lattice points of a hexagonal grid enclosed in a regular hexagon and has the same axes of symmetry as the enclosing hexagon. It is shown that the DFT on such an array is amenable to a standard Fast Fourier Transform and can be computed as a one dimensional DFT. We also provide an efficient method for evaluating the DFT of a function defined on that array based on the corresponding one dimensional standard DFT.

The second sequence is called a Pyxis structure, which originated with Pyxis Innovation Inc. to create an efficient sampling scheme for the earth. Each lattice point in the \( n^{th} \) array of the Pyxis structure is assigned a special label for quick data retrieval. We provide a recursive definition of the Pyxis structure, and show how such a label is assigned based on a certain unique algebraic representation of the corresponding lattice point. Also, we implement an efficient algorithm to determine the label of the vector sum of any
two lattice points whose labels are given. The recursive definition and algebraic labeling
scheme is used to show that, for any integer $n > 2$, the DFT on the $n^{th}$ array of the Pyxis
structure is not amenable to any standard DFT. Furthermore, the fractal dimension of the
limit boundary of the Pyxis structure is shown to be $\frac{\ln 4}{\ln 3}$.
CHAPTER 1
INTRODUCTION

Traditional image processing algorithms and digital image transforms are usually
carried out on pixels of square grids. However, as shown in Allen [1], physical pixels
such as printer dots and electron beams usually have circular shapes and thus operate
more effectively on hexagonal grids, where a hexagonal grid is a tessellation of the plane
by regular hexagons. The pixels of hexagonal grids also provide for higher packing
density of discs and give a more accurate approximation of circular regions than that
of square grids. Furthermore the pixels of hexagonal grids are uniformly connected
in the sense that the distance from a given pixel to any adjacent pixel is the same.
Hence, hexagonal grids are used by a wide variety of researchers in areas such as image
processing (Middleton and Sivaswamy [30], Balasubramaniyam et al. [3], and Strand [39]),
computer graphics (Tytkowski [42]), geoscience (Carr et al. [6]) and ecology (Jurasiński
and Beierkuhnlein [20]). For example, they are being used in the soil moisture and
ocean salinity space mission (Anterrieu et al. [2], and Camps et al. [5]). F. Morgan and
R. Bolton have shown in [32] that, for the efficiency of the distribution of centers of
production, regular hexagons are superior to any other collection of shapes.

The set consisting of all centers of pixels of a hexagonal grid is called a \textit{hexagonal
lattice}. In general, a \textit{d}-dimensional lattice in $\mathbb{R}^d$ is the set of all integer linear combinations
of \textit{d} independent vectors. The elements of a lattice are called \textit{lattice points}. The Voronoi
cell of a lattice point of a \textit{d}-dimensional lattice consists of those points of $\mathbb{R}^d$ which are
closest to that point than any other lattice point of the lattice. Because of the obvious
one to one correspondence between Voronoi cells and lattice points of a lattice, sometimes
we treat a set of lattice points as the corresponding set of Voronoi cells in the plotting to
get a better visualization effect which can be seen in Figure 1-1. An \textit{array} of a lattice is
a finite subset of the lattice and an \textit{array structure} is a sequence of arrays. The $n^{th}$ array
of an array structure is called the $n^{th}$ \textit{level of the array structure}. For example, the array
structure whose $n^{th}$ level is a square shaped array of $n \times n$ lattice points of a square lattice is used for image zooming on square grids. An array (structure) of the hexagonal lattice is called a hexagonal array (structure). This research concerns the following two hexagonal array structures which are used for hexagonal image processing.

A regular hexagonal structure (RHS) is a hexagonal array structure such that, at any given level, the underlying hexagonal lattice is a disjoint union of translated copies, and the set of lattice points whose coordinates are the same as the coordinates of those translations also forms a hexagonal lattice. An algebraic definition of a RHS as well as two particular types of RHSs, called type A and type B, are provided in Chapter 4. Each array within the type A (type B) RHS consists of all hexagonal lattice points enclosed in a regular hexagon and has the same centroid and axes of symmetry as the enclosing hexagon. Figure 1-1 shows one array of each type. Pyxis structure, denoted $P$, is a hexagonal array structure together with a natural method for labeling the lattice points (or hexagons). Let $P(n)$ denote the $n^{th}$ level of $P$ for any integer $n \geq 0$. The precise definition of $P$ and $P(n)$ are given in Chapter 6, and $P(1)$ through $P(4)$ are shown in Figure 1-2. The name and concept of the Pyxis structure originated with PYXIS Innovation Inc. (Peterson [33]), a Canada based company whose goal is an efficient
Figure 1-2. The Pyxis structure at level one through four where \( P(1) \) consists of seven red hexagons, \( P(2) \) consists of 13 blue hexagons, \( P(3) \) consists of 55 green hexagons, and \( P(4) \) consists of 133 black hexagons. The three dashed vectors show the label addition 0506 \( \oplus \) 2005 = 1040

sampling scheme for the surface of the earth (Figure 1-3). The following shows the structure of this efficient sampling scheme. Consider a sphere which is tessellated by 20 regular hexagons and 12 regular pentagons. Figure 1-4 shows the sphere with those 32 polygons flattened onto the plane. For each side of those polygons, make a line segment (as the dashed blue lines in Figure 1-5) with length being equal to the length of the given side divided by \( \sqrt{3} \) such that this line segment and the given side are perpendicular to and bisect each other. After any two neighboring ends are connected using lines as the dashed red lines in Figure 1-5, we get a set of polygons at the next level which consist of 80 hexagons and 12 pentagons dividing the sphere into smaller polygonal regions. Figure 1-6 displays the flattened versions of those 80 hexagons and 12 pentagons (taken as hexagons with one direction empty) on a plane. If the division rule shown in Figure 1-5 is
Figure 1-3. Tessellating sphere using hexagons and 12 pentagons in multiresolutions.

Figure 1-4. Flattened polygons used to tessellate the sphere. (a) shows the 20 hexagons and 12 pentagons in the tessellation. (b) displays each pentagon in Figure (a) as a hexagon with one of its six directions empty.

applied recursively, then the sphere is divided into smaller and smaller polygonal regions but the number of pentagonal regions is always 12. The division of the sphere by applying such recursion \( n \) times is called the division of the sphere at level \( n \). Figure 1-6 and 1-7
Figure 1-5. The (dashed) division lines of the next level are generated from those (solid) division lines of the previous level. (a) and (b) show the division lines near a hexagon and a pentagon of the previous level respectively.

Figure 1-6. The green polygons obtained from the division of the sphere at level one using the scheme of Figure 1-5.

show the flattened versions of such divisions at level one and level two, respectively. For any integer $n > 1$, the set of all cells of the sphere at the $n^{th}$ level is a disjoint union of 20 copies of $P(n - 1)$ and 12 copies of $P(n)$ by omitting one of its six directions. For example,
Figure 1-7. The red polygons obtained from the division of the sphere at level two using scheme of Figure 1-5.

Figure 1-7 shows that the set of all cells of the sphere at level two is a disjoint union of 20 copies of $P(1)$ which are in blue and 12 copies of $P(2)$ which are in red by omitting one of the six directions.

Certain properties of those two hexagonal array structures, in particular the discrete Fourier transform (DFT), are studied. The DFT on arrays of hexagonal lattices is an important topic in hexagonal image processing as explained in Middleton and Sivaswamy [30]. We provide an efficient method to compute the DFT on any array of type A or type B RHS and show that the computational complexity is $O(N \log N)$, where $N$ is the number of the lattice points of the array. We also provide a recursive definition of the Pyxis structure and use it to show that the DFT as defined in Chapter 2 cannot be applied to any array of the Pyxis structure when the level is larger than two.

As shown in Figure 1-2, the cells of $P(1)$ are labeled 0,1,2,...,6, and arranged in a certain order. The cells of $P(2)$ are labeled $ij$ where $i, j = 0, 1, 2,..., 6$ and either $i$ or $j$
is 0. Furthermore, the cell of \( P(2) \) labeled \( i0 \) has the same centroid as the one of \( P(1) \) labeled \( i \) for any \( i = 0, 1, 2, ..., 6 \), and the cells labeled \( 0i \) for \( i = 1, 2, ..., 6 \) surround the one labeled 00. The labeling system of \( P(n) \) for any integer \( n \geq 0 \) appears in Chapter 6. Those labels are important for quick data retrieval. The vector addition of any two Pyxis labels as defined in Chapter 6 (Figure 1-2) is useful in proving some properties of the Pyxis structure in Chapter 6, and a research topic proposed by the Pyxis Innovation Inc. We provide an efficient algorithm to perform such addition. Finally we compute the fractal dimension of the boundary of the limit of the Pyxis structure \( P(n) \) as \( n \to \infty \) since it measures how convoluted that boundary is. Our computation shows that the dimension is \( \frac{\ln 4}{\ln 3} \).
CHAPTER 2
DISCRETE FOURIER TRANSFORM (DFT)

2.1 Discrete Fourier Transform on the Quotient Group of Two Lattices

Throughout this research, \( \mathbb{N}, \mathbb{Z}, \mathbb{R}, \) and \( \mathbb{C} \) denote the set of positive integers, integers, real numbers, and complex numbers, respectively. For any \( d \in \mathbb{N} \), let \( \mathbb{Z}^d, \mathbb{R}^d, \) and \( \mathbb{C}^d \) denote the set of \( d \)-dimensional column vectors whose components are integers, real numbers, and complex numbers, respectively. An abelian group is a set \( G \) together with a closed binary operation \(+\) defined on \( G \) with the following properties:

1. (Associativity) For any \( x, y, z \in G \), \( (x + y) + z = x + (y + z) \).
2. (Existence of a zero) There exists an element \( 0 \in G \) such that, for each \( x \in G \), \( x + 0 = x = 0 + x \).
3. (Existence of an inverse) For each \( x \in G \), there exists an element of \( G \), denoted \(-x\), such that \( x + (-x) = 0 = (-x) + x \).
4. (Commutativity) For any \( x, y \in G \), \( x + y = y + x \).

In this research, all groups will be abelian and we write \( x + (-y) \) as \( x - y \) for any two elements \( x \) and \( y \) of a group \( G \). Let \( G_0 \) be a nonempty subset of a group \( G \). If \( G_0 \) also forms a group under the operation \(+\) of \( G \), then \( G_0 \) is called a subgroup of \( G \). Now assume that \( G_0 \) be a subgroup of a group \( G \). For any \( p, g \in G \), if \( p - g \in G_0 \), then we say that \( p \) is congruent to \( g \) modulo \( G_0 \), denoted by \( p \equiv g \mod G_0 \). For any \( p \in G \), let \( \bar{p} = \{ u \in G : u \equiv p \mod G_0 \} \). Obviously \( \bar{p} = p + G_0 \) where \( p + G_0 \) denotes the set \( \{ p + y \in G : y \in G_0 \} \). The set \( \bar{p} \) is called a coset of \( G_0 \) in \( G \). For any pair \( p, g \in G \), it is easy to show that either \( \bar{p} = \bar{g} \) or \( \bar{p} \cap \bar{g} = \emptyset \). Define \( \bar{p} + \bar{g} = \bar{p + g} \) and let \( G/G_0 = \{ \bar{p} : p \in G \} \). It is easy to check that the set \( G/G_0 \) together with the binary operation \(+\) defined on \( G/G_0 \) is an abelian group. This group is called the quotient group of the group \( G \) by the subgroup \( G_0 \). By choosing one representative from each coset of \( G_0 \) in \( G \), we get a set of coset representatives of the quotient group \( G/G_0 \).
If \( v_1, v_2, \ldots, v_d \) are linearly independent vectors in \( \mathbb{R}^d \), then the set defined by
\[
L = \left\{ \sum_{j=1}^{d} n_j v_j : n_j \in \mathbb{Z}, 1 \leq j \leq d \right\}
\]
is a \( d \)-dimensional lattice and \( \{v_1, v_2, \ldots, v_d\} \) is called a set of generators of \( L \). If \( d = 2 \), \( v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), then \( L \) is the standard square lattice commonly used in image processing. A hexagonal lattice has a set of generators \( \{v_1, v_2\} \) such that \( \|v_1\| = \|v_2\| \) and the angle between \( v_1 \) and \( v_2 \) is \( \frac{\pi}{3} \), where the notation \( \|x\| \) denotes the norm of the \( x \) for any \( x \in \mathbb{R}^2 \). Let \( L_0 \) be a nonempty subset of a lattice \( L \). If \( L_0 \) itself is a lattice, then \( L_0 \) is called a sublattice of \( L \). Obviously a lattice is an abelian group and a sublattice is a subgroup. For any \( \emptyset \neq T \subseteq L \) and \( x \in L \), to avoid confusion in some expressions, we let \( T_x = x + T \). If \( T \) is a set of coset representatives of the quotient group \( L/L_0 \), then \( T \) tiles the lattice \( L \) by translations by the sublattice \( L_0 \) in the sense that \( \bigcup \{ T_p : p \in L_0 \} = L \) and \( T_p = T_g \) whenever \( T_p \cap T_g \neq \emptyset \) for \( p, g \in L_0 \). Hence \( T \) is called a tile of \( L \). Each tiling (tile) involved in this research is a tiling (tile) by translations by a sublattice. Also we just consider those sublattices of \( L \) which have the same dimension as \( L \). In this case, the quotient group \( L/L_0 \) is finite by Lemma 1 of Section I.2.2 of Cassels [7]. The notation \( L_0 \prec L \) is used to denote that \( L_0 \) is a sublattice of \( L \) which has the same dimension as \( L \). The inner product of two vectors \( r, s \in \mathbb{C}^d \) is defined as \( \langle r, s \rangle = \sum_{j=1}^{d} r_j^* s_j \) where \( r_j \) and \( s_j \) are the \( j \)-th component of \( r \) and \( s \) respectively, and \( r_j^* \) denotes the complex conjugate of \( r_j \). If \( r, s \in \mathbb{R}^d \), then \( \langle r, s \rangle = r^T s = s^T r \). The dual of a lattice \( L \) is \( L^* = \{ s \in \mathbb{R}^d : \langle r, s \rangle \in \mathbb{Z} \text{ for all } r \in L \} \).

In this research, the superscript \( * \) means the dual when it is applied to a lattice, and means the complex conjugate when it is applied to a complex number or vector. The cardinality of a set \( S \) is denoted \( |S| \). Furthermore, for two groups \( A \) and \( B \), the symbol \( \cong \) means that \( A \) and \( B \) are isomorphic. The next lemma, whose proof appears in Zapata [47], and Conway and Sloane [9], provides some useful properties of dual lattices.

**Lemma 2.1.1.** Let \( L_0 \) and \( L \) be two lattices. Then we have the following results.

1. \( L_0 \prec L \) if and only if \( L^* \prec L_0^* \).
2. $(L^*)^* = L$ for any lattice $L$.

3. If $L \preceq L_0$, then $L/L_0 \cong L_0^*/L^*$ and hence $|L/L_0| = |L_0^*/L^*|.$

For any finite abelian group $G$, let $C^G$ denote the set of all complex-valued functions defined on $G$. Let $L_0 \preceq L$, $G = L/L_0$, and $\hat{G} = L_0^*/L^*$. The discrete Fourier transform (DFT) on the quotient group $G$ is a function $F : C^G \to C^\hat{G}$ defined by

$$\hat{a} (\bar{s}) = F(a)(\bar{s}) := \sum_{\bar{r} \in G} a(\bar{r}) \cdot e^{-2\pi i (r, s)}.$$  \hspace{1cm} (2–1)

The inverse Fourier transform is the function $F^{-1} : C^\hat{G} \to C^G$ defined by

$$F^{-1}(\hat{a})(\bar{r}) = \frac{1}{|\hat{G}|} \sum_{\bar{s} \in \hat{G}} \hat{a}(\bar{s}) \cdot e^{2\pi i (r, s)}.$$  \hspace{1cm} (2–2)

In the definition of the discrete Fourier transform, the domains $G$ and $\hat{G}$ are called the spatial and frequency domain of the Fourier transform $F$ respectively.

**Proposition 2.1.2.** The discrete Fourier transform is well defined.

**Proof.** If $\bar{r}_1, \bar{r}_2 \in G$, $\bar{s}_1, \bar{s}_2 \in \hat{G}$ such that $\bar{r}_1 = \bar{r}_2$ and $\bar{s}_1 = \bar{s}_2$, then $\bar{r}_1 - \bar{r}_2 = \bar{0} \in G$ and $\bar{s}_1 - \bar{s}_2 = \bar{0} \in \hat{G}$. It follows that $\bar{r}_1 - \bar{r}_2 \in L_0$ and $\bar{s}_1 - \bar{s}_2 \in L^*$. Obviously $\langle \bar{r}_1, \bar{s}_1 \rangle = \langle \bar{r}_2, \bar{s}_2 \rangle + \langle \bar{r}_2, \bar{s}_1 - \bar{s}_2 \rangle + \langle \bar{r}_1 - \bar{r}_2, \bar{s}_1 \rangle$. Since $\bar{s}_1 - \bar{s}_2 \in L^*$ and $\bar{r}_2 \in L$, we have $\langle \bar{r}_2, \bar{s}_1 - \bar{s}_2 \rangle \in \mathbb{Z}$. Also $\bar{r}_1 - \bar{r}_2 \in L_0$ and $\bar{s}_1 \in L_0^*$ imply that $\langle \bar{r}_1 - \bar{r}_2, \bar{s}_1 \rangle \in \mathbb{Z}$. Hence $e^{-2\pi i (r_1, s_1)} = e^{-2\pi i (r_2, s_2)}$. Therefore the DFT is well defined. \hfill \Box

To prove that the DFT is invertible, i.e., $F^{-1} \circ F(a) = a$ for any $a \in C^G$, we need the following lemma:

**Lemma 2.1.3.** Let $L_0 \preceq L$, $G = L/L_0$ and $\hat{G} = L_0^*/L^*$. If $\bar{r} \in G$ and $h(\bar{r}) = \sum_{\bar{s} \in \hat{G}} e^{2\pi i (r, s)}$, then

$$h(\bar{r}) = \begin{cases} |\hat{G}|, & \text{if } \bar{r} = \bar{0}, \\ 0, & \text{otherwise.} \end{cases}$$
Proof. Similar to Proposition 2.1.2, \( h(\bar{r}) \) is well defined. If \( \bar{r} = \bar{0} \), then

\[
\sum_{\bar{s} \in \hat{G}} e^{2\pi i(r,s)} = \sum_{\bar{s} \in \hat{G}} e^{2\pi i(0,s)} = |\hat{G}|.
\]

If \( \bar{r} \neq \bar{0} \), then \( \bar{r} \not\in L_0 = (L_0^*)^* \). It follows that there exists \( s_0 \in L_0^* \) such that \( \langle r, s_0 \rangle \not\in \mathbb{Z} \) by the definition of dual lattices. Hence \( e^{2\pi i(r,s_0)} \neq 1 \). Let \( c = e^{2\pi i(r,s_0)} \). Since \( h(\bar{r}) = \sum_{\bar{s} \in \hat{G}} e^{2\pi i(r,s)} = \sum_{\bar{s} \in \hat{G}} e^{2\pi i(r,s_0+s)} = e^{2\pi i(r,s_0)} \sum_{\bar{s} \in \hat{G}} e^{2\pi i(r,s)} = c(h(\bar{r})) \), we have \( h(\bar{r})(1-c) = 0 \).

Since \( c \neq 1 \), \( h(\bar{r}) = 0 \).

Theorem 2.1.4. (Inversion Theorem) If \( L_0 \prec L \), \( G = L/L_0 \), and \( \hat{G} = L^*_0/L^* \), then \( \mathcal{F}^{-1} \circ \mathcal{F}(a) = a \) for any \( a \in C^G \).

Proof. For any \( \bar{t} \in G \) where \( t \in L \), we have

\[
\mathcal{F}^{-1}(\hat{a})(\bar{t}) = \frac{1}{|\hat{G}|} \sum_{\bar{s} \in \hat{G}} \hat{a}(\bar{s}) \cdot e^{2\pi i(t,s)}
\]

\[
= \frac{1}{|\hat{G}|} \sum_{\bar{s} \in \hat{G}} \left\{ \sum_{\bar{r} \in \hat{G}} a(\bar{r}) \cdot e^{-2\pi i(r,s)} \right\} \cdot e^{2\pi i(t,s)} \tag{2-3}
\]

\[
= \frac{1}{|\hat{G}|} \sum_{\bar{r} \in \hat{G}} a(\bar{r}) \left\{ \sum_{\bar{s} \in \hat{G}} e^{2\pi i(t-r,s)} \right\}
\]

By Lemma 2.1.3, we have

\[
\sum_{\bar{r} \in \hat{G}} a(\bar{r}) \left\{ \sum_{\bar{s} \in \hat{G}} e^{2\pi i(t-r,s)} \right\} = a(\bar{t}) \cdot |\hat{G}|. \tag{2-4}
\]

It follows that \( \mathcal{F}^{-1}(\hat{a})(\bar{t}) = a(\bar{t}) \) for any \( \bar{t} \in G \). Hence \( \mathcal{F}^{-1}(\hat{a}) = a \). \qed

2.2 Convolution and Correlation

In this section, we consider two Fourier transform relationships that constitute a basic link between the spatial and frequency domains. These relationships, called convolution and correlation, are of fundamental importance in image processing techniques based on the Fourier transform.
Let $G$ be a finite abelian group and let $a, b \in \mathbb{C}^G$. The 
convolution $c \in \mathbb{C}^G$ of $a$ and $b$, written $c = a \ast b$, is defined by
\[
c(t) = \sum_{r \in G} a(r)b(t - r) \text{ for all } t \in G.
\]
The importance of the concept of convolution lies in the fact that convolution in the
spatial domain corresponds to multiplication in the frequency domain and vice versa. The
proof of the following Convolution Theorem appears in Zapata and Ritter [48].

**Theorem 2.2.1.** (Convolution Theorem) Let $L_0 \prec L$ and $G = L/L_0$. If $a, b \in \mathbb{C}^G$, then
\[
\hat{a} \ast \hat{b} = \hat{a} \cdot \hat{b}.
\]

A principle application of correlation in image processing is in the area of template
or prototype matching, where the problem is to find the closest match between a given
unknown image and a set of images of known origin. One approach to this problem is
to compute the correlation between the unknown image and each of the known images.
Since the resultant correlations are 2-dimensional functions, this involves searching for the
largest amplitude of each function. As in the case of convolution, the computation of the
correlation is often more efficiently carried out in the frequency domain using the Fourier
transform. Let $G$ be a finite abelian group and $a, b \in \mathbb{C}^G$. The correlation $c \in \mathbb{C}^G$ of $a$ and
$b$, written $c = a \circ b$, is defined as $c(t) = \sum_{r \in G} a^*(r)b(t + r)$ for all $t \in G$. If $a = b$, then the
correlation is called an autocorrelation.

**Theorem 2.2.2.** (Correlation Theorem) Let $L_0 \prec L$ and $G = L/L_0$. If $a, b \in \mathbb{C}^G$ and
\[
c = a \circ b, \text{ then } \hat{c} = \hat{a}^* \cdot \hat{b}.
\]

**Proof.** For all $t \in G$, $c(t) = \sum_{r \in G} a^*(r)b(t + r) = \sum_{s \in G} a^*(s)(-s)b(t - s)$. Define
g \in \mathbb{C}^G by $g(s) = a^*(-s)$ for any $s \in G$. Then $c(t) = \sum_{s \in G} g(s)b(t - s) = (g \ast b)(t)$. Hence $\hat{c} = \hat{g} \cdot \hat{b}$ by Theorem 2.2.1. Since $g(s) = a^*(-s)$ for any $s \in G$, we have
\[
\hat{g}(t) = \sum_{s \in G} g(s) \cdot e^{-2\pi i(s,t)} = \sum_{s \in G} g(-s) \cdot e^{2\pi i(s,t)}
\]
\[
= \sum_{s \in G} a^*(s) \cdot e^{2\pi i(s,t)} = (\hat{a}(t))^*. \text{ It follows that } \hat{g} = \hat{a}^* \text{. Therefore } \hat{c} = \hat{a}^* \cdot \hat{b}. \]
2.3 DFT on a Lattice and the Corresponding Periodicity Matrix

In this section, we discuss the DFT on a tile of a lattice. To make the DFT easier to be computed, we write the DFT in terms of an integer matrix and vectors which have integer components.

A matrix $V$ is called a sampling matrix of a lattice $L$ if its columns are a set of generators for $L$. Obviously in such a case, $L = \{Vn : n \in \mathbb{Z}^d\} = V\mathbb{Z}^d$. It follows easily from the following lemma that the dual of a hexagonal lattice is also hexagonal.

**Lemma 2.3.1.** Let $L$ be a lattice. Then $V$ is a sampling matrix of $L$ if and only if $(V^T)^{-1}$ is a sampling matrix of the dual lattice $L^*$.  

*Proof.* Let $V$ be a sampling matrix of $L$. Then $L = \{Vn : n \in \mathbb{Z}^d\}$. Hence $s \in L^*$ if and only if $\langle Vn, s \rangle = (Vn)^T s = n^T (V^T s) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$, if and only if $V^T s \in \mathbb{Z}^d$, if and only if $s \in (V^T)^{-1} \mathbb{Z}^d$. It follows that $(V^T)^{-1}$ is a sampling matrix of the dual lattice $L^*$. Similarly, if $(V^T)^{-1}$ is a sampling matrix of $L^*$, then $V$ is a sampling matrix of $L$. \hfill $\Box$

A non-singular matrix $M$ with integer entries is called a periodicity matrix. A set of coset representatives of the quotient group $\mathbb{Z}^d/M\mathbb{Z}^d$ is also called a set of coset representatives associated with the periodicity matrix $M$.

**Lemma 2.3.2.** Let $L_0 \prec L$ and $V$ be a sampling matrix of $L$. Then there is a periodicity matrix $M$ such that $VM$ is a sampling matrix of $L_0$. Furthermore $\hat{V} = [(VM)^T]^{-1}$ is a sampling matrix of $L_0^*$.  

*Proof.* Let $B$ be a sampling matrix of $L_0$ whose columns are $b_1, b_2, \ldots, b_d$, and let $v_r$ be the $r^{th}$ column of $V$ for $r = 1, 2, \ldots, d$. Since $b_r \in L$ for each $r = 1, 2, \ldots, d$, there are integers $l_{jr}$ such that $b_r = \sum_{j=1}^{d} l_{jr} v_j$. Let $M$ be the matrix whose entries are $l_{jr}$. Then we have $B = VM$. It follows that $VM$ is a sampling matrix of $L_0$. Hence $\hat{V}$ is a sampling matrix of $L_0^*$ since $\hat{V} = (B^T)^{-1}$.

*Proof.* Let $B$ be a sampling matrix of $L_0$ whose columns are $b_1, b_2, \ldots, b_d$, and let $v_r$ be the $r^{th}$ column of $V$ for $r = 1, 2, \ldots, d$. Since $b_r \in L$ for each $r = 1, 2, \ldots, d$, there are integers $l_{jr}$ such that $b_r = \sum_{j=1}^{d} l_{jr} v_j$. Let $M$ be the matrix whose entries are $l_{jr}$. Then we have $B = VM$. It follows that $VM$ is a sampling matrix of $L_0$. Hence $\hat{V}$ is a sampling matrix of $L_0^*$ since $\hat{V} = (B^T)^{-1}$.

Let $L_0 \prec L$, $G = L/L_0$, $\hat{G} = L_0^*/L^*$, and $\mathcal{F} : \mathbb{C}^G \to \mathbb{C}^{\hat{G}}$ be the DFT. If $V$ is a sampling matrix of $L$ and $\hat{V}$ a sampling matrix of $L_0^*$, then $V$ is called a sampling matrix in the
spatial domain and $\hat{V}$ is called a sampling matrix in the frequency domain of the DFT $\mathcal{F}$. For any $a \in \mathbb{C}^G$, let $f \in \mathbb{C}^L$ be defined by $f(u) = a(u + L_0)$ for any $u \in L$. Then $f$ is called the periodic extension of $a$. Let $P$ and $Q$ be any set of coset representatives of the quotient group $L/L_0$ and $L^*_0/L^*$ respectively. Then Equation 2–1 for computing the DFT on $L/L_0$ becomes the following equation for computing the DFT on $P$.

$$\hat{f}(s) = \sum_{r \in P} f(r) \cdot e^{-2\pi i (r,s)}$$ for all $s \in Q$. \hfill (2–5)

Now for any $r \in P$ and $s \in Q$, let $[r]$ be the coordinates of $r$ with respect to the basis that are the generators of $L$, and let $[s]$ be the coordinates of $s$ with respect to the basis that are the generators of $L^*_0$. Then $r = V[r]$ and $s = \hat{V}[s]$. By Lemma 2.3.2, there is a periodicity matrix $M$ such that the sampling matrix of $L_0$ is $VM$ and the sampling matrix of $L^*_0$ is $\hat{V} = [(VM)^T]^{-1}$. Let $I_P = \{ [p] : p \in P \} \subset \mathbb{Z}^d$ and $I_Q = \{ [q] : q \in Q \} \subset \mathbb{Z}^d$. Then $\langle r, s \rangle = s^T r = (\hat{V}[s])^T V[r] = [s]^T \hat{V}^T V[r] = [s]^T (VM)^{-1} V[r] = [s]^T M^{-1} V^{-1} V[r] = [s]^T M^{-1} [r]$. Hence Equation 2–5 becomes the following equation.

$$\hat{f}(\hat{V}[s]) = \sum_{r \in P} f(V[r]) \cdot e^{-2\pi i [s]^T M^{-1} [r]}$$ for all $s \in Q$. \hfill (2–6)

By replacing $[r]$ and $[s]$ with $m$ and $k$ respectively, Equation 2–6 becomes the following simpler equation.

$$\hat{f}(\hat{V}k) = \sum_{m \in I_P} f(Vm) \cdot e^{-2\pi i k^T M^{-1} m}$$ for all $k \in I_Q$. \hfill (2–7)

### 2.4 Relation Between the DFT and the Continuous Fourier Transform

In this section, we show that the relation between the continuous Fourier Transform and the DFT for the 1-dimensional case as appears in Cartwright [8] also holds for the 2-dimensional case. For any complex-valued, Lebesgue integrable function $f(x)$ with $x \in \mathbb{R}^2$, by Pinsky [34] its Fourier transform is the complex-valued function $\hat{f}(y)$ defined
by the integral
\[
\hat{f}(y) = \mathcal{F}(f)(y) := \int_{\mathbb{R}^2} f(x) \cdot e^{-2\pi i (x,y)} dx, \text{ for any } y \in \mathbb{R}^2. \quad (2-8)
\]

Let \( L \) be a 2-dimensional lattice, \( L_0 \prec L \), and \( P \) a set of coset representatives of the quotient group \( L/L_0 \). Let \( \tilde{P} \) be the union of the Voronoi cells of the lattice points of \( P \) and \( A \) the area of a Voronoi cell of the lattice \( L \). In the finite analogue, we replace \( \mathbb{R}^2 \) in Equation 2–8 by \( \tilde{P} \). The corresponding discretized version of Equation 2–8 is the following equation.
\[
\hat{f}_P(y) := \sum_{p \in P} f(p) \cdot e^{-2\pi i (p,y)} \cdot A, \text{ for any } y \in \mathbb{R}^2. \quad (2-9)
\]

Now let \( Q \) be a set of coset representatives of the quotient group \( L_0^*/L^* \) and \( \tilde{Q} \) be the union of the Voronoi cells of the lattice points of \( Q \). Since \( P \) is a set of coset representatives of \( L/L_0 \), \( \tilde{P} \) tiles \( \mathbb{R}^2 \) by translations by the sublattice \( L_0 \). Similarly \( \tilde{Q} \) tiles \( \mathbb{R}^2 \) by translations by the sublattice \( L^* \) of \( L_0^* \). We claim that \( \hat{f}_P \) is periodic in the sense that \( \hat{f}_P(y) = \hat{f}_P(y + s_0) \) for any \( s_0 \in L^* \). Since \( s_0 \in L^* \) and \( p \in L \), we have \( \langle p, s_0 \rangle \in \mathbb{Z} \). Then \( \langle p, y + s_0 \rangle = \langle p, y \rangle + \langle p, s_0 \rangle \) follows that \( e^{-2\pi i (p,y+s_0)} = e^{-2\pi i (p,y)} \). Hence, by Equation 2–9, \( \hat{f}_P(y + s_0) = \hat{f}_P(y) \). Therefore \( \hat{f}_P(y) \) is periodic. By the definition of \( \tilde{Q} \), \( Q \) is the set of the sampled points of \( \tilde{Q} \) in the lattice \( L_0^* \). In Equation 2–9, the expression \( \sum_{p \in P} f(p) \cdot e^{-2\pi i (p,y)} \) is the DFT as defined in Equation 2–5 or 2–7.
CHAPTER 3
DFT ON SOME PREVIOUSLY STUDIED HEXAGONAL ARRAYS

The 2-dimensional generalized balanced ternary, denoted $GBT_2$, has been used in Zapata and Ritter [48] through Gibson and Lucas [17] for the processing of hexagonally sampled images. The $n^{th}$ level of the $GBT_2$, denoted $GBT_2(n)$, consists of $7^n$ hexagons. For any $n \in \mathbb{N}$, the $GBT_2(n + 1)$ is the union of 7 translated copies of the $GBT_2(n)$ as shown in Figure 3-1. The $GBT_2$ is perhaps the most studied example because of the elegant method of indexing its cells (lattice points) described in Gibson and Lucas [17], Kitto [21], and Middleton and Sivaswamy [30]. A periodicity matrix for the $GBT_2(n)$ is $B^n$ where $B = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$. The DFT on the $GBT_2(n)$ and its fast algorithms have been successfully developed in Zapata and Ritter [48], and Middleton and Sivaswamy [29]. Some deficiencies of the $GBT_2$ have been pointed out by the people working in the Pyxis Innovation Inc. (the shared document of Pyxis Innovation Inc., 2003). For example, the

Figure 3-1. The first three levels of the $GBT_2$ aggregates.
The number of hexagons of the $GBT_2(n + 1)$ is 7 times that of the $GBT_2(n)$. A factor less than 7 would be better for image zooming in from one level to the next. When $n$ is large, the region occupied by the $GBT_2(n)$ is quite different from the regions occupied by usual images and hence it is not convenient to apply the DFT on usual images.

The DFT on arrays of rectangular shape of a hexagonal lattice (shown in Figure 3-2 (a)) was utilized in Ehrhardt [12], and Fitz and Green [14]. For any such array, the sampling matrix of the lattice $L$ is

\[
\begin{pmatrix}
1 & -\frac{1}{2} \\
0 & \frac{\sqrt{3}}{2}
\end{pmatrix}
\]

and the periodicity matrix is of the form $M = \begin{pmatrix} n_1 & \frac{n_2}{2} \\ 0 & n_2 \end{pmatrix}$, where $n_1$ and $n_2$ are integers with $n_2$ divisible by 2. Hence the sampling matrix in the frequency domain is $\hat{V} = \begin{pmatrix} \frac{1}{n_1} & 0 \\ 0 & \frac{2}{\sqrt{3} n_2} \end{pmatrix}$, from which it follows that the sampling grid in the frequency domain is rectangular. Thus it is convenient to output frequencies on a monitor that uses a rectangular grid. Let $\hat{V}_1$ and $\hat{V}_2$ be the two columns of $\hat{V}$. Since both $n_1$ and $n_2$ are integers, the length of $\hat{V}_1$ is not equal to that of $\hat{V}_2$ for any integers $n_1$ and $n_2$. Hence the sampling lattice in the frequency domain is not square. Some fast algorithms for computing such a DFT were analyzed in Ehrhardt [12].

Figure 3-2. Arrays studied in Ehrhardt [12], and Sun and Yao [40]. (a) Hexagonal sampling of a rectangular region used in [12] and Fitz and Green [14]. (b) The array used in Sun and Yao [40] consisting of $3n^2$ lattice points, where $n = 9$. 

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In Sun and Yao [40], a fast algorithm was developed for computing the DFT on an array which consists of $3n^2$ lattice points of a hexagonal lattice. Figure 3-2 (b) shows this array for $n = 9$ where the top row consists of $n$ lattice points but the bottom row consists of $n + 1$ lattice points. A periodicity matrix for this array is $M = \begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}$.

In Anterrieu et al. [2], the hexagonal array shown in Figure 3-3 (a) was studied. Because the hexagonal array in Figure 3-3 (a) does not tile the underlying hexagonal lattice by translations by any sublattice, the DFT on such an array cannot be formulated using the method in Chapter 2. Hence the authors in Anterrieu et al. [2] remove half of the boundary points to apply a 2-dimensional standard DFT. After half of the boundary points of the array in Figure 3-3 (a) are omitted in a certain way, we get the array shown in Figure 3-3 (b) or Figure 3-3 (c), which consists of $n^2$ lattice points. The arrays in Figure 3-3 (b) or Figure 3-3 (c) do tile the underlying hexagonal lattice by translations by a sublattice. A periodicity matrix for the array in either Figure 3-3 (b) or Figure 3-3 (c) is $M = \begin{pmatrix} n & n \\ 0 & n \end{pmatrix}$. Since $M$ is also a periodicity matrix of the rhombus shaped array shown in Figure 3-3 (d), the DFT on each of the arrays in Figure 3-3 (b) and Figure 3-3 (c) is the same as the DFT on that rhombus shaped array and hence can be computed using the fast algorithms for a 2-dimensional standard DFT.

For any array in Figure 3-2 (b) or Figure 3-3 (b), the number of lattice points on its top row is different from that on the bottom row. Hence the shape of those arrays is not a regular hexagon. In the next chapter, we will consider hexagonal arrays whose 6 sides have the same number of lattice points and have the same centroid and symmetric axes as a regular hexagon.
Figure 3-3. Arrays studied in Anterrieu et al. [2]. (a) An array shown in Figure 3 on Page 2533 of Anterrieu et al. [2]. (b) The array obtained from the array in Figure (a) by omitting the boundary points on the top row and the two upper sides. (c) The array obtained from the array in Figure (a) by omitting one of its two consecutive boundary points. (d) The periodic extension from the array in Figure (c) to an array of a rhombus shape.
CHAPTER 4
REGULAR HEXAGONAL STRUCTURE AND ITS TWO SPECIAL TYPES

4.1 Regular Hexagonal Structures

In algebraic terms, a regular hexagonal structure (RHS) is an array structure such that each array in the structure is a set of coset representatives of the quotient group of two hexagonal lattices. Let \( V \) be a sampling matrix of a hexagonal lattice \( L \) with two columns \( \mathbf{v}_1, \mathbf{v}_2 \) satisfying \( \|\mathbf{v}_1\| = \|\mathbf{v}_2\| \) and such that the angle between \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) is \( \frac{\pi}{3} \).

If \( V \begin{pmatrix} r \\ s \end{pmatrix} \) is an arbitrary point of \( L \), then \( V \begin{pmatrix} r - s \\ r \\ s \end{pmatrix} \) is just \( V \begin{pmatrix} r \\ s \end{pmatrix} \) rotated by an angle of \( \pi/3 \). Hence, a hexagonal sublattice of \( L \) has a sampling matrix of the form \( VM \), where \( M = \begin{pmatrix} r & r - s \\ s & r \end{pmatrix} \) with \( r, s \in \mathbb{Z} \).

If we take \( r = 2n \) and \( s = n \) (\( r = n \) and \( s = 0 \)) in the matrix \( M \) above, then we get a periodicity matrix for the array in Sun and Yao [40] (Anterrieu et al. [2]). Hence the arrays in Sun and Yao [40] or Anterrieu et al. [2] constitute a RHS. In the following, we study arrays in the type \( A \) and type \( B \) RHS which are closely related to those in Sun and Yao [40], and Anterrieu et al. [2].

4.2 Type \( A \) Regular Hexagonal Structure

Throughout this research, we let \( \mathbf{v}_1^A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2^A = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \), and

\[
L^A = \{ n_1(\mathbf{v}_1^A) + n_2(\mathbf{v}_2^A) : n_1, n_2 \in \mathbb{Z} \}.
\]

The lattice \( L^A \) is called the type \( A \) hexagonal lattice. In this section, we consider the type \( A \) RHS whose algebraic definition is provided in the following. For any \( n \in \mathbb{N} \), let

\[
\mathcal{R}_n^A = \{ j(\mathbf{v}_1^A) + k(\mathbf{v}_2^A) : j, k \in \mathbb{Z}, |j| \leq n, |k| \leq n, |j - k| \leq n \}.
\]

We call \( \mathcal{R}_n^A \) the \( n^{th} \) level of the type \( A \) RHS.
Hence, the total number of those hexagons is \(2^{j+1}\) values. If \(j > 0\), then the common solution of the two inequalities \(|k| \leq n\) and \(|j - k| \leq n\) is \(j - n \leq k \leq n\) and hence \(k\) takes \(2n - j + 1\) values. The situation for \(j < 0\) is similar.

Hence, the total number of those hexagons is \((2n + 1) + 2 \sum_{j=1}^{n} (2n - j + 1) = 3n^2 + 3n + 1\).

**Lemma 4.2.2.** If \(a, b \in \mathcal{R}_n^A\), then \(a \pm b \in \mathcal{R}_{2n}^A\).

**Proof.** Let \(a = a_1(v_1^A) + a_2(v_2^A) \in \mathcal{R}_n^A\), \(b = b_1(v_1^A) + b_2(v_2^A) \in \mathcal{R}_n^A\), and \(c = a - b\). By the definition of the type A RHS, \(|a_j| \leq n\) for \(j = 1, 2\) and \(|a_2 - a_1| \leq n\). For the same reason, \(|b_j| \leq n\) for \(j = 1, 2\) and \(|b_2 - b_1| \leq n\). If \(c = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\), then \(|g_j| = |a_j - b_j| \leq |a_j| + |b_j| \leq 2n\).
$n + n = 2n$ for $j = 1, 2$, and $|g_2 - g_1| = |(a_2 - b_2) - (a_1 - b_1)| = |(a_2 - a_1) - (b_2 - b_1)| \leq |a_2 - a_1| + |b_2 - b_1| \leq n + n = 2n$. Hence $c \in \mathcal{R}_{2n}^A$. Obviously $b \in \mathcal{R}_{n}^A$ implies $-b \in \mathcal{R}_{n}^A$ by the definition of the type $A$ RHS. Thus $a - b = a + (-b) \in \mathcal{R}_{2n}^A$. 

\begin{lem}
Lemma 4.2.3. The maximal distance of the lattice points of $\mathcal{R}_{n}^A$ from the origin 0 is $n$.
\end{lem}

\begin{proof}
It follows directly from the fact that all lattice points of $\mathcal{R}_{n}^A$ are enclosed in the regular hexagon with vertices $\pm n(v_1^A), \pm n(v_2^A)$, and $\pm n(v_1^A - v_2^A)$.
\end{proof}

For any $n \in \mathbb{N}$, let $\widetilde{V}_{1,n}^A = (2n + 1)v_1^A + (n + 1)v_2^A$, $\widetilde{V}_{2,n}^A = n(v_1^A) + (2n + 1)v_2^A$, and $\widetilde{L}_n^A = \left\{ n_1(\widetilde{V}_{1,n}^A) + n_2(\widetilde{V}_{2,n}^A) : n_1, n_2 \in \mathbb{Z} \right\}$.

\begin{lem}
Lemma 4.2.4. $\widetilde{L}_n^A \cap \mathcal{R}_{2n}^A = \emptyset$.
\end{lem}

\begin{proof}
By definition, $\widetilde{V}_{1,n}^A = (2n + 1)v_1^A + (n + 1)v_2^A$ and $\widetilde{V}_{2,n}^A = n(v_1^A) + (2n + 1)v_2^A$. It follows that $\widetilde{V}_{1,n}^A \notin \mathcal{R}_{2n}^A$ since $2n + 1 > 2n$. Similarly $\widetilde{V}_{2,n}^A \notin \mathcal{R}_{2n}^A$. Let $\widetilde{V}_{3,n}^A = \widetilde{V}_{1,n}^A - \widetilde{V}_{2,n}^A = (n + 1)v_1^A - n(v_2^A) = l_1(v_1^A) + l_2(v_2^A)$ where $l_1 = n + 1$ and $l_2 = -n$. Since $|l_1 - l_2| = 2n + 1 > 2n$, $\widetilde{V}_{3,n}^A \notin \mathcal{R}_{2n}^A$. By direct computation, we have $|\widetilde{V}_{1,n}^A| = |\widetilde{V}_{2,n}^A|$ and the angle between $\widetilde{V}_{1,n}^A$ and $\widetilde{V}_{2,n}^A$ is $\frac{\pi}{3}$. Since $\widetilde{L}_n^A$ is a hexagonal lattice generated by $\widetilde{V}_{1,n}^A$ and $\widetilde{V}_{2,n}^A$, the set $\left\{ \pm \widetilde{V}_{1,n}^A, \pm \widetilde{V}_{2,n}^A, \pm \widetilde{V}_{3,n}^A \right\}$ contains the only 6 closest nonzero lattice points of $\widetilde{L}_n^A$ from the origin 0. For any other nonzero lattice point of $\widetilde{L}_n^A$, its distance from the origin 0 is at least $|\widetilde{V}_{1,n}^A + \widetilde{V}_{2,n}^A| = |(3n + 1)v_1^A + (3n + 2)v_2^A| = \sqrt{9n^2 + 9n + 3} > 2n$. However, by Lemma 4.2.3, the maximal distance of the lattice points of $\mathcal{R}_{2n}^A$ from 0 is 2n. Hence $\widetilde{L}_n^A \cap \mathcal{R}_{2n}^A = \emptyset$.
\end{proof}

To prove the next theorem, we need the following lemma which follows trivially from Lemma 1 of Section I.2.2 of Cassels [7].

\begin{lem}
Lemma 4.2.5. Let $L$ be a lattice with a sampling matrix $V$. If $M$ is a periodicity matrix and $L_0$ is the sublattice of the lattice $L$ such that $VM$ is a sampling matrix of $L_0$, then the order of the quotient group $L/L_0$ is the absolute value of the determinant of the matrix $M$.
\end{lem}
In the following, we let $M_n^A = \begin{pmatrix} 2n+1 & n \\ n+1 & 2n+1 \end{pmatrix}$. Since the lattice $L_n^A$ is generated by the two vectors $\tilde{V}_{1,n}^A$ and $\tilde{V}_{2,n}^A$ where $\tilde{V}_{1,n}^A = (2n+1)v_1^A + (n+1)v_2^A$ and $\tilde{V}_{2,n}^A = n(v_1^A) + (2n+1)v_2^A$, the matrix $M_n^A$ is a sampling matrix of the lattice $L_n^A$ and hence will be used very often.

**Theorem 4.2.6.** For any $n \in \mathbb{N}$, $\Re_n^A$ is a set of coset representatives of the quotient group $L^A/L_n^A$.

**Proof.** Let $V^A$ be the matrix whose two columns are $v_1^A$ and $v_2^A$, and $\tilde{V}_n^A$ the matrix whose two columns are $\tilde{V}_{1,n}^A$ and $\tilde{V}_{2,n}^A$. Since $\tilde{V}_{1,n}^A = (2n+1)v_1^A + (n+1)v_2^A$ and $\tilde{V}_{2,n}^A = n(v_1^A) + (2n+1)v_2^A$, we have $\tilde{V}_n^A = (V^A)(M_n^A)$. By Lemma 4.2.5 the order of the quotient group $L^A/L_n^A$ is the determinant of $M_n^A$. It follows that the order of the quotient group $L^A/L_n^A$ is $3n^2 + 3n + 1$ since it is the determinant of $M_n^A$. By Lemma 4.2.1, the order of $\Re_n^A$ is also $3n^2 + 3n + 1$. Hence, to show $\Re_n^A$ is a set of coset representatives of $L^A/L_n^A$, it suffices to show that no two distinct lattice points of $\Re_n^A$ are congruent modulo $L_n^A$. For any $a, b \in \Re_n^A$ with $a \neq b$, we have $a - b \in \Re_{2n}^A$ by Lemma 4.2.2. Suppose $a - b \in \tilde{L}_n^A$. By Lemma 4.2.4, $\tilde{L}_n^A \cap \Re_{2n}^A = \emptyset$. It follows that $a - b \in \tilde{L}_n^A \cap \Re_{2n}^A = \emptyset$ and hence $a = b$ which leads to a contradiction. Therefore $\Re_n^A$ is a set of coset representatives of $L^A/L_n^A$. \hfill \square

By Theorem 4.2.6, the DFT on $\Re_n^A$ is invertible. Now we consider its frequency domain. In the proof of Theorem 4.2.6, we have $\tilde{V}_n^A = (V^A)(M_n^A)$ where $V^A$ is a sampling matrix of the lattice $L^A$ and $\tilde{V}_n^A$ is a sampling matrix of the lattice $L_n^A$. Let $\tilde{V}_n^A = [(V_n^A)^T]^{-1}$. Since $\tilde{V}_n^A$ is a sampling matrix of the lattice $L_n^A$, $\hat{V}_n^A$ is a sampling matrix of the dual lattice $(\tilde{L}_n^A)^\ast$ of $L_n^A$ by Lemma 2.3.2.

Let $v_{n,1}^A$ and $v_{n,2}^A$ be the two column vectors of the matrix $\tilde{V}_n^A$. It is easy to check that $v_{n,1}^A = \frac{1}{\sqrt{3(3n^2 + 3n + 1)}} \begin{pmatrix} \sqrt{3}(2n + 1) \\ 1 \end{pmatrix}$, $v_{n,2}^A = \frac{1}{\sqrt{3(3n^2 + 3n + 1)}} \begin{pmatrix} -\sqrt{3}(n + 1) \\ 3n + 1 \end{pmatrix}$. 

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∥v_{n,1}^A∥ = ∥v_{n,2}^A∥ and the angle between them is \(\frac{2\pi}{3}\). For any \(n ∈ \mathbb{N}\), let

\[ \mathcal{R}_n^A = \left\{ j(v_{n,1}^A) + k(v_{n,2}^A) : j, k ∈ \mathbb{Z}, |j| ≤ n, |k| ≤ n, |j - k| ≤ n \right\} \]

where \(v_{n,1}^A\) and \(v_{n,2}^A\) are defined as the above.

**Theorem 4.2.7.** For any \(n ∈ \mathbb{N}\), \(\mathcal{R}_n^A\) is a set of coset representatives of the quotient group \((\tilde{L}_n^A)^*/(L_n^A)^*\).

**Proof.** Let us use the same notation as before. From the proof of Theorem 4.2.6, we have \(\tilde{V}_n^A = (V_n^A)(M_n^A)\). It follows that \([V_n^A]^T = [(V_n^A)^T]^{-1}[(M_n^A)^T]^{-1}\). Hence \([V_n^A]^T = [(V_n^A)^T]^{-1}(M_n^A)^T\). Since \(V_n^A\) is a sampling matrix of the lattice \(L_n^A\), \([(V_n^A)^T]^{-1}\) is a sampling matrix of the dual lattice \((L_n^A)^*\). For a similar reason, \([(V_n^A)^T]^{-1}\) is a sampling matrix of the dual lattice \((\tilde{L}_n^A)^*\). Since \((M_n^A)^T = \begin{pmatrix} 2n + 1 & n + 1 \\ n & 2n + 1 \end{pmatrix}\), by a proof similar to the proof of Theorem 4.2.6 it can be shown that \(\mathcal{R}_n^A\) is a set of coset representatives of the quotient group \((\tilde{L}_n^A)^*/(L_n^A)^*\).

From Theorems 4.2.6 and 4.2.7, and by the definitions of the type A RHS and \(\mathcal{R}_n^A\), it is easy to deduce that the geometric shape of the support in the frequency domain of the DFT on \(\mathcal{R}_n^A\) is similar to that of the support in the spatial domain. Throughout the remainder of this research, we let

\[ C(A, n) = \left\{ \begin{pmatrix} j \\ k \end{pmatrix} ∈ \mathbb{Z}^2 : j(v_1^A) + k(v_2^A) ∈ \mathcal{R}_n^A \right\}. \]

By the definition of \(\mathcal{R}_n^A\), obviously we have

\[ C(A, n) = \left\{ \begin{pmatrix} j \\ k \end{pmatrix} ∈ \mathbb{Z}^2 : |j| ≤ n, |k| ≤ n, |j - k| ≤ n \right\}. \]

This notation will be used to write the DFT on \(\mathcal{R}_n^A\) in terms of a periodicity matrix.

Recall in Chapter 2 that, for a periodicity matrix \(M\), a set of coset representatives of
the quotient group \( \mathbb{Z}^d / \mathbb{M}^d \) is also called a set of coset representatives associated with the periodicity matrix \( \mathbb{M} \). Employing the change of bases as in Section 2.3, the following corollary follows from Theorems 4.2.6 and 4.2.7.

**Corollary 4.2.8.** For any \( n \in \mathbb{N} \), \( C(A, n) \) is a set of coset representatives associated with both \( \mathbb{M}^A_n \) and \( (\mathbb{M}^A_n)^T \).

### 4.3 Type B Regular Hexagonal Structure

Throughout this research, we let

\[
\mathbf{v}_1^B = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \quad \mathbf{v}_2^B = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}
\]

\( L^B = \{ n_1(\mathbf{v}_1^B) + n_2(\mathbf{v}_2^B) : n_1, n_2 \in \mathbb{Z} \} \).

The lattice \( L^B \) is called the type B hexagonal lattice. In this section, we consider the type B RHS which is defined below. Subsequent discussion will show that the indexing of the type B RHS is more complicated than that of the type A RHS. To make the definition of the type B RHS easier, we first give the definition of a half type B RHS. For any \( s, d \in \mathbb{Z} \), the notation \( \text{mod}(s, d) \) denotes the remainder when \( s \) is divided by \( d \). For any integer \( n > 0 \), let

\[
\mathcal{R}^B_{n,h} = \{ x(\mathbf{v}_1^B) + (s-x)\mathbf{v}_2^B : x, s \in \mathbb{Z}, 0 \leq s \leq 3n, -n+r+2q \leq x \leq n+q \}
\]

where \( r = \text{mod}(s, 3) \), \( q = \frac{s-r}{3} \), \( \mathbf{v}_1^B = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \) and \( \mathbf{v}_2^B = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \). The set \( \mathcal{R}^B_{n,h} \) is called a half type B RHS at level \( n \). For any integer \( n > 0 \), we define \( \mathcal{R}^B_n = \mathcal{R}^B_{n,h} \cup -\mathcal{R}^B_{n,h} \) where \( -\mathcal{R}^B_{n,h} = \{ -u : u \in \mathcal{R}^B_{n,h} \} \). We call \( \mathcal{R}^B_n \) the \( n \)th level of the type B RHS.

**Remark:** Let \( x(\mathbf{v}_1^B) + (s-x)\mathbf{v}_2^B \in \mathcal{R}^B_n \). By the definitions of \( \mathcal{R}^B_{n,h} \) and \( \mathcal{R}^B_n \), if \( s > 0 \), then \( x(\mathbf{v}_1^B) + (s-x)\mathbf{v}_2^B \in \mathcal{R}^B_{n,h} \); if \( s < 0 \), then \( -(x(\mathbf{v}_1^B) + (s-x)\mathbf{v}_2^B) \in \mathcal{R}^B_{n,h} \); and if \( s = 0 \), then \( x(\mathbf{v}_1^B) + (s-x)\mathbf{v}_2^B \in \mathcal{R}^B_{n,h} \cap (-\mathcal{R}^B_{n,h}) \). Hence, for any given \( s \) satisfying \(-3n \leq s \leq 3n\), \( s \) corresponds to a horizontal layer of hexagons of \( \mathcal{R}^B_n \) as shown in Figure 4-2. \( \mathcal{R}^B_n \) has a total of \( 3n + 1 \) layers of hexagons. The lattice points and corresponding Voronoi cells of \( \mathcal{R}^B_2 \) and \( \mathcal{R}^B_3 \) are shown in Figure 4-2 (a) and Figure 4-2 (b), respectively. The middle layer
in each figure corresponds to \( s = 0 \) which means that the sum of the two components of each lattice point of that layer is 0. The layers above the middle layer correspond to \( s > 0 \), and the layers below the middle layer correspond to \( s < 0 \).

In the following, we are going to show that \( R_n^B \) is a set of coset representatives of the quotient group of two lattices. To make its proof more concise, we need the following preliminary results.

**Lemma 4.3.1.** For any integer \( n > 0 \), \( R_n^B \) consists of \( 9n^2 + 3n + 1 \) lattice points.

**Proof.** By the previous remark, \( R_n^B \) has a total of \( 3n + 1 \) layers of hexagons. The middle layer of \( R_n^B \) contains \( 2n + 1 \) lattice points. Any upper layer with \( s = 3q \) for some positive integer \( q \leq n \) has \((n + q) - (-n + 2q) + 1 = 2n - q + 1\) hexagons. Any upper layer with \( s = 3q + 1 \) for some positive integer \( q \leq n - 1 \) has \((n + q) - (-n + 1 + 2q) + 1 = 2n - q\) hexagons. Any upper layer with \( s = 3q + 2 \) for some positive integer \( q \leq n - 1 \) has \((n + q) - (-n + 2 + 2q) + 1 = 2n - q - 1\) hexagons. By the definition of \( R_n^B \), the number of hexagons down the middle layer is the same as that of hexagons above the middle layer. Hence the total number of those hexagons is

\[
(2n + 1) + 2 \sum_{q=1}^{n} (2n - q + 1) + 2 \sum_{q=0}^{n-1} (2n - q + 2n - q - 1) = 9n^2 + 3n + 1. \tag{4–1}
\]

\[
\square
\]

**Proposition 4.3.2.** If \( \mathbf{a}_k \in R_{n,h}^B \) for \( k = 1, 2 \), then \( \mathbf{a}_1 + \mathbf{a}_2 \in R_{2n,h}^B \).

**Proof.** By definition of \( R_{n,h}^B \), there exist \( x_k, s_k \in \mathbb{Z} \) such that \( \mathbf{a}_k = x_k (\mathbf{v}_1^B) + (s_k - x_k) \mathbf{v}_2^B \) with \( 0 \leq s_k \leq 3n \) and \(-n + r_k + 2q_k \leq x_k \leq n + q_k \) where \( r_k = \text{mod}(s_k, 3), q_k = \frac{s_k - r_k}{3} \). Let \( \bar{x} = x_1 + x_2 \) and \( \bar{s} = s_1 + s_2 \). Then

\[
\mathbf{a}_1 + \mathbf{a}_2 = x(\mathbf{v}_1^B) + (s - x) \mathbf{v}_2^B. \tag{4–2}
\]

Since \( 0 \leq s_k \leq 3n \) for \( k = 1, 2 \), we have \( 0 \leq \bar{s} = s_1 + s_2 \leq 6n \). The inequality \(-n + r_k + 2q_k \leq x_k \leq n + q_k \) for \( k = 1, 2 \) implies that \(-2n + (r_1 + r_2) + 2(q_1 + q_2) \leq 3n.
Figure 4-2. Arrays of the type $B$ RHS. (a) The lattice points of $\mathbb{R}_2^B$. (b) The lattice points of $\mathbb{R}_3^B$. The coordinates inside each hexagon are for the basis $\{v_1^B, v_2^B\}$.

If $x_1 + x_2 \leq 2n + (q_1 + q_2)$. If $r_1 + r_2 < 3$ let $\tilde{s} = s_1 + s_2$, $\tilde{q} = q_1 + q_2$ and $\tilde{r} = r_1 + r_2$. Then $-2n + \tilde{r} + 2\tilde{q} \leq \tilde{x} \leq 2n + \tilde{q}$ and hence $a_1 + a_2 \in \mathbb{R}_{2n,h}^B$ by Equation 4-2. If $r_1 + r_2 \geq 3$, then $r_1 + r_2 = 3 + \tilde{r}$ for some $0 \leq \tilde{r} < 3$. The equation $s_k = 3q_k + r_k$ for $k = 1, 2$ implies that $s_1 + s_2 = 3(q_1 + q_2) + r_1 + r_2 = 3(q_1 + q_2) + 3 + \tilde{r} = 3(q_1 + q_2 + 1) + \tilde{r}$. Now let $\tilde{q} = q_1 + q_2 + 1$. 
The inequality $x_k \leq n + q_k$ for $k = 1, 2$ implies that $\tilde{x} = x_1 + x_2 \leq 2n + (q_1 + q_2) \leq 2n + \tilde{q}$. On the other hand $-n + r_k + 2q_k \leq x_k$ for $k = 1, 2$ implies that $\tilde{x} = x_1 + x_2 \geq -2n + (r_1 + r_2) + 2(q_1 + q_2) = -2n + (3 + \tilde{r}) + 2(q_1 + q_2) = -2n + (1 + \tilde{r}) + 2(q_1 + q_2 + 1) \geq -2n + \tilde{r} + 2\tilde{q}$. Hence we have

$$-2n + \tilde{r} + 2\tilde{q} \leq \tilde{x} \leq 2n + \tilde{q}. \quad (4-3)$$

Since $\tilde{s} = s_1 + s_2 = (3q_1 + r_1) + (3q_2 + r_2) = 3(q_1 + q_2) + (r_1 + r_2) = 3(q_1 + q_2) + 3 + \tilde{r} = 3(q_1 + q_2 + 1) + \tilde{r} = 3\tilde{q} + \tilde{r}$, we have $\tilde{r} = \text{mod}(\tilde{s}, 3)$ and $\tilde{q} = \frac{\tilde{s} - \tilde{r}}{3}$. Hence, by Equation 4–2 and Inequality 4–3, $a_1 + a_2 \in B_{n,h} \subseteq B_n$.

**Proposition 4.3.3.** Let $a_k = x_k(v^B_1) + (s_k - x_k)v^B_2 \in B_{n,h}$ for $k = 1, 2$. If $s_1 \geq s_2$, then $a_1 - a_2 \in B_{2n,h}$.

**Proof.** If $s_1 = s_2$, then $a_1 - a_2 = (x_1 - x_2)v^B_2 + (0 - (x_1 - x_2))v^B_2$. Let $\tilde{x} = x_1 - x_2$. Since $a_k \in B_{n,h}$ for $k = 1, 2$, we have $-3n \leq x_k \leq 3n$. Hence $-6n \leq \tilde{x} = x_1 - x_2 \leq 6n$ and thus $a_1 - a_2$ lies on the middle layer of $B_{2n,h}$ since $s_1 - s_2 = 0$. If $s_1 > s_2$, let $r_k = \text{mod}(s_k, 3)$ and $q_k = \frac{s_k - r_k}{3}$. It then follows from the definition of $B_{n,h}$ that $-n + r_k + 2q_k \leq x_k \leq n + q_k$. Hence, $\tilde{x} = x_1 - x_2 \leq (n + q_1) - (-n + r_2 + 2q_2) = 2n - r_2 + q_1 - 2q_2$ and $\tilde{x} = x_1 - x_2 \geq (-n + r_1 + 2q_1) - (n + q_2) = -2n + r_1 + 2q_1 - q_2$. Thus we have

$$-2n + r_1 + 2q_1 - q_2 \leq \tilde{x} = x_1 - x_2 \leq 2n - r_2 + q_1 - 2q_2. \quad (4-4)$$

If $r_1 \geq r_2$, let $\tilde{q} = q_1 - q_2$ and $\tilde{r} = r_1 - r_2$. Then Inequality 4–4 implies that $-2n + \tilde{r} + 2\tilde{q} \leq \tilde{x} \leq 2n + \tilde{q}$ and, by the definition of $B_{n,h}$, we have $a_1 - a_2 \in B_{2n,h}$. If $r_1 < r_2$, then $r_2 > r_1 \geq 0$ and hence $r_2 \geq 1$. Let $\tilde{q} = q_1 - q_2 - 1$ and $\tilde{r} = 3 + r_1 - r_2$. Then $2n - r_2 + q_1 - 2q_2 \leq 2n - r_2 + q_1 - q_2 \leq 2n - 1 + q_1 - q_2 = 2n + \tilde{q}$, and $-2n + r_1 + 2q_1 - q_2 \geq -2n + r_1 + 2q_1 - q_2 - (q_2 - 2 - r_2) = -2n + \tilde{x} + 2\tilde{q}$. Obviously $s_1 - s_2 = 3(q_1 - q_2) + (r_1 - r_2) = 3(q_1 - q_2 - 1) + (3 + r_1 - r_2) = 3\tilde{q} + \tilde{r}$. Then by Inequality 4–4 it follows that $-2n + \tilde{r} + 2\tilde{q} \leq \tilde{x} \leq 2n + \tilde{q}$. Hence, by the definition of $B_{n,h}$, we have $a_1 - a_2 \in B_{2n,h}$.


Proposition 4.3.4. If $a_k \in \mathbb{R}^B_n$ for $k = 1, 2$, then $a_1 \pm a_2 \in \mathbb{R}^B_{2n}$.

Proof. Since $a_k \in \mathbb{R}_n^B$, either $a_k \in \mathbb{R}_{n,h}^B$ or $-a_k \in \mathbb{R}_{n,h}^B$. If $a_k \in \mathbb{R}_{n,h}^B$ for $k = 1, 2$, then from Proposition 4.3.2 we have that $a_1 + a_2 \in \mathbb{R}_{2n,h}^B \subseteq \mathbb{R}_{2n}^B$. If $-a_k \in \mathbb{R}_{n,h}^B$ for $k = 1, 2$, then $-a_1 - a_2 \in \mathbb{R}_{2n,h}^B$ and hence, by the definition of $\mathbb{R}_{n,h}^B$, we have $a_1 + a_2 \in \mathbb{R}_{2n}^B$. Similarly if just one of $a_1$ or $a_2$ is in $\mathbb{R}_{n,h}^B$, then by Proposition 4.3.3 $a_1 + a_2 \in \mathbb{R}_{2n}^B$. Since $a_2 \in \mathbb{R}_n^B$, we have $-a_2 \in \mathbb{R}_n^B$. Hence $a_1 - a_2 = a_1 + (-a_2) \in \mathbb{R}_{2n}^B$. \hfill \Box

Lemma 4.3.5. The maximal distance of the lattice points of $\mathbb{R}_n^B$ from the origin $0$ is $\sqrt{3}n$.

Proof. Without loss of generality, we only consider the lattice points of $\mathbb{R}_{n,h}^B$. For any $g \in \mathbb{R}_{n,h}^B$, there exist $x, s \in \mathbb{Z}$ with $0 \leq s \leq 3n, -n + r + 2q \leq x \leq n + q$, where $r = \text{mod}(s, 3), q = \frac{s - x}{3}$, such that $g = x(v_1^B) + (s - x)v_2^B$. Let $\psi(g)$ be the square of the distance of $g$ from the origin $0$, and let $\kappa = \max \{\psi(g) : g \in \mathbb{R}_{n,h}^B\}$. Since $g = xv_1^B + (s - x)v_2^B$, we have $\psi(g) = x^2 + (s - x)^2 - x(s - x) = 3x^2 - 3sx + s^2$. By replacing $s$ by $3q + r$ we have $\psi(g) = 3x^2 - 3x(3q + r) + (3q + r)^2$, which is a convex function of $x$. Hence the maximum of the function $\psi(g)$ can only be reached at the two end points of $x$. Since $-n + r + 2q \leq x \leq n + q$ and the values of the function $\psi(g)$ at two end points $x = -n + r + 2q$ and $x = n + q$ are the same, we have

$$\kappa \leq \max \{3(n + q)^2 - 3(n + q)(3q + r) + (3q + r)^2 : 0 \leq 3q + r \leq 3n, 0 \leq r \leq 2\}.$$ 

Let $\alpha(q, r) = 3(n + q)^2 - 3(n + q)(3q + r) + (3q + r)^2$. Obviously $\alpha(q, r) = 3n^2 - 3nq + 3q^2 + 3qr + r^2$ by simplification. If $r = 0$, then $0 \leq q \leq n$ and $\alpha(q, 0) = 3n^2 - 3nq + 3q^2$ which is a convex function of $q$. Hence $\max \{\alpha(q, 0) : q \in [0, n]\} = 3n^2$. If $r = 1$, then $0 \leq q \leq n - 1$ and $\alpha(q, 1) = 3n^2 - 3n(q + 1) + 3q^2 + 3q + 1$ which is also a convex function of $q$. Therefore $\max \{\alpha(q, 1) : q \in [0, n - 1]\} = 3n^2 - 3n + 1 \leq 3n^2$. If $r = 2$, then $0 \leq q \leq n - 1$ and $\alpha(q, 1) = 3n^2 - 3n(q + 2) + 3q^2 + 6q + 4$ which is a convex function of $q$. Hence $\max \{\alpha(q, 2) : q \in [0, n - 1]\} = 3n^2 - 3n + 1 \leq 3n^2$. Thus $\kappa \leq 3n^2$. Let
\( g = n(v_1^B) + 2n(v_2^B) \). It is easy to check that \( g \in \mathbb{R}^{B}_{n,h} \) and that \( \psi(g) = 3n^2 \). Therefore \( \kappa = 3n^2 \).

Let \( V^B \) be the matrix whose two columns are \( v_1^B \) and \( v_2^B \). For any integer \( n > 0 \), let
\[
M_n^B = \begin{pmatrix} 3n + 1 & 3n \\ 1 & 3n + 1 \end{pmatrix}, \quad \tilde{V}_n^B = (V^B)(M_n^B), \quad \text{and } \tilde{L}_n^B \text{ be the lattice generated by the two column vectors of } \tilde{V}_n^B. \text{ The following lemma will be used in Theorem 4.3.7.}

**Lemma 4.3.6.** \( \tilde{L}_n^B \cap \mathbb{R}^B_{2n} = 0. \)

**Proof.** Let \( b_1 \) and \( b_2 \) be the two column vectors of the matrix \( \tilde{V}_n^B \). Since \( \tilde{V}_n^B = (V^B)(M_n^B) = (V^B) \begin{pmatrix} 3n + 1 & 3n \\ 1 & 3n + 1 \end{pmatrix} \), we have \( b_1 = (3n + 1)v_1^B + v_2^B \) and \( b_2 = 3n(v_1^B) + (3n + 1)v_2^B \). Then
\[
b_1 = x(v_1^B) + (s - x)v_2^B \text{ where } x = 3n + 1 \text{ and } s = 3n + 2.
\]
Let \( r = \text{mod}(s, 3), q = \frac{s - r}{3}. \) It follows that \( r = 2 \) and \( q = n. \)

Hence \(-2n + r + 2q = -2n + 2 + 2n = 2 \) and \( 2n + q = 2n + n = 3n. \) Since \( x = 3n + 1 \notin [-2n + r + 2q, 2n + q], \) we have \( b_1 \notin \mathbb{R}^B_{2n,h} \) by the definition of \( \mathbb{R}^B_{n,h}. \)

Also \( s = 3n + 2 > 0 \) implies that \( b_1 \notin -\mathbb{R}^B_{2n,h}. \) Therefore \( b_1 \notin \mathbb{R}^B_{2n}. \) Similarly \( b_2 = x_2^B(v_1^B) + (s_2 - x_2^B)v_2^B \) where \( x_2 = 3n \) and \( s_2 = 6n + 1. \) Since \( s_2 = 6n + 1 > 3(2n), \) we have \( b_2 \notin \mathbb{R}^B_{2n,h} \) by the definition of \( \mathbb{R}^B_{n,h}. \) It follows from \( s_2 > 0 \) that \( b_2 \notin -\mathbb{R}^B_{2n,h}. \)

Therefore \( b_2 \notin \mathbb{R}^B_{2n}. \) Let \( g = b_2 - b_1. \) Then \( g = -v_1^B + 3n(v_2^B). \) It follows that \( g = x_3^B(v_1^B) + (s_3 - x_3^B)v_2^B \) where \( x_3 = -1 \) and \( s_3 = 3n - 1. \) Let \( r_3 = \text{mod}(s_3, 3), q_3 = \frac{s_3 - r_3}{3}. \)

Then we have \( r_3 = 2 \) and \( q_3 = n - 1. \) Hence \(-2n + r_3 + 2q_3 = -2n + 2 + 2(n - 1) = 0 \) and \( 2n + q_3 = 2n + n - 1 = 3n - 1. \) Since \( x_3 = -1 \notin [-2n + r + 2q, 2n + q], \) we have \( g \notin \mathbb{R}^B_{2n,h}. \) Of course, since \( s_3 = 3n - 1 > 0 \) where \( n \) is a positive integer, \( g \notin -\mathbb{R}^B_{2n,h}. \) Therefore \( g \notin \mathbb{R}^B_{2n}. \)

By the definition of \( \mathbb{R}^B_{n}, \mathbb{R}^B_{2n} \) is symmetric with respect to the origin. Hence we have also that \(-b_1, -b_2, -g \notin \mathbb{R}^B_{2n}. \)

Since \( \tilde{L}_n^B \) is a hexagonal lattice generated by \( \{b_1, b_2\} \) and the angle between \( b_1 \) and \( b_2 \) is \( \frac{\pi}{3}, \) the lattice points in the set \( \{\pm b_1, \pm b_2, \pm(b_2 - b_1)\} \) are the only 6 closest nonzero lattice points from the origin \( 0. \) Since \( b_1 = (3n + 1)v_1^B + v_2^B \) and the angle between \( v_1^B \)
and $v_2^B$ is $\frac{2\pi}{3}$, the norm of $b_1$ is $\sqrt{9n^2 + 3n + 1}$. It follows that the distance of any other nonzero lattice point of $\tilde{L}_n^B$ from the origin 0 is at least $\sqrt{3(9n^2 + 3n + 1)} > 5n$ as shown in Figure 4-3. However by Lemma 4.3.5, the maximal distance of the lattice points of $\mathfrak{R}^B_{2n}$ from the origin 0 is $\sqrt{3}(2n) < 4n$. Therefore $\tilde{L}_n^B \cap \mathfrak{R}^B_{2n} = \emptyset$.

Figure 4-3. The lattice points of $\mathfrak{R}^B_{2n}$ (red) and $L_B$ (blue) where $n = 6$.

**Theorem 4.3.7.** For any $n \in \mathbb{N}$, $\mathfrak{R}^B_n$ is a set of coset representatives of the quotient group $L^B/\tilde{L}_n^B$.

**Proof.** Since $V^B$ is a sampling matrix of the lattice $L^B$, $V_n^B$ a sampling matrix of the lattice $\tilde{L}_n^B$, and since $V_n^B = (V^B)(M_n^B)$, by Lemma 4.2.5, the order of the quotient group $L^B/\tilde{L}_n^B$ equals the determinant of $M_n^B$. It follows that the order of the quotient group $L^B/\tilde{L}_n^B$ is $9n^2 + 3n + 1$ since it is the determinant of $M_n^B$. Hence, to show $\mathfrak{R}^B_n$ is a set of coset representatives of $L^B/\tilde{L}_n^B$, it suffices to show that no two distinct lattice points of $\mathfrak{R}^B_n$ are congruent modulo $\tilde{L}_n^B$. For any $a, b \in \mathfrak{R}^B_n$, we have $a - b \in \mathfrak{R}^B_{2n}$ by Proposition 4.3.4. Suppose $a - b \in \tilde{L}_n^B$. By Lemma 4.3.6, we have $\tilde{L}_n^B \cap \mathfrak{R}^B_{2n} = \emptyset$. Hence $a - b \in \tilde{L}_n^B \cap \mathfrak{R}^B_{2n} = \emptyset$. It follows that $a = b$. Therefore $\mathfrak{R}^B_n$ is a set of coset representatives of $L^B/\tilde{L}_n^B$. \qed
It follows from Theorem 4.3.7 that the DFT on $\mathbb{R}_n^B$ is invertible. Next we consider the frequency domain. In the proof of Theorem 4.3.7, we have $\tilde{V}_n^B = (V^B)(M_n^B)$, where $V^B$ is a sampling matrix of the lattice $L^B$ and $\tilde{V}_n^B$ is a sampling matrix of the lattice $\tilde{L}_n^B$. Let $V_n^B = [(V_n^B)^T]^{-1}$. Since $V_n^B$ is a sampling matrix of the lattice $L_n^B$, $\tilde{V}_n^B$ is a sampling matrix of the dual lattice $(\tilde{L}_n^B)^*$ of $\tilde{L}_n^B$ by Lemma 2.3.2. Let $\hat{v}_{n,1}^B$ and $\hat{v}_{n,2}^B$ be the two column vectors of the matrix $\hat{V}_n^B$.

Let $\hat{\mathbb{R}}_{n,h}^B = \left\{ x(\hat{v}_{n,1}^B) + (s - x)\hat{v}_{n,2}^B : x, s \in \mathbb{Z}, 0 \leq s \leq 3n, -n + r + 2q \leq x \leq n + q \right\}$

where $r = \text{mod}(s, 3)$, $q = \frac{s - r}{3}$, $\hat{v}_{n,1}^B$ and $\hat{v}_{n,2}^B$ are defined as the above. For any integer $n > 0$, let $\hat{\mathbb{R}}_n^B = \hat{\mathbb{R}}_{n,h}^B \cup -\hat{\mathbb{R}}_{n,h}^B$ where $-\hat{\mathbb{R}}_{n,h}^B = \left\{ -u : u \in \hat{\mathbb{R}}_{n,h}^B \right\}$.

**Theorem 4.3.8.** For any $n \in \mathbb{N}$, $\hat{\mathbb{R}}_n^B$ is a set of coset representatives of the quotient group $(\tilde{L}_n^B)^*/(L^B)^*$.

**Proof.** The proof is similar to that of Theorem 4.2.7 and 4.3.7, and hence omitted. 

From Theorems 4.3.7 and 4.3.8, and by the definitions of $\mathbb{R}_{n,h}^B$ and $\hat{\mathbb{R}}_{n,h}^B$, it is easy to deduce that the geometric shape of the support in the frequency domain of the DFT on $\mathbb{R}_n^B$ is similar to that of the support in the spatial domain. Throughout this research, we also let

$$C(B, n) = \left\{ \begin{pmatrix} j \\ k \end{pmatrix} \in \mathbb{Z}^2 : j(\hat{v}_1^B) + k(\hat{v}_2^B) \in \mathbb{R}_n^B \right\}.$$ 

It will help to write the DFT on $\mathbb{R}_n^B$ in terms of a periodicity matrix. By the change of bases as in Section 2.3, the following corollary follows from Theorems 4.3.7 and 4.3.8.

**Corollary 4.3.9.** For any $n \in \mathbb{N}$, $C(B, n)$ is a set of coset representatives associated with both $M_n^B$ and $(M_n^B)^T$. 

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4.4 Relation Between the Type A and B RHS and Some Previously Studied Arrays

Consider the hexagonal array shown in Figure 4-4 (a) consisting of $48 = 3 \cdot 4^2$ lattice points. If $\Gamma_n$ denotes the $n^{th}$ level of this structure, then $\Gamma_n$ consists of $3n^2$ lattice points and corresponds to the hexagonal structure investigated in Sun and Yao [40] (also Figure 3-2 (b)). If we delete the points marked in red, then the number of lattice points reduces to $37 = 3 \cdot 4^2 - 3 \cdot 4 + 1$. The reduced array corresponds to the array $\mathcal{R}_3^A$ shown in Figure 4-4 (b). More generally, deleting the lattice points on the bottom and the lower two sides of $\Gamma_n$ reduces the $3n^2$ lattice points to $3n^2 - 3n + 1 = 3(n - 1)^2 + 3(n - 1) + 1$ lattice points. Hence, the remaining lattice points are exactly the lattice points of $\mathcal{R}_n^A$.

![Figure 4-4. Comparison of a previously studied array structure with the type A RHS. (a) The lattice points of $\Gamma_4$. (b) The lattice points of $\mathcal{R}_3^A$.](image)

Similarly, consider the hexagonal array shown in Figure 4-5 (a) consisting of $100 = n^2$ lattice points where $n = 3k + 1$ and $k = 3$. If $\Upsilon_k$ denotes the $k^{th}$ level of this structure, then $\Upsilon_n$ consists of $(3k + 1)^2$ lattice points and corresponds to the hexagonal structure investigated in Anterrieu et al. [2] (also Figure 3-3 (b)). If we delete the points marked in red, then the number of lattice points reduces to $91 = 9 \cdot 3^2 + 3 \cdot 3 + 1$. The reduced array corresponds to the array $\mathcal{R}_3^B$ shown in Figure 4-5 (b). More generally, deleting the lattice points on the bottom and the lower two sides of $\Upsilon_k$ reduces the $n^2$ lattice points to $(3k + 1)^2 - 3k = 9k^2 + 3k + 1$ lattice points. Hence, the remaining lattice points are exactly
the lattice points of $\mathbb{R}^B_k$. Also by comparing Figure 4-2 with Figure 3-3 (c), we can see that if we add $3k$ lattice points along 6 sides of $\mathbb{R}^B_k$, then the expanded array is the kind of array shown in Figure 3-3 (c).

Figure 4-5. Comparison of a previously studied array structure with the type $B$ RHS. (a) The lattice points of $\Upsilon_3$. (b) The lattice points of $\mathbb{R}^B_3$. 
CHAPTER 5
FAST ALGORITHMS FOR COMPUTING THE DFT ON THE TWO SPECIAL TYPES OF THE RHS

5.1 Convert the DFT on a Tile of a Tiling by Translations by a Sublattice to a Standard DFT

In this section we emphasize the correspondence between the domain of computing the original DFT on a lattice and that of computing the standard DFT generally presented in textbooks. Recall that Equation 2–1 for computing the DFT on a tile of a lattice is reduced to Equation 2–7. The use of the Smith normal form to convert Equation 2–7 to a standard DFT appeared previously in Problem 20, Chapter 2, of Dudgeon and Mersereau [10] and in Guessoum and Mersereau [19]. In this research, by a unimodular matrix we mean a square matrix with integer entries and determinant +1 or −1. Applying the Smith normal form to the periodicity matrix $M$ in Equation 2–7, we can obtain unimodular matrices $E$ and $F$ such that $M = EDF$ for some diagonal periodicity matrix $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, where $d_j \geq 0$ for $j = 1, 2$, and $d_1$ divides $d_2$. Let $g_{k,l} = E \begin{pmatrix} k \\ l \end{pmatrix}$, and $h_{k,l} = F^T \begin{pmatrix} k \\ l \end{pmatrix}$. Also, let $G = \{g_{k,l} : 0 \leq k < d_1, 0 \leq l < d_2\}$ and $H = \{h_{k,l} : 0 \leq k < d_1, 0 \leq l < d_2\}$. As in Section 2.3, any function $f$ defined on a set of coset representatives associated with $M$ can be extended periodically to the whole lattice $\mathbb{Z}^2$, where the association is defined in Section 2.3. Then we have the following proposition:

**Proposition 5.1.1.** Let $G, H, M$ and $f$ be as the above. Then $G$ and $H$ are sets of coset representatives associated with $M$ and $M^T$, respectively. Furthermore, the DFT $\hat{f}$ of $f$ satisfies

$$\hat{f}(h_{s,t}) = \sum_{k=0}^{d_1} \sum_{l=0}^{d_2} f(g_{k,l}) \cdot e^{-2\pi i \left(\frac{sk}{d_1} + \frac{tl}{d_2}\right)} \text{ for all } h_{s,t} \in H. \quad (5–1)$$
Proof. Let $E, F$ and $D$ be as the above and let $\Gamma = \{\varepsilon_{k,l} : 0 \leq k < d_1, 0 \leq l < d_2\}$ where

$$
\varepsilon_{k,l} = \begin{pmatrix} k \\ l \end{pmatrix}.
$$

Obviously $\Gamma$ is a set of coset representatives associated with $D$. We claim
that $G$ and $H$ are sets of coset representatives associated with $M$ and $M^T$, respectively.

If there exist $g_{k_1,l_1}, g_{k_2,l_2} \in G$ such that $g_{k_1,l_1} - g_{k_2,l_2} = Mz$ for some $z \in \mathbb{Z}^2$, then

$$
E(\varepsilon_{k_1,l_1} - \varepsilon_{k_2,l_2}) = Mz = EDFz
$$
and hence $\varepsilon_{k_1,l_1} - \varepsilon_{k_2,l_2} = Dz$. Since $\varepsilon_{k_1,l_1}, \varepsilon_{k_2,l_2} \in \Gamma$ and $\Gamma$ is a set of coset representatives associated with $D$, it follows that $\varepsilon_{k_1,l_1} = \varepsilon_{k_2,l_2}$. Thus $G$
is a set of coset representatives associated with $M$. Similarly we can show that $H$
is a set of coset representatives associated with $M^T$. By Equation 2–7 we have

$$
\hat{f}(h_{s,t}) = \sum_{k=0}^{d_1} \sum_{l=0}^{d_2} f(g_{k,l}) \cdot e^{-2\pi i (h_{s,t})^T M^{-1} g_{k,l}} \text{ for all } h_{s,t} \in H. \tag{5-2}
$$

Since $(h_{s,t})^T M^{-1} g_{k,l} = (F^T \varepsilon_{s,t})^T (EDF)^{-1} E \varepsilon_{k,l} = \varepsilon_{s,t}^T D^{-1} \varepsilon_{k,l} = \frac{s_k}{d_1} + \frac{t_l}{d_2}$, we have

$$
\hat{f}(h_{s,t}) = \sum_{k=0}^{d_1} \sum_{l=0}^{d_2} f(g_{k,l}) \cdot e^{-2\pi i \left( \frac{s_k}{d_1} + \frac{t_l}{d_2} \right)} \text{ for all } h_{s,t} \in H. \tag{5-3}
$$

Thus, $\hat{f}$ can be computed using algorithms of the standard DFT.

Corollary 5.1.2. Let $G$, $H$, and $f$ be the same as in Proposition 5.1.1, and $M =$

$$
\begin{pmatrix} r & -s \\ s & r-s \end{pmatrix}
$$

be a periodicity matrix such that $r$ and $s$ are relatively prime. Then there
exist unimodular matrices $E$ and $F$ such that $M = EDF$ where $D = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}$ and

$N = |\det(M)|$. Hence the DFT $\hat{f}$ of $f$ can be converted to a 1-dimensional standard DFT.

Proof. Without loss of generality we assume that $\det(M) \geq 0$. Since $r$ and $s$ are relatively
prime, there exist integers $x$ and $y$ such that $xr + ys = 1$. It follows from $xr + ys = 1$
that $\begin{pmatrix} x & y \\ -s & r \\ s & r-s \end{pmatrix} \begin{pmatrix} r & -s \\ s & r-s \end{pmatrix} \begin{pmatrix} 1 & sx - (r-s)y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}$ and that the matrix

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\(
\begin{pmatrix}
  x & y \\
-\frac{s}{r} & r
\end{pmatrix}
\)

is unimodular. Let
\[
E = \begin{pmatrix}
  x & y \\
-\frac{s}{r} & r
\end{pmatrix}^{-1}
\quad \text{and} \quad
F = \begin{pmatrix}
  1 & -sx + (r-s)y \\
0 & 1
\end{pmatrix}.
\]

Then \( M = EDF \). Hence by Proposition 5.1.1 the DFT \( \hat{f} \) of \( f \) can be converted to a 1-dimensional standard DFT.

\[
\text{5.2 Fast Algorithms for the DFT and its Inverse on the Type A RHS}
\]

Let \( f \) be a complex function defined on \( \mathbb{R}_n^A \). By Equation 2–5, its DFT \( \hat{f} \) satisfies the following equation:
\[
\hat{f}(q) = \sum_{p \in \mathbb{R}_n^A} f(p) \cdot e^{-2\pi i (p,q)}, \quad \text{for all } q \in \hat{\mathbb{R}}_n^A.
\]  

(5–4)

As in Section 2.3, \( f \) can be extended periodically to the whole lattice \( L^A \). It follows from Equation 2–5 that, for any set of coset representatives \( P \) of \( L^A/\tilde{L}_n^A \), the set \( \mathbb{R}_n^A \) can be replaced by \( P \). Similarly, the set \( \hat{\mathbb{R}}_n^A \) can be replaced by a set of coset representatives \( Q \) of the quotient group \( (\tilde{L}_n^A)^*/(L^A)^* \).

To simplify our notation, let \( f \) and \( \hat{f} \) be two functions defined on \( \mathbb{Z}^2 \) such that \( f(m) = f((V^A)m) \) and \( \hat{f}(m) = \hat{f}(V^A m) \) for any \( m \in \mathbb{Z}^2 \), where \( V^A = [(V^A M^A_n)^T]^{-1} \).

Recall in Chapter 4 that
\[
C(A,n) = \left\{ \begin{pmatrix} j \\ k \end{pmatrix} \in \mathbb{Z}^2 : \ j(v_1^A) + k(v_2^A) \in \mathbb{R}_n^A \right\} = \left\{ \begin{pmatrix} j \\ k \end{pmatrix} \in \mathbb{Z}^2 : \ |j| \leq n, |k| \leq n, |j-k| \leq n \right\}.
\]  

(5–5)

By Corollary 4.2.8, \( C(A,n) \) is a set of coset representatives associated with both \( M^A_n \) and \( (M^A_n)^T \), where the matrix \( M^A_n = \begin{pmatrix} 2n+1 & n \\ n+1 & 2n+1 \end{pmatrix} \) is defined in Section 4.2. Hence, by Equation 2–7, the DFT of \( f \) satisfies
\[
\hat{f}(k) = \sum_{m \in C(A,n)} f(m) \cdot e^{-2\pi i k^T (M^A_n)^{-1}m} \quad \text{for all } k \in C(A,n).
\]  

(5–6)
For any integer $n > 0$ let $P_1 = \left\{ \begin{pmatrix} 0 \\ l \end{pmatrix} : 0 \leq l \leq 3n^2 + 3n \right\}$ and $P_2 = \left\{ \begin{pmatrix} 2l \\ l \end{pmatrix} : 0 \leq l \leq 3n^2 + 3n \right\}$. The following proposition shows that the DFT of $f$ on $C(A, n)$ can be converted into a standard DFT.

**Proposition 5.2.1.** For any integer $n > 0$, each of the sets $P_1$ and $P_2$ is a set of coset representatives associated with both $M_n^A$ and $(M_n^A)^T$. Furthermore, $\hat{f}(\begin{pmatrix} 2k \\ k \end{pmatrix}) = \sum_{m=0}^{3n^2+3n} f(\begin{pmatrix} 0 \\ m \end{pmatrix}) \cdot e^{-2 \pi i lm (3n^2+3n+1)}$ for all $0 \leq k \leq 3n^2 + 3n$.

**Proof.** By the Smith normal form of $M_n^A$, we have $M_n^A = E(D_n^A)F$ where $E = \begin{pmatrix} 1 & 0 \\ -(3n+1) & 1 \end{pmatrix}$, $D_n^A = \begin{pmatrix} 1 & 0 \\ 0 & 3n^2+3n+1 \end{pmatrix}$, and $F = \begin{pmatrix} 2n+1 & n \\ 2 & 1 \end{pmatrix}$. By computing the matrix products $E \begin{pmatrix} 0 \\ k \end{pmatrix}$ and $F^T \begin{pmatrix} 0 \\ k \end{pmatrix}$, we have $P_1 = \{ E \begin{pmatrix} 0 \\ k \end{pmatrix} : 0 \leq k \leq 3n^2 + 3n \}$ and $P_2 = \{ F^T \begin{pmatrix} 0 \\ k \end{pmatrix} : 0 \leq k \leq 3n^2 + 3n \}$. It follows from Proposition 5.1.1 that $P_1$ is a set of coset representatives associated with $M_n^A$, $P_2$ is a set of coset representatives associated with $(M_n^A)^T$, and $\hat{f}(\begin{pmatrix} 2k \\ k \end{pmatrix}) = \sum_{m=0}^{3n^2+3n} f(\begin{pmatrix} 0 \\ m \end{pmatrix}) \cdot e^{-2 \pi i lm (3n^2+3n+1)}$, for all $k$ satisfying $0 \leq k \leq 3n^2 + 3n$.

By the Smith normal form of $(M_n^A)^T$, we have $(M_n^A)^T = E(D_n^A)F$, where $E = \begin{pmatrix} 1 & 0 \\ 3n+2 & 1 \end{pmatrix}$ and $F = \begin{pmatrix} 2n+1 & n+1 \\ -2 & -1 \end{pmatrix}$ are unimodular. It now follows from Proposition 5.1.1 that the set $P_1$ is a set of coset representatives associated with $(M_n^A)^T$ and the set $P_2$ is a set of coset representatives associated with $M_n^A$. 

The next Corollary follows from Proposition 5.2.1 and the use of change of bases.
Corollary 5.2.2. For any \( n \in \mathbb{N} \), each of the sets \( \{ k(v^A_2) : 0 \leq k \leq 3n^2 + 3n \} \) and \( \{ 2k(v^A_1) + k(v^A_2) : 0 \leq k \leq 3n^2 + 3n \} \) is a set of coset representatives of the quotient group \( L^A/\widetilde{L}^A_n \). Similarly each of the sets \( \{ k(\hat{v}^A_{n,2}) : 0 \leq k \leq 3n^2 + 3n \} \) and \( \{ 2k(\hat{v}^A_{n,1}) + k(\hat{v}^A_{n,2}) : 0 \leq k \leq 3n^2 + 3n \} \) is a set of coset representatives of the quotient group \( (\widetilde{L}^A_n)^*/(L^A)^* \).

Because \( f \) is originally defined on \( C(A, n) \), to apply Proposition 5.2.1 to get the values of \( \hat{f} \) on \( C(A, n) \), we need an efficient correspondence between \( C(A, n) \) and \( P_1 \) in the spatial domain, and an efficient correspondence between \( C(A, n) \) and \( P_2 \) in the frequency domain.

(a) The correspondence between \( C(A, n) \) and \( P_1 \) in the spatial domain

Since each of \( C(A, n) \) and \( P_1 \) is a set of coset representatives associated with \( M^A_n \), there exists a bijection \( \varphi \) between them such that \( a \equiv \varphi(a) \mod M^A_n \) for any \( a \in C(A, n) \). The bijection \( \varphi \) is called the correspondence between \( C(A, n) \) and \( P_1 \) in the spatial domain. In the following, we give an efficient method to find the correspondence \( \varphi \).

Let \( a = \begin{pmatrix} u \\ v \end{pmatrix} \in C(A, n) \). By the definition of \( C(A, n) \), we have \( |u| \leq n \), \( |v| \leq n \) and \( |u - v| \leq n \). We consider the following three cases for \( u \).

Case 1: \( u = 0 \). Because \( u = 0 \), we have \( a = \begin{pmatrix} 0 \\ v \end{pmatrix} \). If \( 0 \leq v \leq n \), since

\[
P_1 = \left\{ \begin{pmatrix} 0 \\ k \end{pmatrix} : 0 \leq k \leq 3n^2 + 3n \right\},
\]

we have \( a \in P_1 \). Hence \( \varphi(a) = a \). If \( -n \leq v < 0 \), then \( \begin{pmatrix} 0 \\ 3n^2 + 3n + v + 1 \end{pmatrix} \in P_1 \). Let \( b = \begin{pmatrix} 0 \\ 3n^2 + 3n + v + 1 \end{pmatrix} \). Because \( b - a = \begin{pmatrix} 0 \\ 3n^2 + 3n + v + 1 \end{pmatrix} \) and \( M^A_n = \begin{pmatrix} 2n + 1 & n \\ n + 1 & 2n + 1 \end{pmatrix} \), we have \( b - a = M^A_n \begin{pmatrix} -n \\ 2n + 1 \end{pmatrix} \).

Hence \( a \equiv b \mod M^A_n \). It follows that \( \varphi(a) = b = \begin{pmatrix} 0 \\ 3n^2 + 3n + v + 1 \end{pmatrix} \).
Case 2: $-n \leq u < 0$. Let $b = \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$ where $b_2 = (3n+1)(u+n) + v + 2n + 1$. Then
\[
b - a = \begin{pmatrix} -u \\ (3n+1)(u+n) + 2n + 1 \end{pmatrix} = \mathbf{M}_n^A \begin{pmatrix} -(u+n) \\ 2(u+n) + 1 \end{pmatrix}.
\]
Hence $a \equiv b \mod \mathbf{M}_n^A$.

Because $-n \leq u < 0$ and $-n \leq v < u + n$, we have $b_2 \geq (3n+1)(-n+n) - n + 2n + 1 = n + 1$ and $b_2 \leq (3n+1)(-1+n) + (-1+n) + 2n + 1 = 3n^2 + n - 1 \leq 3n^2 + 2n$. Since $b = \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$, it follows that $b \in \mathcal{P}_1$. Thus $\varphi(a) = b = \begin{pmatrix} 0 \\ (3n+1)(u+n) + v + 2n + 1 \end{pmatrix}$.

Case 3: $0 < u \leq n$. Let $b = \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$ where $b_2 = (3n+1)u + v$. Then
\[
b - a = \begin{pmatrix} -u \\ (3n+1)u \end{pmatrix} = \mathbf{M}_n^A \begin{pmatrix} -u \\ 2u \end{pmatrix}.
\]
Hence $a \equiv b \mod \mathbf{M}_n^A$. Since $1 \leq u \leq n$ and $u-n \leq v \leq n$, we have $b_2 = (3n+1)u + v \geq (3n+1)(1+n) = 2n+2$ and $b_2 = (3n+1)u + v \leq (3n+1)n + n = 3n^2 + 2n$. It follows that $b \in \mathcal{P}_1$. Thus $\varphi(a) = b = \begin{pmatrix} 0 \\ (3n+1)u + v \end{pmatrix}$.

(b) **The correspondence between $\mathcal{P}_2$ and $C(A,n)$ in the frequency domain**

By Proposition 5.2.1 and Corollary 4.2.8, each of the sets $\mathcal{P}_2$ and $C(A,n)$ is a set of coset representatives associated with the matrix $(\mathbf{M}_n^A)^T$. Hence there exists a bijection $\chi$ between $\mathcal{P}_2$ and $C(A,n)$ such that $g \equiv \chi(g) \mod (\mathbf{M}_n^A)^T$ for any $g \in \mathcal{P}_2$. Such a bijection is called the correspondence between $\mathcal{P}_2$ and $C(A,n)$ in the frequency domain.

In the following, we show an efficient method to find the bijection $\chi$. To make expressions simpler, we assume that $n$ is odd, and let $n = 2l + 1$ for some integer $l > 0$. When $n$ is even, the correspondence can be similarly shown. The method consists of the following two steps. In Step 1, we find the correspondence between $\mathcal{P}_2$ and a set $U$ which will be defined in Step 1. In Step 2, we find the correspondence between $U$ and $C(A,n)$.
Step 1: The correspondence between \( \mathcal{P}_2 \) and \( U \).

For any \( h \in \mathbb{Z} \) satisfying \( 0 \leq h \leq l \), let \( D_{0,h} = h \left( \begin{array}{cc} 2 \\ 1 \end{array} \right) \). Also for any \( k \in \mathbb{Z} \) satisfying \( 1 \leq k \leq l \), let

\[
A_{k,h} = h \left( \begin{array}{cc} 2 \\ 1 \end{array} \right) + \left( \begin{array}{c} 1 \\ k \end{array} \right)
\]

for any \( h \in \mathbb{Z} \) satisfying \(-(l + 1) \leq h \leq l\), and let \( B_{k,h} = h \left( \begin{array}{cc} 2 \\ 1 \end{array} \right) + \left( \begin{array}{c} 0 \\ k \end{array} \right) \) for any \( h \in \mathbb{Z} \) satisfying \(-l \leq h \leq l\). Furthermore let

\[
D_0 = \{ D_{0,h} : h \in \mathbb{Z}, 0 \leq h \leq l \}, \quad A_k = \{ A_{k,h} : h \in \mathbb{Z}, -(l + 1) \leq h \leq l \}, \quad B_k = \{ B_{k,h} : h \in \mathbb{Z}, -l \leq h \leq l \},
\]

and let \( U = D_0 \cup (\bigcup_{k=1}^{3l+2} A_k) \cup (\bigcup_{k=1}^{3l+2} B_k) \). Then \( U \) contains

\[
l + 1 + (3l + 2)(4l + 3) = \frac{n-1}{2} + 1 + \frac{3n+1}{2}(2n + 1) = 3n^2 + 3n + 1
\]

lattice points each of which will be shown to be congruent to an element of \( C(A, n) \) with respect to the modulo relation associated with the matrix \( (M_n^A)^T \). When \( n = 3 \), the lattice points enclosed by the blue dotted polygon in Figure 5-1 are exactly the lattice points of \( U \). In the following, we show an efficient method to find a bijection \( \Phi \) between \( \mathcal{P}_2 \) and \( U \) such that \( g \equiv \Phi(g) \mod (M_n^A)^T \) for any \( g \in \mathcal{P}_2 \).

Since \( g \in \mathcal{P}_2 \), there exists \( m \in \mathbb{Z} \) satisfying \( 0 \leq m \leq 3n^2 + 3n \) such that \( g = \left( \begin{array}{c} 2m \\ m \end{array} \right) \).

To find \( \Phi(g) \), we consider the following two cases.

**Case 1:** If \( 0 \leq m \leq l \), then \( g \in D_0 \subset U \). Hence \( \Phi(g) = g \).

**Case 2:** If \( l < m \leq 3n^2 + 3n \), since \( l + (4l + 3)(3l + 2) = 3n^2 + 3n \), there exists \( k \in \mathbb{Z} \) satisfying \( 1 \leq k \leq 3l + 2 \) and \( t \in \mathbb{Z} \) satisfying \( 1 \leq t \leq 4l + 3 \) such that \( m = l + (4l + 3)(k - 1) + t \).

When \( 1 \leq t \leq 2l + 2 \), let \( h = t - l - 2 \). Then \( t = h + l + 2 \). Since \( m = l + (4l + 3)(k - 1) + t \), it follows that \( m = 2l + 2 + (4l + 3)(k - 1) + h \). Hence

\[
g - A_{k,h} = \left( \begin{array}{c} 2m \\ m \end{array} \right) - \left( \begin{array}{c} 2h + 1 \\ h + k \end{array} \right) = \left( \begin{array}{c} 4l + 4 + (8l + 6)(k - 1) + 2h \\ 2l + 2 + (4l + 3)(k - 1) + h \end{array} \right)
\].
\[
\begin{pmatrix}
2h + 1 \\
h + k
\end{pmatrix}
= \begin{pmatrix}
4l + 3 + 2(4l + 3)(k - 1) \\
2l + 2 + (4l + 3)(k - 1) - k
\end{pmatrix}
= \begin{pmatrix}
(4l + 3)(2k - 1) \\
(2l + 1)(2k - 1)
\end{pmatrix}
= \begin{pmatrix}
4l + 3 \\
2l + 1
\end{pmatrix}
\begin{pmatrix}
2k - 1 \\
0
\end{pmatrix}
= \begin{pmatrix}
2n + 1 \\
n
\end{pmatrix}
\begin{pmatrix}
n + 1 \\
2n + 1
\end{pmatrix}
= \begin{pmatrix}
2k - 1 \\
0
\end{pmatrix}.
\]
Thus \( g \equiv A_{k,h} \mod (M_A^T)^T \). Since \( A_{k,h} \in U \), it follows that \( \Phi(g) = A_{k,h} \).

When \( 2l + 3 \leq t \leq 4l + 3 \), let \( h = t - 3l - 3 \). Then \( t = h + 3l + 3 \). Since
\[
m = l + (4l + 3)(k - 1) + t,
\]
it follows that \( m = (4l + 3)(k - 1) + h + 4l + 3 = k(4l + 3) + h \).

Hence \( g - B_{k,h} = \begin{pmatrix} 2m \\ m \end{pmatrix} - \begin{pmatrix} 2h \\ h + k \end{pmatrix} = \begin{pmatrix} 2k(4l + 3) + 2h \\ k(4l + 3) + h \end{pmatrix} - \begin{pmatrix} 2h \\ h + k \end{pmatrix} = \begin{pmatrix} 2k(4l + 3) \\ k(4l + 2) \end{pmatrix} = \begin{pmatrix} 4l + 3 \\ 2l + 1 \end{pmatrix} \begin{pmatrix} 2k \\ 0 \end{pmatrix} = \begin{pmatrix} 2n + 1 \\ n \end{pmatrix} \begin{pmatrix} n + 1 \\ 2n + 1 \end{pmatrix} = \begin{pmatrix} 2k \\ 0 \end{pmatrix} \). Thus
\( g \equiv B_{k,h} \mod (M_A^T)^T \). Since \( A_{k,h} \in U \), it follows that \( \Phi(g) = B_{k,h} \).

We have shown that, with respect to the modulo relation associated with the matrix \( (M_A^T)^T \), the elements of \( P_2 \) correspond sequentially to the elements of \( D_0, A_1, B_1, A_2, B_2, \ldots, A_{3l+2}, \) and \( B_{3l+2} \).

**Step 2: The correspondence between \( U \) and \( C(A,n) \) in the frequency domain.**

Recall in Step 1 that \( U = D_0 \cup \bigcup_{k=1}^{3l+2} A_k \cup \bigcup_{k=1}^{3l+2} B_k \). In this step, we show an efficient method to find a bijection \( \Psi \) between \( U \) and \( C(A,n) \) such that \( u \equiv \Psi(u) \mod (M_A^T)^T \) for any \( u \in U \).

By the definitions of \( D_0 \) and \( C(A,n) \), we have \( D_0 \subseteq C(A,n) \). It follows that \( \Psi(u) = u \) for any \( u \in D_0 \).

For any \( u \in A_k \), by the definition of \( A_k \), there is \( h \in \mathbb{Z} \) satisfying \(-l - 1 \leq h \leq l\) such that \( u = A_{k,h} \) where \( A_{k,h} = \begin{pmatrix} 2h + 1 \\ h + k \end{pmatrix} \). We consider the following three cases for \( k \) to find \( \Psi(u) \).
Case 1: $1 \leq k \leq l + 1$. Let $a = 2h + 1$ and $b = h + k$. Since $-l - 1 \leq h \leq l$, we have $-2l - 1 \leq a \leq 2l + 1$. Because $n = 2l + 1$, it follows that $-n \leq a \leq n$. Hence $|a| \leq n$.

Since $-l - 1 \leq h \leq l$ and $1 \leq k \leq l + 1$, we have $-l \leq b \leq 2l + 1 = n$. Since $-l > -n$, it follows that $-n \leq b \leq n$. To ensure $A_{k,h} \subset C(A,n)$, we also need to show that $|a - b| \leq n$.

Since $-l - 1 \leq h \leq l$ and $1 \leq k \leq l + 1$, we have $h + 1 - k \leq l + 1 - 1 = l < n$ and $h + 1 - k \geq (-l - 1) + 1 - (l + 1) = -2l - 1 = -n$. Since $a - b = h + 1 - k$, it follows that $|a - b| \leq n$. Therefore $A_{k,h} \subset C(A,n)$. It follows that $\Psi(u) = \Psi(A_{k,h}) = A_{k,h}$.

Case 2: $l + 2 \leq k \leq 2l + 1$. Since $h$ satisfies $-(l + 1) \leq h \leq l$, we consider the following three sub-cases.
Sub-case 1: If \(-(l + 1) \leq h \leq k - 2l - 3\), let \(a = 2h + n + 1\), \(b = h + k - n - 1\) and \(\tilde{u} = u - \begin{pmatrix}
-n \\
n + 1
\end{pmatrix}
\). Since \(u = A_{k,h} = \begin{pmatrix}
2h + 1 \\
h + k
\end{pmatrix}
\), we have \(\tilde{u} = \begin{pmatrix}
2h + 1 \\
h + k
\end{pmatrix}
\) – \(\begin{pmatrix}
-n \\
n + 1
\end{pmatrix}
\) = \(\begin{pmatrix}
2h + n + 1 \\
h + k - n - 1
\end{pmatrix}
\). We claim that \(\tilde{u} \in C(A,n)\). Since \(l + 2 \leq k \leq 2l + 1\) and \(-l + 1 \leq h \leq k - 2l - 3\), we have \(a = 2h + n + 1 \leq 2(k - 2l - 3) + n + 1 \leq 2(2l + 1 - 2l - 3) + n + 1 = n - 3 < n\) and \(a \geq -2(l + 1) + n + 1 = -2(l + 1) + 2l + 1 + 1 = 0 > -n\). Hence \(|a| < n\).

For the same reason, we have \(b = h + k - n - 1 \leq (k - 2l - 3) + k - n - 1 = 2k - 2l - n - 4 \leq 2(2l + 1) - 2l - n - 4 = 2l - n - 2 = 2l - (2l + 1) - 2 = -3 < n\), and \(b \geq -(l + 1) + (l + 2) - n - 1 = -n\). Hence \(|b| \leq n\). Also we have \(a - b = (2h + n + 1) - (h + k - n - 1) = h + 2n + 2 - k \leq (k - 2l - 3) + 2n + 2 - k = 2n - 2l - 1 = 2n - n = n\), and \(a - b = h + 2n + 2 - k \geq -(l + 1) + 2n + 2 - (2l + 1) = 2n - 3l = 2(2l + 1) - 3l = l + 2 > 0 > -n\). Hence \(|a - b| \leq n\). Therefore \(\tilde{u} \in C(A,n)\).

Because \(u - \tilde{u} = \begin{pmatrix}
-n \\
n + 1
\end{pmatrix}
\) = \(\begin{pmatrix}
2n + 1 \\
n 2n + 1
\end{pmatrix}
\) \(\begin{pmatrix}
-1 \\
1
\end{pmatrix}
\), we have \(u \equiv \tilde{u} \mod (M_n^A)^T\). It follows that \(\Psi(u) = \tilde{u}\).

Sub-case 2: If \(k - 2l - 2 \leq h \leq 2l - k + 1\), we claim that \(u \in C(A,n)\). Let \(a = 2h + 1\) and \(b = h + k\). Since \(u = A_{k,h} = \begin{pmatrix}
2h + 1 \\
h + k
\end{pmatrix}
\), we need to show that \(|a| \leq n\), \(|b| \leq n\), and \(|a - b| \leq n\). Because \(l + 2 \leq k \leq 2l + 1\) and \(k - 2l - 2 \leq h \leq 2l - k + 1\), we have \(2h + 1 \leq 2(2l - k + 1) + 1 = 4l - 2k + 3 \leq 4l - 2(2l + 2) = 2l - 2 < 2l + 1 = n\), and \(2h + 1 \geq 2(k - 2l - 2) + 1 = 2k - 4l - 3 \geq 2(l + 2) - 4l - 3 = -2l + 1 > -2l - 1 = -n\). Hence \(|2h + 1| < n\), i.e., \(|a| < n\). For the same reason, we have \(h + k \leq (2l - k + 1) + k = 2l + 1 = n\), and \(h + k \leq (k - 2l - 2) + k = 2k - 2l - 2 \geq 2(l + 2) - 2l - 2 = 2 > -n\). Hence \(|h + k| < n\), i.e., \(|b| < n\). We also have \(h + 1 - k \leq (2l - k + 1) + 1 - k = 2l + 2 - 2k \leq 2l + 2 - 2(l + 2) = -2 < n\), and \(h + 1 - k \geq (k - 2l - 2) + 1 - k = -2l - 1 = -n\). Hence \(|h + 1 - k| \leq n\). Since
\[ a - b = h + 1 - k, \] it follows that \(|a - b| \leq n. Therefore \( u \in C(A, n) \). It follows that \( \Psi(u) = u \).

**Sub-case 3:** If \( 2l - k + 2 \leq h \leq l \), let \( a = 2h - n, b = h + k - 2n - 1 \) and
\[
\tilde{u} = u - \begin{pmatrix}
  n + 1 \\
  2n + 1 
\end{pmatrix}.
\]
Since \( u = A_{k,h} = \begin{pmatrix} 2h + 1 \\ h + k \end{pmatrix} \), we have \( \tilde{u} = \begin{pmatrix} 2h + 1 \\ h + k \end{pmatrix} - \begin{pmatrix}
  n + 1 \\
  2n + 1 
\end{pmatrix} = \begin{pmatrix} 2h - n \\ h + k - 2n - 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \)

We claim that \( \tilde{u} \in C(A, n) \). Since \( l + 2 \leq k \leq 2l + 1 \) and \( 2l - k + 2 \leq h \leq l \), we have \( a = 2h - n \leq 2l - n < 2l < n \) and \( a \geq 2(2l - k + 2) - n = 4l - 2k + 4 - n \geq 4l - 2(2l + 1) + 4 - n = 2 - n > -n \). Thus \(|a| < n. For the same reason, we have \( b = h + k - 2n - 1 \leq l + (2l + 1) - 2n - 1 = 3l - 2n - 1 = 3n - 2n = n, \) and \( b \geq (2l - k + 2) + k - 2n - 1 = 2l + 2n = n - 2n = -n \). Thus \(|b| \leq n. Also we have \( a - b = (2h - n) - (h + k - 2n - 1) = h + n + 1 - k \leq l + n + 1 - k \leq l + n + 1 - (l + 2) = n - 1 < n, \) and \( a - b = h + n + 1 - k \geq (2l - k + 2) + n + 1 - k = 2l + n - 2k + 3 = 2l + n - 2(2l + 1) + 3 = n - 2l + 1 > -2l > -2l - 1 = -n. Thus \(|a - b| \leq n. Therefore \( \tilde{u} \in C(A, n). \)

Because \( u - \tilde{u} = \begin{pmatrix}
  n + 1 \\
  2n + 1 
\end{pmatrix} = \begin{pmatrix} 2n + 1 \\ n \\ 2n + 1 
\end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), we have \( u \equiv \tilde{u} \mod (M^A_n)^T. \) It follows that \( \Psi(u) = \tilde{u}. \)

**Case 3:** \( 2l + 2 \leq k \leq 3l + 2. \) Since \( h \) satisfies \(-(l + 1) \leq h \leq l, \) we consider the following two sub-cases.

**Sub-case 1:** If \(-(l + 1) \leq h \leq -1, \) let \( a = 2h + n + 1, b = h + k - n - 1 \) and
\[
\tilde{u} = u - \begin{pmatrix}
  -n \\
  n + 1 
\end{pmatrix}.
\]
Since \( u = A_{k,h} = \begin{pmatrix} 2h + 1 \\ h + k \end{pmatrix} \), we have \( \tilde{u} = \begin{pmatrix} 2h + 1 \\ h + k \end{pmatrix} - \begin{pmatrix}
  -n \\
  n + 1 
\end{pmatrix} = \begin{pmatrix} 2h + n + 1 \\ h + k - n - 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \)

We claim that \( \tilde{u} \in C(A, n). \) Since \( 2l + 2 \leq k \leq 3l + 2 \) and \(-(l + 1) \leq h \leq -1, \) we have \( a = 2h + n + 1 \leq -2 + n + 1 < n \) and \( a \geq -2(l + 1) + n + 1 = -2l + 1 + 2l + 1 + 1 = 0 > -n. Hence \(|a| < n. For the same reason, we have \( b = h + k - n - 1 \leq -1 + k - n - 1 = k - n - 2 \leq -1. \) Therefore \( \tilde{u} \in C(A, n). \)

\[ n > -1. \] Therefore \( \tilde{u} \in C(A, n). \)

Because \( u - \tilde{u} = \begin{pmatrix}
  n + 1 \\
  2n + 1 
\end{pmatrix} = \begin{pmatrix} 2n + 1 \\ n \\ 2n + 1 
\end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), we have \( u \equiv \tilde{u} \mod (M^A_n)^T. \) It follows that \( \Psi(u) = \tilde{u}. \)
\((3l + 2) - n - 2 = 3l - n = 3l - (2l + 1) = l - 1 < l < n, \) and \(b \geq -(l + 1) + (2l + 2) - n - 1 = l - n \geq -n.\) Hence \(|b| \leq n.\) Also we have \(a - b = (2h + n + 1) - (h + k - n - 1) = h + 2n + 2 - k \leq -1 + 2n + 2 - k = 2n + 1 - k \leq 2n + 1 - (2l + 2) = 2n - 2l - 1 = 2n - n = n, \) and \(a - b = h + 2n + 2 - k \geq -(l + 1) + 2n + 2 - (3l + 2) = 2n - 4l - 1 = 2(2l + 1) - 4l - 1 = 1 > -n.\) Hence \(|a - b| \leq n.\) Therefore \(\tilde{u} \in C(A, n).\)

Because \(u - \tilde{u} = \begin{pmatrix} -n \\ n + 1 \end{pmatrix} = \begin{pmatrix} 2n + 1 \\ n + 1 \\ n \\ 2n + 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \) we have \(u \equiv \tilde{u} \text{ mod } (M_n^A)^T.\) It follows that \(\Psi(u) = \tilde{u}.\)

**Sub-case 2:** If \(0 \leq h \leq l,\) let \(a = 2h - n, b = h + k - 2n - 1\) and \(\tilde{u} = u - \begin{pmatrix} n + 1 \\ 2n + 1 \end{pmatrix} \begin{pmatrix} 2h + 1 \\ h + k \end{pmatrix}, \) we have \(\tilde{u} = \begin{pmatrix} h - k \\ h + k - 2n - 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.\)

We claim that \(\tilde{u} \in C(A, n).\) Since \(2l + 2 \leq k \leq 3l + 2\) and \(0 \leq h \leq l,\) we have \(a = 2h - n \leq 2l - n < 2l < n\) and \(a \geq 0 - n = -n.\) Hence \(|a| < n.\) For the same reason, we have \(b = h + k - 2n - 1 \leq l + (3l + 2) - 2n - 1 = 4l - 2n + 1 = 4l - 2(2l + 1) + 1 = -1 < n,\) and \(b = h + k - 2n - 1 \geq 0 + k - 2n - 1 \geq 2l + 2 - 2n - 1 = 2l - 2n + 1 = 2l - 2(2l + 1) + 1 = -2l - 1 = -n.\) Hence \(|b| \leq n.\) Also we have \(a - b = (2h - n) - (h + k - 2n - 1) = h + n + 1 - k \leq l + n + 1 - k \leq l + n + 1 - (2l + 2) = n - l - 1 < n,\) and \(a - b = h + n + 1 - k \geq 0 + n + 1 - k = 2l + 2 - k \geq 2l + 2 - (3l + 2) = -l > -n.\) Hence \(|a - b| \leq n.\) Therefore \(\tilde{u} \in C(A, n).\)

Because \(u - \tilde{u} = \begin{pmatrix} n + 1 \\ 2n + 1 \end{pmatrix} = \begin{pmatrix} 2n + 1 \\ n + 1 \\ n \\ 2n + 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \) we have \(u \equiv \tilde{u} \text{ mod } (M_n^A)^T.\) It follows that \(\Psi(u) = \tilde{u}.\)

We have shown that \(\Psi(u)\) is either \(u,\) or \(u - \begin{pmatrix} -n \\ n + 1 \end{pmatrix},\) or \(u - \begin{pmatrix} n + 1 \\ 2n + 1 \end{pmatrix}.\) For any \(k \in \mathbb{Z}\) satisfying \(1 \leq k \leq 3l + 2\) and \(h \in \mathbb{Z}\) satisfying \(-l \leq h \leq l,\) by the definition
of $A_{k,h}$ and $B_{k,h}$, we have $A_{k,h} - B_{k,h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. As shown in Figure 5-1, the method for finding $\Psi(A_{k,h})$ can be modified easily to find $\Psi(B_{k,h})$. Finally let $\chi$ be the composition of $\Phi$ and $\Psi$. Then $\chi$ is the required correspondence between $\mathcal{P}_2$ and $C(A,n)$ in the frequency domain.

### 5.3 Fast Algorithms for the DFT and its Inverse on the Type $B$ RHS

In this section, we show an efficient method to convert the DFT and its inverse on the type $B$ RHS to a 1-dimensional standard DFT and a 1-dimensional standard inverse DFT, respectively. It turns out that the DFT on $\mathbb{R}^B_n$ can be similarly computed to the DFT on $\mathbb{R}^A_n$. For any complex function $f$ defined on $\mathbb{R}^B_n$, by Equation 2–5, its DFT $\hat{f}$ satisfies the following equation:

$$\hat{f}(q) = \sum_{p \in \mathcal{R}^B_n} f(p) \cdot e^{-2\pi i \langle p, q \rangle}, \text{ for all } q \in \mathcal{R}^B_n. \quad (5–7)$$

As in the previous section, the set $\mathcal{R}^B_n$ can be replaced by any set of coset representatives $P$ of $L^B_n/\tilde{L}^B_n$, and $\mathcal{R}^B_n$ can be replaced by a set of coset representatives $Q$ of the quotient group $(\tilde{L}^B_n)^*/(L^B_n)^*$. Let $f$ be a function defined on $\mathbb{Z}^2$ such that $f(m) = f((V^B)m)$ for any $m \in \mathbb{Z}^2$. By Corollary 4.3.9, $C(B,n)$ is a set of coset representatives associated with both $\mathcal{M}^B_n$ and $(\mathcal{M}_n^B)^T$, where the matrix $\mathcal{M}^B_n = \begin{pmatrix} 3n + 1 & 3n \\ 1 & 3n + 1 \end{pmatrix}$ is defined in Section 4.3.

Hence, by Equation 2–7, the DFT of $f$ satisfies

$$\hat{f}(k) = \sum_{m \in C(B,n)} f(m) \cdot e^{-2\pi i k^T (\mathcal{M}_n^B)^{-1} m}, \text{ for all } k \in C(B,n). \quad (5–8)$$

The following proposition shows that the DFT of $f$ on $C(B,n)$ can be similarly converted to a standard DFT.

**Proposition 5.3.1.** For any integer $n > 0$, let $S_1 = \left\{ \begin{pmatrix} 0 \\ l \end{pmatrix} : 0 \leq l \leq 9n^2 + 3n \right\}$ and $S_2 = \left\{ \begin{pmatrix} l \\ l \end{pmatrix} : 0 \leq l \leq 9n^2 + 3n \right\}$. Then the set $S_1$ is a set of coset representatives
associated with $M^B_n$ and the set $S_2$ is a set of coset representatives associated with $(M^B_n)^T$.

Furthermore, $\hat{f}(\begin{pmatrix} l \\ l \end{pmatrix}) = \sum_{m=0}^{9n^2+3n} f(\begin{pmatrix} 0 \\ m \end{pmatrix}) \cdot e^{\frac{-2\pi ilm}{9n^2+3n+1}}$ for all $l$ satisfying $0 \leq l \leq 9n^2+3n$.

Proof. By the Smith normal form of $M^B_n$ we have $M^B_n = E(D^B_n)F$, where $E = \begin{pmatrix} 1 & 0 \\ -3n & 1 \end{pmatrix}$, $D^B_n = \begin{pmatrix} 1 & 0 \\ 0 & 9n^2+3n+1 \end{pmatrix}$, and $F = \begin{pmatrix} 3n+1 & 3n \\ 1 & 1 \end{pmatrix}$. Obviously

$S_1 = \{ E \begin{pmatrix} 0 \\ l \end{pmatrix} : 0 \leq l \leq 9n^2+3n \}$ and $S_2 = \{ F^T \begin{pmatrix} 0 \\ l \end{pmatrix} : 0 \leq l \leq 9n^2+3n \}$.

Hence, it now follows from Proposition 5.1.1 that $S_1$ is a set of coset representatives associated with $M^B_n$, $S_2$ is a set of coset representatives associated with $(M^B_n)^T$, and

$\hat{f}(\begin{pmatrix} l \\ l \end{pmatrix}) = \sum_{m=0}^{9n^2+3n} f(\begin{pmatrix} 0 \\ m \end{pmatrix}) \cdot e^{\frac{-2\pi ilm}{9n^2+3n+1}}$ for all $l$ satisfying $0 \leq l \leq 9n^2+3n$. \qed

The correspondence between $C(B, n)$ and $S_1$ in the spatial domain can be established in a likewise fashion.

### 5.4 Computational Complexity and Cooley-Tukey Factorization for the DFT on the Type A and Type B RHS

For the $n^{th}$ level of the type A RHS $\mathfrak{R}^A_n$, let $N$ be its size. By Lemma 4.2.1, $N = 3n^2+3n+1$. From the results of Section 5.2, the computational complexity of the DFT on $\mathfrak{R}^A_n$ is the complexity of the correspondences (from $C(A, n)$ to $P_1$, and from $P_2$ to $C(A, n)$) plus the computational complexity of the 1-dimensional standard Fourier transform whose input size is $N$. The computational complexity of those two correspondences is obviously $O(N)$ as shown in the algorithms. It was shown in Frigo and Johnson [15] that the computational complexity for the 1-dimensional standard Fourier transform of an arbitrary input size $N$ is $O(N \log N)$ although the sizes that are products of small factors are transformed most efficiently. Thus the computational complexity of the DFT on $\mathfrak{R}^A_n$ is still $O(N \log N)$. Similarly the computational complexity of the DFT on $\mathfrak{R}^B_n$ is $O(N \log N)$ where $N = 9n^2+3n+1$. 

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Let $f$ be a complex function defined on $\mathbb{R}_n^A$. By Equation 2–5, its DFT $\hat{f}$ satisfies the following equation:

$$\hat{f}(q) = \sum_{p \in \mathbb{R}_n^A} f(p) \cdot e^{-2\pi i \langle p, q \rangle}, \text{ for all } q \in \mathbb{R}_n^A. \quad (5–9)$$

We generated a function $f$ defined on $\mathbb{R}_{298}^A$ using a random function with values ranging from 0 to 100. The set $\mathbb{R}_{298}^A$ consists of $N = 267307 > 512^2 = 262144$ lattice points. Although the number 267307 is prime, on a 2.8 GHz PC, the Matlab computational time of the DFT of $f$ is less than four seconds using the method provided in this chapter.

According to Moler [31], Version 6 of Matlab uses the Fast Fourier Transform in the West (FFTW) whose webpage is www.fftw.org. As shown in Frigo and Johnson [15], FFTW computes the DFT of arbitrary input size and utilizes Rader’s algorithm (Rader [35]) for prime sizes. The Rader’s algorithm computes the DFT of size $N$ by way of two DFTs and one inverse DFT of size $N - 1$. For the DFT on $\mathbb{R}_n^A$, its size is $N = 3n^2 + 3n + 1$. Since $N - 1 = 3n(n + 1)$, when $n > 2$, the product $n(n + 1)$ has at least three non trivial integer factors and hence $N - 1$ has at least four non trivial integer factors. Therefore, when $N$ is prime, the Rader’s algorithm can be applied efficiently for computing the DFT on $\mathbb{R}_n^A$ using the Cooley-Tukey algorithm for two DFTs and one inverse DFT of size $N - 1$.

When the determinant of the matrix $M_n^A$ can be factored, we can also apply Theorem 2.6 (the Cooley-Tukey factorization of matrix form) in Zapata and Ritter [48]. Let us recall that theorem in the following form:

**Theorem 5.4.1.** Let $N$ be a $d$-dimensional periodicity matrix and $I_N$ a set of coset representatives associated with $N$. Let $f : I_N \rightarrow \mathbb{C}$. Suppose there exist periodicity matrices $P$ and $Q$ and there exist sets of coset representatives $I_P$ and $I_Q$ associated with $P$ and $Q$ respectively such that $N = PQ$, $|\text{det}(P)| > 1$, $|\text{det}(Q)| > 1$, $I_P \subseteq I_N$ and $I_Q \subseteq I_N$. Let $\hat{I}_N$, $\hat{I}_P$, $\hat{I}_Q$ be a set of coset representatives associated with $N^T$, $P^T$ and $Q^T$ respectively such that $\hat{I}_P \subseteq \hat{I}_N$ and $\hat{I}_Q \subseteq \hat{I}_N$. Then the DFT $\hat{f}$ of $f$ is given by

$$\hat{f}(Q^T m + l) = \sum_{p \in I_P} C(p, l) \cdot e^{-2\pi i \langle l, N^{-1} p \rangle} \cdot e^{-2\pi i \langle m, P^{-1} p \rangle}, \quad (5–10)$$
for all \( l \in \hat{I}_{Q_T}, m \in \hat{I}_{P_T} \), where \( C(p, l) = \sum_{q \in I_q} f(Pq + p) \cdot e^{-2\pi i \frac{l}{T} Q^{-1} q} \).

For our case, the determinant of the matrix \( M_n^A \) is \( 3n^2 + 3n + 1 \). Suppose that there exist integers \( 1 < a, b \in \mathbb{Z} \) such that \( 3n^2 + 3n + 1 = ab \). Let \( T_n^A = \begin{pmatrix} 1 & 0 \\ -3n - 1 & ab \end{pmatrix} \) and \( F = \begin{pmatrix} 2n + 1 & n \\ 2 & 1 \end{pmatrix} \). Since \( M_n^A = T_n^A F \) and \( F \) is unimodular, \( M_n^A \) and \( T_n^A \) have the same periods.

Since \( 3n^2 + 3n + 1 = ab \), we have \( T_n^A = \begin{pmatrix} 1 & 0 \\ -3n - 1 & ab \end{pmatrix} = GH \) where \( G = \begin{pmatrix} 1 & 0 \\ -3n - 1 & a \end{pmatrix} \) and \( H = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \). It is easy to check that \( P = \left\{ \begin{pmatrix} 0 \\ j \end{pmatrix} : 0 \leq j < a \right\} \) is a set of coset representatives associated with \( G \) and that \( Q = \left\{ \begin{pmatrix} 0 \\ k \end{pmatrix} : 0 \leq k < b \right\} \) is a set of coset representatives associated with \( H \). Let \( E = \{ Gq + p : p \in P, q \in Q \} \). Then \( E \) is a set of coset representatives associated with \( T_n^A \). By direct computation, we have

\[
E = \left\{ \begin{pmatrix} 1 & 0 \\ -3n - 1 & a \end{pmatrix} \begin{pmatrix} 0 \\ k \end{pmatrix} + \begin{pmatrix} 0 \\ j \end{pmatrix} : 0 \leq j < a, 0 \leq k < b \right\} = \left\{ \begin{pmatrix} 0 \\ j + ak \end{pmatrix} : 0 \leq j < a, 0 \leq k < b \right\}
\]

(5-11)

It is easy to verify that all conditions of Theorem 5.4.1 are satisfied.
6.1 Definition and Labeling of the Pyxis Structure

6.1.1 The Definition of the Pyxis Structure

The Pyxis structure \( P \) was proposed in Peterson [33] and mentioned in Chapter 1. To give a recursive definition based on vector additions, we need the following notation. First recall from Chapter 4 that

\[
\begin{align*}
\mathbf{v}_1^A &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
\mathbf{v}_2^A &= \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \\
\mathbf{v}_1^B &= \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \\
\mathbf{v}_2^B &= \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix},
\end{align*}
\]

\( L^A = \{ n_1(\mathbf{v}_1^A) + n_2(\mathbf{v}_2^A) : n_1, n_2 \in \mathbb{Z} \} \) and \( L^B = \{ n_1(\mathbf{v}_1^B) + n_2(\mathbf{v}_2^B) : n_1, n_2 \in \mathbb{Z} \} \).

Let \( \rho_1 \) be a fixed positive number. For any \( n \) satisfying \( 1 < n \in \mathbb{N} \), let \( \rho_n = (\frac{1}{\sqrt{3}})^n \rho_1 \), and let \( \mathbf{v}_{2n-1,k} = \rho_{2n-1} \mathbf{v}_k^A \) and \( \mathbf{v}_{2n,k} = \rho_{2n} \mathbf{v}_k^B \) for \( k \in \{1, 2\} \). Now we define a sequence of lattices by \( L_n = \{ n_1 \mathbf{v}_{n,1} + n_2 \mathbf{v}_{n,2} : n_1, n_2 \in \mathbb{Z} \} \) for any \( n \in \mathbb{N} \). From this definition we can see that, if we rotate the lattice \( L_{2n+1} \) by 30° about the origin and multiply the resulting lattice by \( \sqrt{3} \), then we get the lattice \( L_{2n} \). The following lemma shows the inclusion relation of those lattices.

To give a recursive definition of the Pyxis structure, for any \( n \in \mathbb{N} \), we let \( \beta_{n,1} = \mathbf{v}_{n,1} + \mathbf{v}_{n,2}, \beta_{n,2} = \mathbf{v}_{n,2}, \beta_{n,3} = -\mathbf{v}_{n,1}, \beta_{n,j} = -\beta_{n,j-3} \) for \( j = 4, 5, 6 \), and \( \beta_n = \{ \beta_{n,j} : j = 1, 2, ..., 6 \} \). Figure 6-1 shows the generators of the lattices \( L_1 \) and \( L_2 \) as well as the elements of \( \beta_1 \) and \( \beta_2 \). Let \( \bar{\beta}_n = \beta_n \cup \{ \mathbf{0} \} \) where \( \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). Recall from Chapter 2 that, for any lattice \( L \) and \( \emptyset \neq X, Y \subseteq L \), \( X + Y \) denotes the set \( \{ \mathbf{x} + \mathbf{y} \in L : \mathbf{x} \in X, \mathbf{y} \in Y \} \).

Let \( P(0) = \mathbf{0}, P(1) = \bar{\beta}_1 \), and for any integer \( n > 1 \) let

\[
P(n) = P(n - 1) \cup (P(n - 2) + \beta_n).
\]
Figure 6-1. The generators of the lattices \( L_1 \) and \( L_2 \), and the lattice points contained in \( \beta_1 \) and \( \beta_2 \). In this figure, the two red vectors are \( \mathbf{v}_{1,1} \) and \( \mathbf{v}_{1,2} \), the two dashed green vectors are \( \mathbf{v}_{2,1} \) and \( \mathbf{v}_{2,2} \), \( \beta_1 \) consists of the six black * points, and \( \beta_2 \) consists of the six blue o points.

For any integer \( n \geq 0 \), the set \( P(n) \) is called the Pyxis structure at level \( n \). We will show in Theorem 6.1.4 that \( P(n-1) \) and \( P(n-2) + \beta_n \) are disjoint. The next lemma shows the containment relationship among \( P(n) \) and \( L_n \) for any \( n \in \mathbb{N} \).

**Lemma 6.1.1.** For any \( n \in \mathbb{N} \), we have \( L_n \subset L_{n+1} \) and \( P(n) \subset L_n \).

**Proof.** First we prove \( L_n \subset L_{n+1} \). Because the proof when \( n \) is even can be similarly done, we assume that \( n \) is odd, i.e., \( n = 2k - 1 \) for some \( k \in \mathbb{N} \). Since \( \mathbf{v}_1^A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \),

\[
\mathbf{v}_2^A = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad \mathbf{v}_1^B = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2^B = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix},
\]

we have \( \mathbf{v}_1^A = \frac{1}{\sqrt{3}}(\mathbf{v}_1^B - \mathbf{v}_2^B) \) and \( \mathbf{v}_2^A = \frac{1}{\sqrt{3}}(\mathbf{v}_1^B + 2\mathbf{v}_2^B) \). Then for any \( n_1, n_2 \in \mathbb{Z} \), we have \( n_1 \mathbf{v}_{2k-1,1} + n_2 \mathbf{v}_{2k-1,2} = n_1 \rho_{2k-1} \mathbf{v}_1^A + n_2 \rho_{2k-1} \mathbf{v}_2^A = \rho_{2k-1}(n_1 \mathbf{v}_1^A + n_2 \mathbf{v}_2^A) = \frac{1}{\sqrt{3}} \rho_{2k-1}(n_1 (\mathbf{v}_1^B - \mathbf{v}_2^B) + n_2 (\mathbf{v}_1^B + 2\mathbf{v}_2^B)) = \rho_{2k}((n_1 + n_2) \mathbf{v}_1^B + (2n_2 - n_1) \mathbf{v}_2^B) = (n_1 + n_2) \mathbf{v}_{2k,1} + (2n_2 - n_1) \mathbf{v}_{2k,2} \in L_{2k} = L_{n+1} \). Thus \( L_n \subset L_{n+1} \). Obviously the length of \( \mathbf{v}_{2k-1,1} \) and \( \mathbf{v}_{2k-1,2} \) is \( \rho_{2k-1} \) and the angle between \( \mathbf{v}_{2k-1,1} \) and \( \mathbf{v}_{2k-1,2} \) is \( \frac{2\pi}{3} \). It follows that the length of any vector in the set \( L_{2k-1} \) is at

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least \( \rho_{2k-1} \) which is bigger than the length of \( \mathbf{v}_{2k, 1} \). Hence \( \mathbf{v}_{2k, 1} \not\in L_{2k-1} \) and \( L_{2k} \neq L_{2k-1} \). Therefore \( L_{2k-1} \subset L_{2k} \), i.e., \( L_n \subset L_{n+1} \).

By the definition of \( \beta_n \), we have \( \beta_n \subset L_n \) for any \( n \in \mathbb{N} \). Also by the definition of \( P(1) \), we have \( P(1) = \bar{\beta}_1 \). It follows that \( P(1) \subset L_1 \). Obviously \( P(0) = 0 \subset L_0 \). Using induction, assume that \( P(k) \subset L_k \) for any \( k \) satisfying \( 0 \leq k < n \). By the definition of \( P(n) \), for any \( n \geq 2 \), we have \( P(n) = P(n-1) \cup (P(n-2) + \beta_n) \). By the assumption of this induction, we have \( P(n-2) \subset L_{n-2} \) and \( P(n-1) \subset L_{n-1} \). By the result of the previous paragraph, we have \( L_{n-2} \subset L_{n-1} \subset L_n \). Because \( \beta_n \subset L_n \) and \( P(n) = P(n-1) \cup (P(n-2) + \beta_n) \), it follows that \( P(n) \subset L_n \). Since \( P(n) \) contains only finite number of lattice points, \( P(n) \neq L_n \). Thus \( P(n) \subset L_n \). □

The first 4 levels of the Pyxis structure are shown in Figure 6-2. Notice that the area of a cell of \( P(n) \) is three times that of a cell of \( P(n+1) \). The factor three is a suitable scaling factor for re-sampling an image from one resolution into another. This is one of the advantages of the Pyxis structure over the \( GBT_2 \) where, as we mentioned in Chapter 3, the scaling factor seven in \( GBT_2 \) would make an image change rapidly if we re-sample it into the grid of the previous level. The following two lemmas will be used to prove Theorem 6.1.4, which provides some properties of the Pyxis structure.

**Lemma 6.1.2.** If \( n \geq 2 \) and \( f : L_{n-2} \times \beta_n \to L_{n-2} + \beta_n \) is the map defined by \( f(a, b) = a + b \) for any \( a \in L_{n-2} \) and \( b \in \beta_n \), then \( f \) is a bijection.

**Proof.** Obviously \( f \) is onto. We will show that \( f \) is also one to one. If \( f(a, b) = f(\tilde{a}, \tilde{b}) \) for some \( (a, b), (\tilde{a}, \tilde{b}) \in L_{n-2} \times \beta_n \), then \( a + b = \tilde{a} + \tilde{b} \). Hence \( b - \tilde{b} = \tilde{a} - a \). Since \( a, \tilde{a} \in L_{n-2} \) and \( L_{n-2} \) is a lattice, we have \( \tilde{a} - a \in L_{n-2} \). It follows that \( b - \tilde{b} \in L_{n-2} \). Hence there are \( l_1, l_2 \in \mathbb{Z} \) such that

\[
b - \tilde{b} = l_1 \mathbf{v}_{n-2,1} + l_2 \mathbf{v}_{n-2,2}.
\] \hspace{1cm} (6-1)
Figure 6-2. The first 4 levels of the Pyxis structure where $P(1)$ consists of 7 blue hexagons, $P(2)$ consists of 13 red hexagons, $P(3)$ consists of 55 green hexagons, and $P(4)$ consists of 133 black hexagons. In this figure, the black vector is the sum of the two blue vectors. It follows that the coordinates of the lattice point labeled 0205 is the sum of the coordinates of the two lattice points labeled 0030 and 0001, respectively.

Let $\mathcal{D} = \{(1,1), (0,1), (-1,0), (-1,-1), (0,-1), (1,0)\}$. It follows easily from the definition of $\beta_n$ that

$$\beta_n = \{n_1 \mathbf{v}_{n,1} + n_2 \mathbf{v}_{n,2} : (n_1, n_2) \in \mathcal{D}\}.$$

Since $\mathbf{b}, \tilde{\mathbf{b}} \in \beta_n$, there exist $(n_1, n_2), (\tilde{n}_1, \tilde{n}_2) \in \mathcal{D}$ such that $\mathbf{b} = n_1 \mathbf{v}_{n,1} + n_2 \mathbf{v}_{n,2}$ and $\tilde{\mathbf{b}} = \tilde{n}_1 \mathbf{v}_{n,1} + \tilde{n}_2 \mathbf{v}_{n,2}$. It follows that

$$\mathbf{b} - \tilde{\mathbf{b}} = (n_1 - \tilde{n}_1) \mathbf{v}_{n,1} + (n_2 - \tilde{n}_2) \mathbf{v}_{n,2}.$$  

(6–2)

By combining Equations 6–1 and 6–2, we have

$$(n_1 - \tilde{n}_1) \mathbf{v}_{n,1} + (n_2 - \tilde{n}_2) \mathbf{v}_{n,2} = l_1 \mathbf{v}_{n-2,1} + l_2 \mathbf{v}_{n-2,2}.$$
Because $v_{n-2,j} = 3v_{n,j}$ for $j = 1, 2$, it follows that

$$(n_1 - \tilde{n}_1)v_{n,1} + (n_2 - \tilde{n}_2)v_{n,2} = 3(l_1v_{n,1} + l_2v_{n,2}).$$

Hence $n_1 - \tilde{n}_1 = 3l_1$. However $(n_1, n_2), (\tilde{n}_1, \tilde{n}_2) \in D$ implies that $|n_1 - \tilde{n}_1| \leq 2$. Thus $n_1 - \tilde{n}_1 = 0$. Similarly $n_2 - \tilde{n}_2 = 0$. Hence $b = \tilde{b}$ and $a = \tilde{a}$. Therefore $f$ is one to one. □

**Lemma 6.1.3.** For any $n$ satisfying $0 \leq n \in \mathbb{Z}$, we have $L_n \cap \beta_{n+1} = \emptyset$.

**Proof.** We will just give a proof when $n$ is even since the proof is similar when $n$ is odd.

Suppose that $\beta_{n+1} \cap L_n \neq \emptyset$. Since $\beta_{n+1} = \{n_1v_{n+1,1} + n_2v_{n+1,2} : (n_1, n_2) \in D\}$ where $D = \{(1,1), (0,1), (-1,0), (-1,-1), (0,-1),(1,0)\}$, there exist $(n_1, n_2) \in D$ and $l_1, l_2 \in \mathbb{Z}$ such that

$$n_1v_{n+1,1} + n_2v_{n+1,2} = l_1v_{n,1} + l_2v_{n,2}. \tag{6-3}$$

Since $n$ is assumed to be even, $v_{n+1,k} = \rho_{n+1}v_k^A$ and $v_{n,k} = \rho_nv_k^B$ for $k = 1, 2$. Then Equation 6–3 becomes

$$n_1\rho_{n+1}v_1^A + n_2\rho_{n+1}v_2^A = l_1\rho_nv_1^B + l_2\rho_nv_2^B. \tag{6-4}$$

Since $\rho_n = \sqrt{3}\rho_{n+1}$, it follows that $n_1v_1^A + n_2v_2^A = \sqrt{3}(l_1v_1^B + l_2v_2^B)$. Hence

$$n_1v_1^A + n_2v_2^A = \sqrt{3}(\frac{l_1}{\sqrt{3}}(2v_1^A + v_2^A) + \frac{l_2}{\sqrt{3}}(-v_1^A + v_2^A)) = (2l_1 - l_2)v_1^A + (l_1 + l_2)v_2^A.$$

Then $n_1 = 2l_1 - l_2$ and $n_2 = l_1 + l_2$. Thus $n_1 + n_2 = 3l_1$. However $(n_1, n_2) \in D$ implies that $n_1 + n_2$ equals $\pm 1$ or $\pm 2$, which contradicts that $n_1 + n_2 = 3l_1$ for some $l_1 \in \mathbb{Z}$. Therefore $\beta_{n+1} \cap L_n = \emptyset$. □

**Theorem 6.1.4.** For any $n$ satisfying $0 \leq n \in \mathbb{Z}$, we have

1. $P(n) \subset P(n+1)$;
2. $P(n+1) \cap (P(n) + \beta_{n+2}) = \emptyset$;
3. $|P(n)| = \frac{1}{3}[3^{n+2} - (-2)^{n+2}]$. 

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We also have initial conditions \( \beta_i \) any \( \lambda_i \). Lemma 6.1.5.

6.1.2 The Labeling of the Pyxis Structure

To define the labeling, we prove that, for any \( \lambda_i \) satisfying \( 1 \leq i < n \), we have \( \lambda_i \leq 6 \) for each \( 1 \leq i \leq n \) and either \( \lambda_i = 0 \) or \( \lambda_{i+1} = 0 \) for any \( i \) satisfying \( 1 \leq i < n \). Because labels of lattice points of \( P(n) \), as opposed to their coordinates, consist of integers, such an assignment is useful for quick data retrieval of the Pyxis structure.

Let \( \beta_{n,0} = 0 \) for any \( n \in \mathbb{N} \). Since \( \bar{\beta}_n = \beta_n \cup \{0\} \), we have \( \bar{\beta}_n = \{ \beta_{n,j} : j = 0, 1, 2, ..., 6 \} \).

To define the labeling, we prove that, for any \( a \in P(n) \), there exists uniquely determined \( \lambda_i \in \{0, 1, 2, ..., 6\} \) for each \( 1 \leq i \leq n \) such that \( a = \sum_{i=1}^{n} \beta_{i,\lambda_i} \), and either \( \lambda_i = 0 \) or \( \lambda_{i+1} = 0 \) for each \( i \) satisfying \( 1 \leq i < n \). The next lemma will be used in the proof of the following theorem.

**Lemma 6.1.5.** Let \( \lambda_i \in \{0, 1, 2, ..., 6\} \) for \( i = 1, 2, ..., n \). If either \( \lambda_i = 0 \) or \( \lambda_{i+1} = 0 \) for any \( i \) satisfying \( 1 \leq i < n \), then \( \sum_{i=1}^{n} \beta_{i,\lambda_i} \in P(n) \).
**Proof.** If \( n = 1 \), then this lemma is obviously true. Suppose that it is true for any \( n < t \) for some integer \( t \). When \( n = t \), let \( \mathbf{a} = \sum_{i=1}^{t} \beta_{i,\lambda_i} \). If \( \lambda_t = 0 \), then \( \beta_{t,\lambda_t} = 0 \). Hence \( \mathbf{a} = \sum_{i=1}^{t-1} \beta_{i,\lambda_i} \in P(t-1) \) by the assumption of this induction. By Item 1 of Theorem 6.1.4, we have \( P(t-1) \subset P(t) \). Hence \( \mathbf{a} \in P(t) \). If \( \lambda_t \neq 0 \), then \( \lambda_{t-1} = 0 \) by the condition of this lemma. Hence

\[
\mathbf{a} = \beta_{t,\lambda_t} + \sum_{i=1}^{t-2} \beta_{i,\lambda_i}. \tag{6-5}
\]

The assumption of this induction follows that \( \sum_{i=1}^{t-2} \beta_{i,\lambda_i} \in P(t-2) \). Since \( \lambda_t \neq 0 \), we have \( \beta_{t,\lambda_t} \in \beta_t \). Hence by Equation 6-5, we have \( \mathbf{a} \in \beta_t + P(t-2) \subset P(t) \).

**Theorem 6.1.6.** For any \( n \in \mathbb{N} \) and \( \mathbf{a} \in P(n) \), there exists uniquely determined \( \lambda_i \in \{0, 1, 2, ..., 6\} \) for each \( i = 1, 2, ..., n \) such that \( \mathbf{a} = \sum_{i=1}^{n} \beta_{i,\lambda_i} \) and either \( \lambda_i = 0 \) or \( \lambda_{i+1} = 0 \) for each \( i \) satisfying \( 1 \leq i < n \).

**Proof.** When \( n = 1 \), it is obviously true. Suppose that it is true for any \( n < t \) for some integer \( t \). When \( n = t \), by definition we have \( P(t) = P(t-1) \cup (P(t-2) + \beta_t) \). Hence \( \mathbf{a} \in P(t) \) implies that either \( \mathbf{a} \in P(t-1) \) or \( \mathbf{a} \in P(t-2) + \beta_t \). If \( \mathbf{a} \in P(t-1) \), then by the assumption of this induction, \( \mathbf{a} \) has such an expression in \( P(t-1) \), which is \( \mathbf{a} = \sum_{i=1}^{t-1} \beta_{i,\lambda_i} \). Let \( \lambda_t = 0 \). Then \( \beta_{t,\lambda_t} = 0 \). It follows that \( \mathbf{a} = \sum_{i=1}^{t} \beta_{i,\lambda_i} = \sum_{i=1}^{n} \beta_{i,\lambda_i} \), which is the required expression of \( \mathbf{a} \) in \( P(n) \). If \( \mathbf{a} \in P(t-2) + \beta_t \), then \( \mathbf{a} = \mathbf{b} + \mathbf{c} \) for some \( \mathbf{b} \in P(t-2) \) and \( \mathbf{c} \in \beta_t \). By the assumption of this induction, \( \mathbf{b} \) has the required expression \( \mathbf{b} = \sum_{i=1}^{t-2} \beta_{i,\lambda_i} \in P(t-2) \). Since \( \mathbf{c} \in \beta_t \), there is \( \lambda_t \in \{1, 2, ..., 6\} \) such that \( \mathbf{c} = \beta_{t,\lambda_t} \). Let \( \lambda_{t-1} = 0 \). Then \( \mathbf{a} = \mathbf{b} + \mathbf{c} = \sum_{i=1}^{t} \beta_{i,\lambda_i} = \sum_{i=1}^{n} \beta_{i,\lambda_i} \) is the required expression of \( \mathbf{a} \) in \( P(n) \).

To show the uniqueness, suppose that \( \mathbf{a} \) has another such expression \( \mathbf{a} = \sum_{i=1}^{n} \beta_{i,\tilde{\lambda}_i} \), where \( \tilde{\lambda}_i \in \{0, 1, 2, ..., 6\} \) for each \( i = 1, 2, ..., n \) and either \( \tilde{\lambda}_i = 0 \) or \( \tilde{\lambda}_{i+1} = 0 \) for any \( i \) satisfying \( 1 \leq i < n \). It follows that

\[
\sum_{i=1}^{n} \beta_{i,\lambda_i} = \sum_{i=1}^{n} \beta_{i,\tilde{\lambda}_i}. \tag{6-6}
\]
If \( \lambda_n = \tilde{\lambda}_n \), then Equation 6–6 becomes
\[
\sum_{i=1}^{n-1} \beta_{i,\lambda_i} = \sum_{i=1}^{n-1} \beta_{i,\tilde{\lambda}_i}. \tag{6–7}
\]
By Lemma 6.1.5, we have \( \sum_{i=1}^{n-1} \beta_{i,\lambda_i} \in P(n-1) \). By the assumption of this induction, it follows from Equation 6–7 that \( \lambda_i = \tilde{\lambda}_i \) for any \( i \) satisfying \( 1 \leq i \leq n-1 \). Thus the expression of \( \mathbf{a} \) is unique in this case. If \( \lambda_n \neq 0 \) but \( \tilde{\lambda}_n = 0 \), then it follows from Equation 6–5 that
\[
\beta_{n,\lambda_n} = \sum_{i=1}^{n-1} \beta_{i,\lambda_i} - \sum_{i=1}^{n-1} \beta_{i,\tilde{\lambda}_i}. \tag{6–8}
\]
By Lemma 6.1.1, we have \( \bar{\beta}_i \subset L_{n-1} \) for each \( i = 1, 2, ..., n-1 \). By Equation 6–8, it follows that \( \beta_{n,\lambda_n} \in L_{n-1} \). Since \( \lambda_n \neq 0 \), we have \( \beta_{n,\lambda_n} \in \beta_n \). Hence \( \beta_{n,\lambda_n} \in \beta_n \cap L_{n-1} = \emptyset \) by Lemma 6.1.3, which is a contradiction. If \( \lambda_n = 0 \) but \( \tilde{\lambda}_n \neq 0 \), then a contradiction can be similarly shown. If \( \lambda_n \neq 0 \) and \( \tilde{\lambda}_n \neq 0 \), then \( \lambda_{n-1} = 0 \) and \( \tilde{\lambda}_{n-1} = 0 \) by the requirement of such expressions. By Lemma 6.1.5, we have \( \sum_{i=1}^{n-2} \beta_{i,\lambda_i} \in P(n-2) \) and \( \sum_{i=1}^{n-2} \beta_{i,\tilde{\lambda}_i} \in P(n-2) \). Since \( \beta_{n,\lambda_n} \in \beta_n \) and \( \beta_{n,\tilde{\lambda}_n} \in \beta_n \), by Lemma 6.1.2, it follows from Equation 6–6 that \( \sum_{i=1}^{n-2} \beta_{i,\lambda_i} = \sum_{i=1}^{n-2} \beta_{i,\tilde{\lambda}_i} \) and \( \beta_{n,\lambda_n} = \beta_{n,\tilde{\lambda}_n} \). Since \( \sum_{i=1}^{n-2} \beta_{i,\lambda_i} = \sum_{i=1}^{n-2} \beta_{i,\tilde{\lambda}_i} \), by the assumption of this induction, we have \( \lambda_i = \tilde{\lambda}_i \) for each \( i \) satisfying \( 1 \leq i \leq n-2 \). Furthermore, since \( \beta_{n,\lambda_n} = \beta_{n,\tilde{\lambda}_n} \), we have \( \lambda_n = \tilde{\lambda}_n \). Therefore the uniqueness is proved for all cases. \( \square \)

For any \( n \in \mathbb{N} \) and \( \mathbf{a} \in P(n) \), the unique expression \( \mathbf{a} = \sum_{i=1}^{n} \beta_{i,\lambda_i} \) in Theorem 6.1.6 is called the standard expression of \( \mathbf{a} \) in \( P(n) \). The string \( \lambda_1 \lambda_2 ... \lambda_n \) of integers in that standard expression is called the label of \( \mathbf{a} \) in \( P(n) \). If \( V_{\mathbf{a}} \) denotes the Voronoi cell of \( \mathbf{a} \in L_n \), then the string \( \lambda_1 \lambda_2 ... \lambda_n \) is also called the label of \( V_{\mathbf{a}} \). The label of each lattice point of \( P(n) \) also serves as the label of the Voronoi cell of that lattice point. It is easy to verify that, in Figure 6-2 of the previous subsection, the center hexagon of \( P(1) \) is labeled 0, and other six hexagons of \( P(1) \) are labeled 1,2,3,4,5,6 going counter-clockwise. For any
$n$ satisfying $0 \leq n \in \mathbb{Z}$, let $\Lambda_n$ denote the set of the labels of lattice points of $P(n)$. The following corollary shows the structure of $\Lambda_n$.

**Corollary 6.1.7.** For any $n$ satisfying $2 < n \in \mathbb{Z}$, we have

1. $\Lambda_n = \{\lambda_1 \lambda_2 \ldots \lambda_n : \lambda_i \in \{0, 1, 2, \ldots, 6\} \text{ for each } i, \text{ and } \lambda_i \cdot \lambda_{i+1} = 0 \text{ for each } i < n\}$.

2. $\Lambda_n = \{\lambda_0 j : \lambda \in \Lambda_{n-2}, j = 1, 2, 3, 4, 5, 6\} \cup \{\lambda_0 : \lambda \in \Lambda_{n-1}\}$.

**Proof.** 1. Since $\Lambda_n$ is the set of the labels of lattice points of $P(n)$, by Lemma 6.1.5 and Theorem 6.1.6, we have

$$\Lambda_n = \{\lambda_1 \lambda_2 \ldots \lambda_n : \lambda_i \in \{0, 1, 2, \ldots, 6\} \text{ for each } i, \text{ and } \lambda_i \cdot \lambda_{i+1} = 0 \text{ for each } i < n\}.$$ 

2. It follows easily from Item 1 that

$$\{\lambda_0 j : \lambda \in \Lambda_{n-2}, j = 1, 2, 3, 4, 5, 6\} \cup \{\lambda_0 : \lambda \in \Lambda_{n-1}\} \subseteq \Lambda_n.$$ 

Conversely, for any $\lambda \in \Lambda_n$, let $\lambda = \lambda_1 \lambda_2 \ldots \lambda_n$. If $\lambda_n = 0$, then $\lambda = \tilde{\lambda} 0$ where $\tilde{\lambda} = \lambda_1 \lambda_2 \ldots \lambda_{n-1} \in \Lambda_{n-1}$. If $\lambda_n \neq 0$, by Theorem 6.1.6, we have $\lambda_{n-1} = 0$. Hence $\lambda = \tilde{\lambda} 0 j$ where $\tilde{\lambda} = \lambda_1 \lambda_2 \ldots \lambda_{n-2} \in \Lambda_{n-2}$ and $j = \lambda_n \in \{1, 2, \ldots, 6\}$. Therefore $\Lambda_n \subseteq \{\lambda_0 j : \lambda \in \Lambda_{n-2}, j = 1, 2, 3, 4, 5, 6\} \cup \{\lambda_0 : \lambda \in \Lambda_{n-1}\}$. \hfill \Box

### 6.1.3 Addition of the Labels of the Pyxis Structure

In this subsection, we first define the sum of labels of two lattice points in $P(n)$ to be the label of the sum of these two lattice points which are taken as the vectors of $\mathbb{R}^2$.

The addition of Pyxis labels will be used in Section 6.2. Our algorithm for adding any two labels in $\Lambda_n$ uses the fact that $\frac{1}{3} \mathbf{a} + \frac{1}{3} \mathbf{b} \in P(n + 2)$ for any $n \in \mathbb{N}$ and $\mathbf{a}, \mathbf{b} \in P(n)$ which is proved in this subsection. Tables 6-1 and 6-2 are also used in the algorithm. The computational complexity of our algorithm is shown to be of order $n$.

Let $\mathbf{a}, \mathbf{b} \in P(n)$ be labeled $a_1 a_2 \ldots a_n$ and $b_1 b_2 \ldots b_n$, respectively. If $\mathbf{a} + \mathbf{b} \in P(n)$ and the label of $\mathbf{a} + \mathbf{b}$ in $P(n)$ is $c_1 c_2 \ldots c_n$, then the label $c_1 c_2 \ldots c_n$ is called the *sum of the labels* $a_1 a_2 \ldots a_n$ and $b_1 b_2 \ldots b_n$, and we write $a_1 a_2 \ldots a_n \oplus b_1 b_2 \ldots b_n = c_1 c_2 \ldots c_n$ in $\Lambda_n$. For example, the three dashed vectors in Figure 1-2 show that $0506 \oplus 2005 = 1040$ in $\Lambda_4$. 

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Partial addition table for \( P(3) \).

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Partial addition table for \( P(4) \).

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By the plot of the lattice points of \( P(3) \) (\( P(4) \) respectively) in Figure 6-2, it is easy to verify Tables 6-1 (6-2 respectively) which gives partial addition tables for \( P(3) \) (\( P(4) \) respectively) and will be used in our algorithm. The following lemma gives some properties of labels of the Pyxis structure which will be used very often for the addition of labels.

**Lemma 6.1.8.** For any \( n \in \mathbb{N} \) we have the following:

1. If \( \lambda_1 \lambda_2 \ldots \lambda_n \in \Lambda_n \) and \( k < n \), then \( \lambda_1 \lambda_2 \ldots \lambda_n = \lambda_1 \lambda_2 \ldots \lambda_k 00 \oplus 00 \ldots 0 \lambda_{k+1} \lambda_{k+2} \ldots \lambda_n \) in \( \Lambda_n \).

2. If \( 1 \leq k \leq n \), \( A = \lambda_1 \lambda_2 \ldots \lambda_k 00 \ldots 0 \in \Lambda_n \), \( B = 00 \ldots 0 \lambda_{k+1} \lambda_{k+2} \ldots \lambda_n \in \Lambda_n \), and either \( \lambda_k = 0 \) or \( \lambda_{k+1} = 0 \), then \( A \oplus B = \lambda_1 \lambda_2 \ldots \lambda_n \) in \( \Lambda_n \).

3. If \( a \in P(n) \) with label \( \lambda_1 \lambda_2 \ldots \lambda_n \in \Lambda_n \), then \( \frac{1}{3} a \in P(n+2) \) with label \( 00 \lambda_1 \lambda_2 \ldots \lambda_n \in \Lambda_{n+2} \), and the label of \( a \) in \( P(n+1) \) is \( \lambda_1 \lambda_2 \ldots \lambda_n 0 \in \Lambda_{n+1} \).
4. For any \( a_1a_2...a_n \in \Lambda_n, \) \( b_1b_2...b_n \in \Lambda_n \) and \( c_1c_2...c_n \in \Lambda_n \), we have \( a_1a_2...a_n \oplus b_1b_2...b_n = c_1c_2...c_n \) in \( \Lambda_n \) if and only if \( a_1a_2...a_n \oplus 0b_1b_2...b_n = 0c_1c_2...c_n \) in \( \Lambda_{n+1} \) if and only if \( a_1a_2...a_n \oplus b_1b_2...b_n = 0 \) in \( \Lambda_{n+1} \).

Proof. 1. Let \( a = \sum_{i=1}^{n} \beta_{i,\lambda_i}, \) \( b = \sum_{i=1}^{k} \beta_{i,\lambda_i}, \) and \( c = \sum_{i=k+1}^{n} \beta_{i,\lambda_i}. \) Since \( \lambda_1\lambda_2...\lambda_n \in \Lambda_n, \) by Item 1 of Corollary 6.1.7, we have either \( \lambda_i = 0 \) or \( \lambda_{i+1} = 0 \) for any \( i \) satisfying \( 1 \leq i < n. \) By Lemma 6.1.5, it follows that \( a, b, c \in P(n). \) By the definition of labels of the elements in \( P(n), \) the labels of \( a, b \) and \( c \) are \( \lambda_1\lambda_2...\lambda_n, \lambda_1\lambda_2...\lambda_k00...0 \) and \( 00...0\lambda_{k+1}\lambda_{k+2}...\lambda_n, \) respectively. Since \( a = b + c, \) we have \( \lambda_1\lambda_2...\lambda_n = \lambda_1\lambda_2...\lambda_k00...0 \oplus 00...0\lambda_{k+1}\lambda_{k+2}...\lambda_n \) in \( \Lambda_n. \)

2. Since \( A, B \in \Lambda_n, \) there exist \( a, b \in P(n) \) such that \( A \) and \( B \) are the labels of \( a \) and \( b, \) respectively. Because \( A = \lambda_1\lambda_2...\lambda_k00...0 \in \Lambda_n, \) we have \( a = \sum_{i=1}^{k} \beta_{i,\lambda_i}. \) Similarly \( B = 00...0\lambda_{k+1}\lambda_{k+2}...\lambda_n \in \Lambda_n \) implies that \( b = \sum_{i=k+1}^{n} \beta_{i,\lambda_i}. \) Since \( \lambda_1\lambda_2...\lambda_k00...0 \in \Lambda_n, \) we have either \( \lambda_i \) or \( \lambda_{i+1} \) for any \( i \) satisfying \( 1 \leq i < k. \) Similarly it follows from \( 00...0\lambda_{k+1}\lambda_{k+2}...\lambda_n \in \Lambda_n \) that either \( \lambda_i \) or \( \lambda_{i+1} \) for any \( i \) satisfying \( k + 1 \leq i < n. \) Furthermore, by the given condition of this Item, we have either \( \lambda_k = 0 \) or \( \lambda_{k+1} = 0. \) Hence either \( \lambda_i = 0 \) or \( \lambda_{i+1} = 0 \) for any \( i \) satisfying \( 1 \leq i < n. \) By Lemma 6.1.5, it follows that \( \sum_{i=1}^{n} \beta_{i,\lambda_i} \in P(n) \) and the label of \( \sum_{i=1}^{n} \beta_{i,\lambda_i} \) in \( P(n) \) is \( \lambda_1\lambda_2...\lambda_n. \) Since \( a + b = \sum_{i=1}^{n} \beta_{i,\lambda_i}, \) we have \( A \oplus B = \lambda_1\lambda_2...\lambda_n \) in \( \Lambda_n. \)

3. Since \( a \in P(n) \) and the label of \( a \) is \( \lambda_1\lambda_2...\lambda_n, \) the standard expression of \( a \) is \( a = \sum_{i=1}^{n} \beta_{i,\lambda_i}. \) It follows that \( \frac{1}{3}a = \sum_{i=1}^{n} \frac{1}{3} \beta_{i,\lambda_i}. \) Because \( \frac{1}{3} \beta_{i,\lambda_i} = \beta_{i+2,\lambda_i}, \) we have \( \frac{1}{3}a = \sum_{i=1}^{n} \beta_{i+2,\lambda_i} \in P(n+2). \) It follows that the label of \( \frac{1}{3}a \) in \( P(n+2) \) is \( 00\lambda_1\lambda_2...\lambda_n \in \Lambda_{n+2}. \)

If we let \( \lambda_{n+1} = 0, \) then \( a = \sum_{i=1}^{n+1} \beta_{i,\lambda_i}. \) It follows that the label of \( a \) in \( P(n+1) \) is \( \lambda_1\lambda_2...\lambda_n\lambda_{n+1}, \) i.e., \( \lambda_1\lambda_2...\lambda_n0. \)

4. Since \( a_1a_2...a_n, b_1b_2...b_n, c_1c_2...c_n \in \Lambda_n, \) there exist \( a, b, c \in P(n) \) such that \( a_1a_2...a_n, b_1b_2...b_n, c_1c_2...c_n \) are the labels of \( a, b \) and \( c, \) respectively. Then the standard expressions of \( a, b \) and \( c \) are \( a = \sum_{i=1}^{n} \beta_{i,a_i}, \) \( b = \sum_{i=1}^{n} \beta_{i,b_i}, \) and \( c = \sum_{i=1}^{n} \beta_{i,c_i}, \) respectively.
If \( a_1a_2...a_n \oplus b_1b_2...b_n = c_1c_2...c_n \) in \( \Lambda_n \), then \( a + b = c \), i.e., \( \sum_{i=1}^{n} \beta_{i,a_i} + \sum_{i=1}^{n} \beta_{i,b_i} = \sum_{i=1}^{n} \beta_{i,c_i} \). It follows that \( \sum_{i=1}^{n} \frac{1}{\sqrt{3}} \beta_{i,a_i} + \sum_{i=1}^{n} \frac{1}{\sqrt{3}} \beta_{i,b_i} = \frac{1}{\sqrt{3}} \sum_{i=1}^{n} \beta_{i,c_i} \). By rotating those vectors by an angle of \( \frac{\pi}{6} \) as shown in Figure 6-1, it follows that \( \sum_{i=1}^{n} \beta_{i+1,a_i} + \sum_{i=1}^{n} \beta_{i+1,b_i} = \sum_{i=1}^{n} \beta_{i+1,c_i} \). Hence \( 0a_1a_2...a_n \oplus 0b_1b_2...b_n = 0c_1c_2...c_n \) in \( \Lambda_{n+1} \). If \( 0a_1a_2...a_n \oplus 0b_1b_2...b_n = 0c_1c_2...c_n \) in \( \Lambda_{n+1} \), similarly we can show that \( a_1a_2...a_n \oplus b_1b_2...b_n = c_1c_2...c_n \) in \( \Lambda_n \).

Let \( a_{n+1} = b_{n+1} = c_{n+1} = 0 \). Then \( \sum_{i=1}^{n+1} \beta_{i,a_i} + \sum_{i=1}^{n+1} \beta_{i,b_i} = \sum_{i=1}^{n+1} \beta_{i,c_i} \) if and only if \( \sum_{i=1}^{n+1} \beta_{i,a_i} + \sum_{i=1}^{n+1} \beta_{i,b_i} = \sum_{i=1}^{n+1} \beta_{i,c_i} \). It follows that \( a_1a_2...a_n \oplus b_1b_2...b_n = c_1c_2...c_n \) if and only if \( a_1a_2...a_{n+1} \oplus b_1b_2...b_{n+1} = c_1c_2...c_{n+1} \) if and only if \( a_1a_2...a_n0 \oplus b_1b_2...b_n0 = c_1c_2...c_n0 \).

Let \( L \) be a hexagonal lattice. For any \( a, b \in L \), if their Voronoi cells share exactly one side, then we say that \( a \) and \( b \) are next to each other in the lattice \( L \), and each of them is called a neighbor of the another in the lattice \( L \). The following three lemmas will be applied in subsequent lemmas.

**Lemma 6.1.9.** Let \( 1 < n \in \mathbb{N} \). If \( a, b \in \beta_n \) which are next to each other in the lattice \( L_n \), then \( a + b \in \beta_{n-1} \).

**Proof.** It follows directly from Figure 6-1.

**Lemma 6.1.10.** For any \( n \in \mathbb{N} \) and \( a, b \in L_n \), \( a \) is next to \( b \) in the lattice \( L_n \) if and only if \( b - a \in \beta_n \).

**Proof.** If \( a \) is next to \( b \) in the lattice \( L_n \), then their Voronoi cells shares exactly one side. After a translation of \( -a \), it follows that the Voronoi cell of \( a - a \) shares exactly one side with the Voronoi cell of \( b - a \). Hence \( a - a \), i.e., \( 0 \), is next to \( b - a \) in the lattice \( L_n \). Obviously \( \beta_n = \{ x \in L_n : x \text{ is next to } 0 \text{ in the lattice } L_n \} \). Thus \( b - a \in \beta_n \). Conversely if \( b - a \in \beta_n \), then \( b - a \) is next to \( 0 \) in the lattice \( L_n \). After a translation of \( a \), it follows that \( a \) is next to \( b \) in the lattice \( L_n \).

**Lemma 6.1.11.** For any \( a, b \in \beta_n \), we have:

1. If \( a \neq b \), then either \( a + b = 0 \) or \( a + b \in \beta_{n-1} \cup \beta_n \).
2. If $a \neq \pm b$, then $a$ is next to either $b$ or $-b$ in the lattice $L_n$.

Proof. 1. Obviously $\beta_n$ contains exactly six lattice points which are all neighbors of 0 in the lattice $L_n$. Since $a, b \in \beta_n$ and $a \neq b$, it follows that $b = -a$, or $b$ is next to either $a$ or $-a$ in the lattice $L_n$. If $b = -a$, then $a + b = 0$. If $b$ is next to $a$, then by Lemma 6.1.9, $a + b \in \beta_{n-1}$. If $b$ is next to $-a$ in the lattice $L_n$, then by Lemma 6.1.10, we have $b - (-a) \in \beta_n$. Hence $a + b \in \beta_n$.

2. Since $a, b \in \beta_n$ and $\beta_n$ contains exactly six lattice points which are all neighbors of 0 in the lattice $L_n$, if $a \neq \pm b$, then $a$ must be next to either $b$ or $-b$ in the lattice $L_n$. $\square$

The following two lemmas will be used in Theorem 6.1.14, which is one of the main results in this subsection.

Lemma 6.1.12. For any $a, b, c \in P(1)$, we have $\frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c \in P(3)$.

Proof. Let $u = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c$ and let the labels of $a, b$ and $c$ be $a_1, b_1$, and $c_1$, respectively. By applying Item 3 of Lemma 6.1.8, it follows that the labels of $\frac{1}{3}a, \frac{1}{3}b$ and $\frac{1}{3}c$ are 00$a_1$, 00$b_1$, and 00$c_1$, respectively.

If either $a = 0$, or $b = 0$, or $c = 0$, without loss of generality, we assume that $c = 0$. It follows that $u = \frac{1}{3}a + \frac{1}{3}b$ and the label of $u$ is 00$a_1 \oplus 00b_1$. By applying Table 6-1, we have 00$a_1 \oplus 00b_1 = d_10d_3$ for some label $d_10d_3 \in \Lambda_3$. Hence the label of $u$ is $d_10d_3 \in \Lambda_3$ and thus $u \in P(3)$.

Otherwise, if $a = b = c$, then $u = a \in P(1) \subset P(3)$.

Otherwise, if either $a = -b$, or $a = -c$, or $b = -c$, without loss of generality, we assume $a = -b$. It follows that $u = \frac{1}{3}c \in P(3)$.

Otherwise, if either $a = b$, or $a = c$, or $b = c$, without loss of generality, we assume that $a = b$. Since $c \neq \pm a$ and $a, c \in \beta_1$, it follows that $a$ is next to either $c$ or $-c$ in $L_1$. If $a$ is next to $c$ in $L_1$, then $\frac{1}{3}a$ is next to $\frac{1}{3}c$ in $L_3$. By lemma 6.1.9, it follows that $\frac{1}{3}a - \frac{1}{3}c \in \beta_3$. Hence $u = a + \frac{1}{3}(c - a) \in P(1) + \beta_3 \subset P(3)$. If $a$ is next to $-c$ in $L_1$, then
$\frac{1}{3}a$ is next to $-\frac{1}{3}c$ in $L_3$. It follows that $\frac{1}{3}a + \frac{1}{3}c \in \beta_3$, and $\frac{1}{3}a$ is next to $\frac{1}{3}a + \frac{1}{3}c$ in $L_3$.

Hence, by Lemma 6.1.9, $u = \frac{1}{3}a + \frac{1}{3}(c + a) \in \beta_2 \subset P(2) \subset P(4)$.

Otherwise, $a, b$ and $c$ are distinct and each of those three elements is next to at least one of the other two elements in $L_1$. Without loss of generality, we assume that $a$ is next to $b$ and $b$ is next to $c$ in $L_1$. It follows that $a + c = b$. Hence $u = \frac{1}{3}(a + b + c) = \frac{2}{3}b = b - \frac{1}{3}b \in P(1) + \beta_3 \subset P(3)$. Therefore this lemma is valid for all cases. $\square$

**Lemma 6.1.13.** For any $a, b, c \in P(2)$, we have $\frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c \in P(4)$.

**Proof.** Let $u = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c$ and let the labels of $a, b$ and $c$ be $a_1a_2$, $b_1b_2$, and $c_1c_2$, respectively. By applying Item 3 of Lemma 6.1.8, it follows that the labels of $\frac{1}{3}a, \frac{1}{3}b$ and $\frac{1}{3}c$ are $00a_1a_2$, $00b_1b_2$, and $00c_1c_2$, respectively. Since $a, b, c \in P(2)$ and $P(2) = \beta_1 \cup \beta_2 \cup \{0\}$, we verify this lemma by considering the following cases.

**Case 1:** If $a, b, c \in \beta_1$, then $a_2 = b_2 = c_2 = 0$. By Lemma 6.1.12, we have $00a_1 \oplus 00b_1 \oplus 00c_1 \in \Lambda_3$. By Item 4 of Lemma 6.1.8, it follows that $00a_10 \oplus 00b_10 \oplus 00c_10 \in \Lambda_4$, i.e., $00a_1a_2 \oplus 00b_1b_2 \oplus 00c_1c_2 \in \Lambda_4$. Hence $u \in P(4)$.

**Case 2:** If $a, b, c \in \beta_2$, then $a_1 = b_1 = c_1 = 0$. Similar to the proof of Case 1, we have $u \in P(4)$.

**Case 3:** If either $a = 0$, or $b = 0$, or $c = 0$, without loss of generality, we assume that $c = 0$. It follows that $u = \frac{1}{3}a + \frac{1}{3}b$ and the label of $u$ is $00a_1a_2 \oplus 00b_1b_2$. If $a_1 \neq 0$ and $b_1 \neq 0$, then by Item 1 of Corollary 6.1.7 we have $a_2 = b_2 = 0$. By applying Table 6-1, there exists $d_1d_3 \in \Lambda_3$ such that $00a_1 \oplus 00b_1 = d_1d_3$ in $\Lambda_3$. By Item 4 of Lemma 6.1.8, it follows that $00a_10 \oplus 00b_10 = d_1d_30$ in $\Lambda_4$. If either $a_1 = 0$ or $b_1 = 0$, then by Table 6-2, we have $00a_1a_2 \oplus 00b_1b_2$ in $\Lambda_4$. It follows that $u \in P(4)$.

**Case 4:** If $a, b, c \in \beta_1 \cup \beta_2$ and the set $\{a, b, c\} \cap \beta_1$ contains one element, without loss of generality, we assume $a \in \beta_1$. It follows that $b, c \in \beta_2$. Let the labels of $b$ and $c$ be $0b_2$ and $0c_2$, respectively.

If $b_2 \neq c_2$, then $b \neq c$. By Item 1 of Lemma 6.1.11, it follows that $b + c \in P(2)$. Hence either $b + c \in \beta_2$ or $b + c \in \beta_1$ or $b + c = 0$. By Item 3 of Lemma 6.1.8, it follows that
either \( \frac{1}{3}(b + c) \in \beta_4 \) or \( \frac{1}{3}(b + c) \in \beta_3 \) or \( \frac{1}{3}(b + c) = 0 \). Because \( a \in \beta_1 \), we have \( \frac{1}{3}a \in \beta_3 \).

Since \( u = \frac{1}{3}a + \frac{1}{3}(b + c) \), it follows that either \( u \in \beta_3 + \beta_4 \) or \( u \in \beta_3 + \beta_3 \) or \( u \in \beta_3 \subset P(4) \).

By Table 6-2, we have \( \beta_3 + \beta_4 \subseteq P(4) \). By Table 6-1, we have \( \beta_3 + \beta_3 \subseteq P(3) \subseteq P(4) \).

Hence \( u \in P(4) \).

If \( b_2 = c_2 \), without loss of generality, we assume that \( b_2 = c_2 = 1 \). Then, by Table 6-2, the label of \( \frac{1}{3}(b + c) \) is 0001 \( \oplus \) 0001 = 0104 in \( \Lambda_4 \). Since \( a \in \beta_1 \), the label of \( a \) in \( \Lambda_2 \) is \( a_20 \). Hence the label of \( u \) is 00a20 \( \oplus \) 0104 in \( \Lambda_4 \). By Table 6-2, there exists 0d10d3 \( \in \Lambda_4 \) such that 00a20 \( \oplus \) 0004 = 0d10d3 in \( \Lambda_4 \) where \( d_1 \in \{0, 3, 4, 5\} \). Since \( d_1 \in \{0, 3, 4, 5\} \), by Table 6-1, we have 00d1 \( \oplus \) 001 = 00t1 in \( \Lambda_3 \) for some \( t_1 \in \{0, 1, 2, 6\} \). By Item 4 of Lemma 6.1.8, it follows that 0d100 \( \oplus \) 0100 = 0t100 in \( \Lambda_4 \). Hence the label of \( u \) is 00a20 \( \oplus \) 0104 = 0d10d3 \( \oplus \) 0100 = 0t100 \( \oplus \) 000d3 = 0t10d3 in \( \Lambda_4 \). Thus \( u \in P(4) \).

**Case 5:** If \( a, b, c \in \beta_1 \cup \beta_2 \) and the set \( \{a, b, c\} \cap \beta_1 \) contains two elements, without loss of generality, we assume that \( a, b \in \beta_1 \). It follows that \( c \in \beta_2 \). Hence, by Item 3 of Lemma 6.1.8, we have \( \frac{1}{3}a, \frac{1}{3}b \in \beta_3 \) and \( \frac{1}{3}c \in \beta_4 \).

If \( a \neq b \), then, by Item 1 of Lemma 6.1.11, we have either \( \frac{1}{3}a + \frac{1}{3}b \in \beta_2 \), or \( \frac{1}{3}a + \frac{1}{3}b \in \beta_3 \), or \( \frac{1}{3}a + \frac{1}{3}b = 0 \). Hence either \( u \in \beta_2 + \beta_4 \subset P(2) + \beta_4 \subset P(4) \), or \( u \in \beta_3 + \beta_4 \subset P(4) \) by Table 6-2, or \( u \in \beta_4 \subset P(4) \). Thus \( u \in P(4) \).

If \( a = b \), without loss of generality, we assume that \( a_1 = b_1 = 1 \). Then the label of \( \frac{1}{3}a + \frac{1}{3}b \) in \( P(4) \) is 0010 \( \oplus \) 0010 = 1040 in \( \Lambda_4 \). Let the label of \( c \) be 0c2. It follows that the label of \( u \) is 1040 \( \oplus \) 000c2 = 1000 \( \oplus \) 0040 \( \oplus \) 000c2 in \( \Lambda_4 \). By Table 6-2, we have 0040 \( \oplus \) 000c2 = 0d10d2 in \( \Lambda_4 \) where \( d_1 \in \{0, 3, 4\} \). Hence the label of \( u \) becomes 1000 \( \oplus \) 0d10d2 = 1000 \( \oplus \) 0d100 \( \oplus \) 000d2 in \( \Lambda_4 \). Since \( d_1 \in \{0, 3, 4\} \), by Table 6-2, we have 10 \( \oplus \) 0d1 \( \in \{10, 01, 06\} \) \( \subset \Lambda_2 \). By applying Item 4 of Lemma 6.1.8 twice, it follows that 1000 \( \oplus \) 0d100 \( \in \{1000, 0100, 0600\} \) \( \subset \Lambda_4 \). Hence the label of \( u \) belongs to the set \( \{1000 \oplus 000d_2, 0100 \oplus 000d_2, 0600 \oplus 000d_2\} = \{100d_2, 010d_2, 060d_2\} \subset \Lambda_4 \). Thus \( u \in P(4) \).

\( \square \)
The following theorem will play an important part in the algorithm for adding any two labels.

**Theorem 6.1.14.** For any \( a, b \in P(n) \), we have \( \frac{1}{3}a + \frac{1}{3}b \in P(n + 2) \).

**Proof.** We will just give a proof when \( n \) is even since the proof for odd \( n \) is similar. Let the labels of \( a \) and \( b \) be \( a_1a_2...a_n \) and \( b_1b_2...b_n \), respectively. It follows from Item 3 of Lemma 6.1.8 that \( \frac{1}{3}a, \frac{1}{3}b \in P(n + 2) \), and the labels of \( \frac{1}{3}a \) and \( \frac{1}{3}b \) are \( 00a_1a_2...a_n \in \Lambda_{n+2} \) and \( 00b_1b_2...b_n \in \Lambda_{n+2} \), respectively. Let \( A = 00a_1a_2...a_n \) and \( B = 00b_1b_2...b_n \). In the following, we use induction to show that \( A \oplus B \in \Lambda_{n+2} \).

If \( n = 2 \), then this statement follows from Lemma 6.1.13. Assume this statement is true for all integers less than \( n \). We then have \( 00a_3a_4...a_n \oplus 00b_3b_4...b_n \in \Lambda_n \). Let \( c_1c_2...c_n = 00a_3a_4...a_n \oplus 00b_3b_4...b_n \in \Lambda_n \). It follows that, in \( \Lambda_{n+2} \), we have

\[
00c_1c_2...c_n = 0000a_3a_4...a_n \oplus 0000b_3b_4...b_n. \tag{6–9}
\]

By Item 1 of Lemma 6.1.8, in \( \Lambda_{n+2} \), we have

\[
A \oplus B = 00a_1a_200...0 \oplus 0000a_3a_4...a_n \oplus 00b_1b_200...0 \oplus 0000b_3b_4...b_n \\
= 00a_1a_200...0 \oplus 00b_1b_200...0 \oplus 0000a_3a_4...a_n \oplus 0000b_3b_4...b_n. \tag{6–10}
\]

By applying Equation 6–9, Equation 6–10 becomes

\[
A \oplus B = 00a_1a_2000...0 \oplus 00b_1b_2000...0 \oplus 00c_1c_2c_3c_4c_5...c_n \\
= 00a_1a_2000...0 \oplus 00b_1b_2000...0 \oplus 00c_1c_2000...0 \oplus 0000c_3c_4c_5...c_n \\
= 00a_1a_2000...0 \oplus 00b_1b_2000...0 \oplus 00c_1c_2000...0 \oplus 0000c_3c_40...0 \oplus 000000c_5...c_n. \tag{6–11}
\]

To finish the proof, we consider the following cases.

**Case 1:** \( c_3 = 0 \). By Lemma 6.1.13, we have \( 00a_1a_2 \oplus 00b_1b_2 \oplus 00c_1c_2 \in \Lambda_4 \). Let \( d_1d_2d_3d_4 = 00a_1a_2 \oplus 00b_1b_2 \oplus 00c_1c_2 \). By applying Item 4 of Lemma 6.1.8 for \( n – 2 \) times, it follows that \( d_1d_2d_3d_4000...0 = 00a_1a_2000...0 \oplus 00b_1b_2000...0 \oplus 00c_1c_2000...0 \) in \( \Lambda_{n+2} \). Hence
Equation 6–11 becomes $A \oplus B = d_1d_2d_3d_4000...0 \oplus 0000c_3c_4c_5...c_n$ in $\Lambda_{n+2}$. Since $c_3 = 0$, by Item 2 of Lemma 6.1.8, it follows that $A \oplus B = d_1d_2d_3d_4c_3c_4c_5...c_n$ in $\Lambda_{n+2}$.

**Case 2:** $c_3 \neq 0$. By Item 1 of Corollary 6.1.7, we have $c_2 = c_4 = 0$ in this case. In Equation 6–11, we claim that we can assume $a_2 = b_2 = 0$ as well.

Suppose $a_2 \neq 0$. By Item 1 of Corollary 6.1.7, it follows that $a_1 = 0$. Hence $00a_1a_200 \oplus 0000c_3c_4 = 00a_200 \oplus 0000c_30$ in $\Lambda_6$. By Table 6-1, in $\Lambda_3$, we have $0a_20 \oplus 00c_3 = x0y$ in $\Lambda_3$ for some $x, y \in \{0, 1, 2, ..., 6\}$ with $y \neq 0$. By applying Item 4 of Lemma 6.1.8 for $n – 1$ times, it follows that $00a_2000...0 \oplus 0000c_300...0 = 00x0y00...0$ in $\Lambda_{n+2}$, i.e., $00a_1a_2000...0 \oplus 0000c_3c_40...0 = 00x0y00...0$ in $\Lambda_{n+2}$. Hence Equation 6–11 becomes

$$A \oplus B = 00x0000...0 \oplus 00b_1b_20000...0 \oplus 00c_1c_20000...0 \oplus 0000y0c_3...c_n$$

$$= 00x0000...0 \oplus 00b_1b_20000...0 \oplus 00c_1c_20000...0 \oplus 0000y0c_4c_5...c_n$$

in $\Lambda_{n+2}$, which is what we want. Similarly we can assume $b_2 = 0$. Hence Equation 6–11 becomes

$$A \oplus B = 00a_10000...0 \oplus 00b_10000...0 \oplus 00c_10000...0 \oplus 0000c_3c_4c_5...c_n$$

in $\Lambda_{n+2}$. 

By Lemma 6.1.12 we have $00a_1 \oplus 00b_1 \oplus 00c_1$ in $\Lambda_3$. Let $xyz = 00a_1 \oplus 00b_1 \oplus 00c_1 \in \Lambda_3$. By applying Item 4 of Lemma 6.1.8 for $n – 1$ times, it follows that $xyz0000...0 = 00a_10000...0 \oplus 00b_10000...0 \oplus 00c_10000...0$ in $\Lambda_{n+2}$. Hence Equation 6–13 becomes

$A \oplus B = xyz0000...0 \oplus 0000c_3c_4c_5...c_n = xyz0c_3c_4c_5...c_n$ in $\Lambda_{n+2}$ by Item 2 of Lemma 6.1.8. Thus $A \oplus B \in \Lambda_{n+2}$. \hfill \Box

The following proposition tells whether or not $a + b \in P(n)$ based on the label of $\frac{1}{3}a + \frac{1}{3}b \in P(n + 2)$.

**Proposition 6.1.15.** For any $a, b \in P(n)$, let $s = \frac{1}{3}a + \frac{1}{3}b$. If $s_1s_2...s_n s_{n+1}s_{n+2} \in \Lambda_{n+2}$ is the label of $s$, then $a + b \in P(n)$ if and only if $s_1 = s_2 = 0$.

**Proof.** If $a + b \in P(n)$, then by Theorem 6.1.6 there exists uniquely determined $c_i \in \{0, 1, 2, ..., 6\}$ for each $i = 1, 2, ..., n$ such that $a + b = \sum_{i=1}^{n} \beta_i c_i$ and either $c_i = 0$ or

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\(c_{i+1} = 0\) for any \(i\) satisfying \(1 \leq i < n\). It follows that \(\frac{1}{3}a + \frac{1}{3}b = \sum_{i=1}^{n} \frac{1}{3}\beta_{i,c_i} = \sum_{i=1}^{n} \beta_{i+2,c_i}\). Since \(s = \frac{1}{3}a + \frac{1}{3}b\) and \(s_1s_2...s_n\) is the label of \(s\), it follows that \(s_1 = s_2 = 0\).

Conversely if \(s_1 = s_2 = 0\), since \(s_1s_2...s_n\) is the label of \(s\), we have \(s = \sum_{i=1}^{n+2} \beta_{i,s_i} = \sum_{i=3}^{n+2} \beta_{i,s_i} = \sum_{i=3}^{n+2} \frac{1}{3}\beta_{i-2,s_i} = \frac{1}{3} \sum_{i=3}^{n+2} \beta_{i-2,s_i}\). Since \(s = \frac{1}{3}a + \frac{1}{3}b\), it follows that \(a + b = \sum_{i=3}^{n+2} \beta_{i-2,s_i} = \sum_{j=1}^{n} \beta_{j,s_j+2}\). Because \(s_1s_2...s_n\) is the label of \(s\), either \(s_i = 0\) or \(s_{i+1} = 0\) for any \(i\) satisfying \(1 \leq i < n + 2\). By Lemma 6.1.5, it follows that \(\sum_{i=3}^{n+2} \beta_{i-2,s_i} \in \mathbb{P}(n)\). Thus \(a + b \in \mathbb{P}(n)\).

Let \(n \in \mathbb{N}\) and \(a, b \in \mathbb{P}(n)\), and let \(a_1a_2...a_n \in \Lambda_n\) and \(b_1b_2...b_n \in \Lambda_n\) be the labels of \(a\) and \(b\), respectively. By applying Item 3 of Lemma 6.1.8, it follows that the labels of \(\frac{1}{3}a\) and \(\frac{1}{3}b\) are \(00a_1a_2...a_n \in \Lambda_{n+2}\) and \(00b_1b_2...b_n \in \Lambda_{n+2}\), respectively. By Theorem 6.1.14, we have \(\frac{1}{3}a + \frac{1}{3}b \in \mathbb{P}(n + 2)\). Hence there exists \(c_1c_2...c_n\) such that \(00a_1a_2...a_n + 00b_1b_2...b_n = c_1c_2...c_n\) in \(\Lambda_{n+2}\). We call \(c_1c_2\) the carry and the remainder of adding the two labels \(a_1a_2...a_n\) and \(b_1b_2...b_n\), respectively.

For example, by Table 6-2, we have \(0001 \oplus 0030 = 0205\) in \(\Lambda_4\). It follows that the carry and the remainder of adding the two labels 01 and 30 are 02 and 05, respectively. By Proposition 6.1.15, \(c_1c_2 = 00\) if and only if \(a + b \in \mathbb{P}(n)\).

For any integer \(n > 0\), we use the following algorithm to add any two labels \(a_1a_2...a_n \in \Lambda_n\) and \(b_1b_2...b_n \in \Lambda_n\). From the discussion of the previous paragraph, we need to compute \(00a_1a_2...a_n + 00b_1b_2...b_n\) to get the sum \(c_1c_2...c_n\). Recall that any label of the Pyxis structure has the property that, for any two consecutive digits, at least one of them is zero. If the addition proceeds from the right digits \(a_n\) and \(b_n\) to the left digits \(a_1\) and \(b_1\), the idea of remainder, carry and sum of adding two digits is similar to that of the usual addition except that, for the addition of \(a_i\) and \(b_i\), we use Table 6-1 when \(i\) is odd and use Table 6-2 when \(i\) is even. Furthermore the carry of adding \(a_i\) and \(b_i\) consists of two digits and the sum of two labels obtained in the usual way may no longer have the property that, for any two consecutive digits, at least one of them is zero. For example, if we add two labels 10 and 05 in \(\Lambda_2\) using the usual digit by
digit addition, the obtained sum 15 is no longer a legitimate Pyxis label because there are two nonzero consecutive digits. Hence the addition of two elements in $\Lambda_n$ is quite complicated. However, we have successfully developed the following algorithm which output the required sum and has the computational complexity of order $n$ when two labels in $\Lambda_n$ are added.

**Algorithm addLabels**

*Input:* An integer $n > 0$, and two labels $a_1a_2...a_n \in \Lambda_n$ and $b_1b_2...b_n \in \Lambda_n$.

*Output:* The $Sum = 00a_1a_2...a_n \oplus 00b_1b_2...b_n$ in $\Lambda_{n+2}$.

*Step 1.* If $n = 1$ or $n = 2$, then apply Table 6-1 or Table 6-2, respectively, to return $Sum = 00a_1 \oplus 00b_1$ in $\Lambda_3$ or $Sum = 00a_1a_2 \oplus 00b_1b_2$ in $\Lambda_4$. Otherwise do the following Step 2 through Step 7.

*Step 2.* If $n$ is odd, then let $N = n + 1$ and adjoin a 0 to the right end of the two labels $a_1a_2...a_n$ and $b_1b_2...b_n$, i.e., let $a_N = b_N = 0$. Otherwise let $N = n$.

*Step 3.* Let $M = \frac{N}{2}$. Also, for $i = 1$ to $M$, let $aPair\{i\} = a ua_v$ and $bPair\{i\} = b_ub_v$ where $u = 2(M - i) + 1$ and $v = 2(M - i) + 2$.

*Step 4.* Let $Sum$ be the empty set, and add $aPair\{1\}$ and $bPair\{1\}$ to get a temporary carry denoted $tempCarry$ and a temporary remainder denoted $tempRem$.

*Step 5.* For $i = 2$ to $M$, do the following.

- Let $carrySet$ consist of elements of the set $\{aPair\{i\}, bPair\{i\}, tempCarry\}$ that are different from 00, and let $carrySet_1$ consist of elements of $carrySet$ whose last digit is nonzero and let $carrySet_2$ consist of elements of $carrySet$ whose first digit is nonzero. Also let $k_j$ denote the size of the set $carrySet_j$ for $j = 1, 2$.

  1. If the first digit of $tempRem$ is 0, let $remd$ (means remainder) be $tempRem$.
     
     (a) If $k_1 = 0$, apply Subroutine 1 to add the elements in the set $carrySet_2$ to obtain a carry $c_1c_2$ and a remainder $r_10$. Let $tempCarry = c_1c_2$ and $tempRem = r_10$.
     
     (b) If $k_1 = 1$, let $0c_2$ denote the element in the set $carrySet_1$.  

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i. If $k_2 = 0$, let $tempCarry = 00$ and $tempRem = 0c_2$.

ii. Else if $k_2 = 1$, let $d_10$ denote the element in the set $carrySet_2$. Apply Table 6-2 to compute $000c_2 \oplus 00d_10 = 0e_10c_2$. Let $tempCarry = 0e_1$ and $tempRem = 0e_2$.

iii. Else, let $d_10$ and $e_10$ denote the two elements in the set $carrySet_2$. By Table 6-2, $000c_2 \oplus 00d_10 = 0f_10f_2$ for some $0f_10f_2 \in \Lambda_4$ with $f_2 \neq 0$. Similarly, we have $000f_2 \oplus 00e_10 = 0g_10g_2$ for some $0g_10g_2 \in \Lambda_4$. By Theorem 6.1.14, we have $0f_1 \oplus 0g_1 = h_1h_2 \in \Lambda_2$. Let $tempCarry = h_1h_2$ and $tempRem = 0g_2$.

(c) If $k_1 = 2$, let $0c_1$ and $0d_1$ denote the two elements in the set $carrySet_1$.

i. If $k_2 = 0$, by Table 6-2, $000c_1 \oplus 00d_1 = 0f_1f_2f_3$ for some $0f_1f_2f_3 \in \Lambda_4$.

Let $tempCarry = 0f_1$ and $tempRem = f_2f_3$.

ii. Else, let $e_10$ denote the element in the set $carrySet_2$.

A. If $c_1 = d_1$, by Table 6-2, we have $000c_1 \oplus 00d_1 = 0f_10f_3$ for some $0f_10f_3 \in \Lambda_4$. By Table 6-2 again, we have $000f_3 \oplus 00e_10 = 0g_10g_3$ for some $0g_10g_3 \in \Lambda_4$. By Theorem 6.1.14, we have $0f_1 \oplus 0g_1 = h_1h_2 \in \Lambda_2$. Let $tempCarry = h_1h_2$ and $tempRem = 0g_3$.

B. Else, since $c_1 \neq d_1$, by Lemma 6.1.11 we have $0c_1 \oplus 0d_1 = f_1f_2 \in \Lambda_2$.

By Table 6-2, we have $00f_1f_2 \oplus 00e_10 = g_1g_2g_3g_4$ for some $g_1g_2g_3g_4 \in \Lambda_4$. Let $tempCarry = g_1g_2$ and $tempRem = g_3g_4$.

(d) If $k_1 = 3$, add the three elements in the set $carrySet_1$ using the following Subroutine 2 to obtain $0h_2h_3h_4 \in \Lambda_4$. Let $tempCarry = 0h_2$ and $tempRem = h_3h_4$.

2. Else (now the first digit of $remd$ is nonzero),

(a) if $k_1 = 0$, let $remd = tempRem$ and add the elements of the set $carrySet_2$ using Subroutine 1 to get a carry $c_1c_2$ and a remainder $r_1r_2$.

Let $tempCarry = c_1c_2$ and $tempRem = r_1r_2$. 

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(b) Else let $e_10$ denote $tempRem$ and let $0f_2$ denote the first element of the set $carrySet_1$. Compute $00e_10f_20$ using table 6-1 to get $g_10g_3 \in \Lambda_3$. Let $carrySet_2 = carrySet_2 \cup \{g_10\}$.

i. If $k_1 = 1$, let $remd = g_30$ and add the elements of the set $carrySet_2$ using Subroutine 1 to get a carry $c_1c_2$ and a remainder $r_10$. Let $tempCarry = c_1c_2$ and $tempRem = r_10$.

ii. Else, let $0y_2$ denote the second element of the set $carrySet_1$ and compute $00g_30y_20$ using Table 6-1 to get $h_10h_3 \in \Lambda_3$. Let $carrySet_2 = carrySet_2 \cup \{h_10\}$.

A. If $k_1 = 2$, let $remd = h_30$ and add the elements of the set $carrySet_2$ using Subroutine 1 to get a carry $c_1c_2$ and a remainder $r_10$. Let $tempCarry = c_1c_2$ and $tempRem = r_10$.

B. Else, let $0z_2$ denote the third element of the set $carrySet_1$ and compute $00h_30z_20$ using Table 6-1 to get $l_10l_3 \in \Lambda_3$. Let $carrySet_2 = carrySet_2 \cup \{l_10\}$, $remd = l_30$, and add the elements of the set $carrySet_2$ using Subroutine 1 to get a carry $c_1c_2$ and a remainder $r_10$. Let $tempCarry = c_1c_2$ and $tempRem = r_10$.

• Let $Sum$ be the concatenation of $remd$ and $Sum$, and increase $i$.

**Step 6.** Let $Sum$ be the concatenation of the $tempCarry$, $tempRem$, and $Sum$. If $n$ is odd, delete the 0 in the right side of $Sum$. Return $Sum$.

**Subroutine 1:** Input: A set $\Omega$ (containing at most three elements) of labels in $\Lambda_2$ whose second digit is 0.

Output: The carry $C = x_1x_2 \in \Lambda_2$ and remainder $R = y_10 \in \Lambda_2$ of adding labels in the set $\Omega$.

Let $k$ be the size of the set $\Omega$. Since $\Omega$ contains at most three elements, we consider the following four cases.

• If $k = 0$, return $C = 00$ and $R = 00$. 

• If \( k = 1 \), let \( C = 00 \) and let \( R \) be the element in the set \( \Omega \). Return \( C \) and \( R \).

• If \( k = 2 \), add the two elements in the set \( \Omega \) using Table 6-2 to get a carry \( C = x_1x_2 \in \Lambda_2 \) and a remainder \( R = y_1y_0 \in \Lambda_2 \), and return them.

• If \( k = 3 \), let \( a_10, b_10, \) and \( c_10 \) denote the three elements of the set \( \Omega \).
  1. If \( a_1 = b_1 = c_1 \), then return \( C = a_10 \) and \( R = 00 \).
  2. Else if \( a_1 = b_1 \), then compute \( 00b_1 \oplus 00c_1 \) using Table 6-2 to get a carry \( C = 00 \) and a remainder \( R = y_1y_2 \in \Lambda_2 \), and return \( C \) and \( R \).
  3. Else compute \( 00a_1 \oplus 00b_1 \) using Table 6-2 to get a carry \( C = 00 \) and a remainder \( R = y_1y_2 \in \Lambda_2 \), and return \( C \) and \( R \).

Subroutine 2: Input: A set \( \Omega \) (containing at most three elements) of labels in \( \Lambda_2 \) whose first digit is 0.

Output: The carry \( C = 0x_2 \in \Lambda_2 \) and remainder \( R = y_1y_2 \in \Lambda_2 \) of adding labels in the set \( \Omega \).

Let \( k \) denote the size of the set \( \Omega \). Since \( \Omega \) contains at most three elements, we consider the following four cases.

• If \( k = 0 \), return \( C = 00 \) and \( R = 00 \).

• If \( k = 1 \), let \( C = 00 \) and let \( R \) denote the element in the set \( \Omega \). Return \( C \) and \( R \).

• If \( k = 2 \), add the two elements in the set \( \Omega \) using Table 6-2 to get a carry \( C = 0x_2 \in \Lambda_2 \) and a remainder \( R = y_1y_2 \in \Lambda_2 \), and return them.

• If \( k = 3 \), let \( 0a_2, 0b_2, \) and \( 0c_2 \) denote the three elements of the set \( \Omega \).
  1. If \( a_2 = b_2 = c_2 \), then return \( C = 0a_2 \) and \( R = 00 \).
  2. Else if \( a_2 = b_2 \), then compute \( 0b_2 \oplus 0c_2 \) using Table 6-2 to get a carry \( C = 0x_2 \in \Lambda_2 \) and a remainder \( r_1r_2, \) and compute \( r_1r_2 \oplus 0a_2 \) using Table 6-2 to get a carry \( C = 0x_2 \in \Lambda_2 \) and a remainder \( R = y_1y_2 \in \Lambda_2 \), and return \( C \) and \( R \).
3. Else compute \(0a_2 \oplus 0b_2\) to get a carry 00 and a remainder \(r_1r_2\), and compute \(r_1r_2 \oplus 0c_2\) using Table 6-2 to get a carry \(C = 0x_2 \in \Lambda_2\) and a remainder \(R = y_1y_2 \in \Lambda_2\), and return \(C\) and \(R\).

At each iteration in Step 5 of Algorithm `addLabels`, the set `carrySet` contains at most three elements. It follows that the computational complexity of adding any two labels in \(\Lambda_n\) using Algorithm `addLabels` is of order \(n\). In the following, we give two examples showing the application of Algorithm `addLabels`. In Example 1, for each iteration in Step 5 of the algorithm, the first digit of `tempRem` is 0. In Example 2, for each iteration in Step 5 of the algorithm, the first digit of `tempRem` is nonzero. Hence the addition in Example 2 is actually more complicated than that in Example 1.

**Example 1:** Let \(n = 6\), \(A = 500104\), and \(B = 403020\).

- In Step 1, since \(n > 2\) and \(n\) is even, go to Step 2.
- In Step 2, let \(N = n = 6\).
- In Step 3, let \(M = \frac{N}{2} = 3\). Also let \(aPair\{1\} = 04\), \(bPair\{1\} = 20\), \(aPair\{2\} = 01\), \(bPair\{2\} = 30\), \(aPair\{3\} = 50\), and \(bPair\{3\} = 40\).
- In Step 4, let `Sum` be the empty set. By Table 6-2, we have \(aPair\{1\} \oplus bPair\{1\} = 04 \oplus 20 = 02\) in \(\Lambda_2\). It follows that the carry and the remainder of this addition are 00 and 02, respectively. Hence `tempCarry` = 00 and `tempRem` = 02.
- In Step 5, for \(i = 2\) to \(M\) do the following.
  1. When \(i = 2\), let `carrySet` consist of elements of the set

\[
\{aPair\{2\}, bPair\{2\}, tempCarry\}
\]

that are different from 00, i.e., let `carrySet` = \(\{01, 30\}\). Then let `carrySet_1` = \(\{01\}\) and `carrySet_2` = \(\{30\}\). It follows that \(k_1 = k_2 = 1\). Since the first digit of `tempRem` is 0, let `remd` = `tempRem` = 02. Since \(k_1 = 1\), let 0\(c_2\) denote the element in the set `carrySet_1`, i.e., let 0\(c_2\) = 01. Since \(k_2 = 1\), let \(d_10\) denote the element in the set `carrySet_2`, i.e., let \(d_10\) = 30. Apply Table 6-2
to compute $000c_2 \oplus 00d_10 = 0001 \oplus 0030 = 0205$, which is denoted $0e_10e_2$.

Let $tempCarry = 0e_1 = 02$ and $tempRem = 0e_2 = 05$. Let $Sum$ be the concatenation of $remd = 02$ and $Sum = \emptyset$, i.e., let $Sum = 02$, and increase $i$.

2. When $i = 3$, let $carrySet$ consist of elements of the set

$$\{aPair\{3\}, bPair\{3\}, tempCarry\}$$

that are different from 00, i.e., let $carrySet = \{50, 40, 02\}$. Then let

$carrySet_1 = \{02\}$, and $carrySet_2 = \{50, 40\}$. It follows that $k_1 = 1$ and $k_2 = 2$.

Since the first digit of $tempRem$ is 0, let $remd = tempRem = 05$. Since $k_1 = 1$, let $0c_2$ denote the element in the set $carrySet_1$, i.e., let $0c_2 = 02$. Since $k_2 = 2$, let $d_10$ and $e_10$ denote the two elements in the set $carrySet_2$, i.e., let $d_10 = 50$ and $e_10 = 40$. By Table 6-2, we have $000c_2 \oplus 00d_10 = 0002 \oplus 0050 = 0004$, which is denoted $0f_10f_2$. Similarly, we have $000f_2 \oplus 00e_10 = 0004 \oplus 0040 = 0402$, which is denoted $0g_10g_2$. By Table 6-2, we have $0f_1 \oplus 0g_1 = 00 \oplus 04 = 04$, which is denoted $h_1h_2$. Let $tempCarry = h_1h_2 = 04$ and $tempRem = 0g_2 = 02$. Let $Sum$ be the concatenation of $remd = 05$ and $Sum = 02$, i.e., let $Sum = 0502$, and increase $i$.

- In Step 6, let $Sum$ be the concatenation of the $tempCarry = 04$, $tempRem = 02$, and $Sum = 0502$, i.e., let $Sum = 04020502$.

Notice that, in Example 1, the first digit of $tempRem$ is 0 for each iteration in Step 5 of the algorithm. In the next, we show another example where the first digit of $tempRem$ is nonzero for each iteration in Step 5 of the algorithm.

**Example 2:** Let $n = 6$, $A = 050101$, and $B = 401006$.

- In Step 1, since $n > 2$ and $n$ is even, go to Step 2.
- In Step 2, let $N = n = 6$.
- In Step 3, let $M = \frac{N}{2} = 3$. Also let $aPair\{1\} = 01$, $bPair\{1\} = 06$, $aPair\{2\} = 01$, $bPair\{2\} = 10$, $aPair\{3\} = 05$, and $bPair\{3\} = 40$. 

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• In Step 4, let $Sum$ be the empty set. By Table 6-2, we have $aPair\{1\} \oplus bPair\{1\} = 01 \oplus 06 = 10$ in $\Lambda_2$. It follows that the carry and the remainder of this addition are 00 and 10, respectively. Hence $tempCarry = 00$ and $tempRem = 10$.

• In Step 5, for $i = 2$ to $M$ do the following.
  
  1. When $i = 2$, let $carrySet$ consist of elements of the set

  \[
  \{aPair\{2\}, bPair\{2\}, tempCarry\}
  \]

  that are different from 00, i.e., let $carrySet = \{01, 10\}$. Then let $carrySet_1 = \{01\}$, and $carrySet_2 = \{10\}$. It follows that $k_1 = k_2 = 1$. Since the first digit of $tempRem$ is nonzero and $k_1 \neq 0$, let $e_10$ denote $tempRem$ and let $0f_2$ denote the element of the set $carrySet_1$. It follows that $e_1 = 1$ and $f_2 = 1$. By Table 6-1, we have $00e_1 \oplus 0f_20 = 001 \oplus 010 = 103$, which is denoted $g_1g_3$, i.e., $g_1 = 1$ and $g_3 = 3$. Let $carrySet_2 = carrySet_2 \cup \{10\} = \{10\} \cup \{10\} = \{10, 10\}$. Since $k_1 = 1$, let $remd = g_30 = 30$ and add the elements in the set $carrySet_2 = \{10, 10\}$ using Table 6-2 to get a carry $c_1c_2 = 10$ and a remainder $r_10 = 40$. Let $tempCarry = c_1c_2 = 10$ and $tempRem = r_10 = 40$. Let $Sum$ be the concatenation of $remd = 30$ and $Sum = 0$, i.e., let $Sum = 30$, and increase $i$.

  2. When $i = 3$, let $carrySet$ consist of elements of the set

  \[
  \{aPair\{3\}, bPair\{3\}, tempCarry\}
  \]

  that are different from 00, i.e., let $carrySet = \{05, 40, 10\}$. Then let $carrySet_1 = \{05\}$, and $carrySet_2 = \{40, 10\}$. It follows that $k_1 = 1$ and $k_2 = 2$. Since the first digit of $tempRem$ is nonzero and $k_1 \neq 0$, let $e_10$ denote $tempRem$ and let $0f_2$ denote the element of the set $carrySet_1$. It follows that $e_1 = 4$ and $f_2 = 5$. By Table 6-1, we have $00e_1 \oplus 0f_20 = 004 \oplus 050 = 502$, which implies that $g_1 = 5$ and $g_3 = 2$. Let $carrySet_2 = carrySet_2 \cup \{50\}$ =
\{40, 10\} \cup \{50\} = \{40, 10, 50\}. Since \( k_1 = 1 \), let \( \text{remd} = g_80 = 20 \) and add the elements in the set \( \text{carrySet}_2 = \{40, 10, 50\} \) using Table 6-2 to get a carry \( c_1c_2 = 00 \) and a remainder \( r_10 = 50 \). Let \( \text{tempCarry} = c_1c_2 = 00 \) and \( \text{tempRem} = r_10 = 50 \). Let \( \text{Sum} \) be the concatenation of \( \text{remd} = 20 \) and \( \text{Sum} = 30 \), i.e., let \( \text{Sum} = 2030 \), and increase \( i \).

- In Step 6, let \( \text{Sum} \) be the concatenation of the \( \text{tempCarry} = 00 \), \( \text{tempRem} = 50 \), and \( \text{Sum} = 2030 \), i.e., let \( \text{Sum} = 00502030 \).

**Theorem 6.1.16.** For any integer \( n > 0 \) and two labels \( a_1a_2...a_n, b_1b_2...b_n \in \Lambda_n \), the Sum output from Algorithm addLabels equals \( 00a_1a_2...a_n \oplus 00b_1b_2...b_n \in \Lambda_{n+2} \).

**Proof.** By Item 4 of Lemma 6.1.8, we just need to consider the case when \( n \) is even. Also when \( n \leq 2 \), this theorem is obviously true. Hence we assume that \( n \) is an even integer which is bigger than 2. Let \( n = 2M \) for some integer \( M > 1 \), \( A = 00a_1a_2...a_{2M} \in \Lambda_{n+2} \), and \( B = 00b_1b_2...b_{2M} \in \Lambda_{n+2} \). For \( i = 1 \) to \( M \), let \( A_i = 00a_{2(M-i)+1}a_{2(M-i)+2}...a_{2M-1}a_{2M} \in \Lambda_{2i+2} \), and \( B_i = 00b_{2(M-i)+1}b_{2(M-i)+2}...b_{2M-1}b_{2M} \in \Lambda_{2i+2} \). By Theorem 6.1.14, we have \( A_i \oplus B_i \in \Lambda_{2i+2} \). Let \( \text{Sum}_1, \text{tempCarry}_1 \) and \( \text{tempRem}_1 \) be the \( \text{Sum}, \text{tempCarry} \) and \( \text{tempRem} \) output from Step 4 of Algorithm addLabels. For \( i = 1 \) to \( M \), let \( T_i = A_i \oplus B_i \), and let \( \text{remd}_i, \text{Sum}_i, \text{tempCarry}_i, \) and \( \text{tempRem}_i \) be the \( \text{remd}, \text{Sum}, \text{tempCarry}, \) and \( \text{tempRem} \) output from the \( i^{th} \) iteration of Step 5 in Algorithm addLabels.

In the following, we use induction to prove that \( T_i \) is the concatenation of \( \text{tempCarry}_i, \text{tempRem}_i, \) and \( \text{Sum}_i \) for any \( i \) satisfying \( 1 \leq i \leq M \). When \( i = 1 \), \( A_1 = 00a_{2M-1}a_{2M} \) and \( B_1 = 00b_{2M-1}b_{2M} \). In Algorithm addLabels, obviously \( aPair\{1\} = a_{2M-1}a_{2M} \) and \( bPair\{1\} = b_{2M-1}b_{2M} \). From Step 4 of Algorithm addLabels, \( \text{Sum}_1 = \emptyset \) and hence \( A_1 \oplus B_1 \) is the concatenation of \( \text{tempCarry}_1, \text{tempRem}_1, \) and \( \text{Sum}_1 \). It follows that \( T_1 \) is the concatenation of \( \text{tempCarry}_1, \text{tempRem}_1, \) and \( \text{Sum}_1 \). Now assume that \( T_{i-1} \) is the concatenation of \( \text{tempCarry}_{i-1}, \text{tempRem}_{i-1}, \) and \( \text{Sum}_{i-1} \) for some \( i \) satisfying
\[ 2 \leq i \leq M. \text{ Since} \]
\[ A_i = 00a_{2(M-i)+1}a_2(M-i)+2a_2(M-i+1)+1a_2(M-i+1)+2\cdots a_2M-1a_2M \]  
\[ = 00a_2(M-i)+1a_2(M-i)+200\cdots 00 \oplus 00A_{i-1} \]  
and
\[ B_i = 00b_{2(M-i)+1}b_2(M-i)+2b_2(M-i+1)+1b_2(M-i+1)+2\cdots b_{2M-1}b_{2M} \]
\[ = 00b_2(M-i)+1b_2(M-i)+200\cdots 00 \oplus 00B_{i-1}, \]
we have
\[ T_i = A_i \oplus B_i = 00a_2(M-i)+1a_2(M-i)+200\cdots 00 \oplus 00b_2(M-i)+1b_2(M-i)+200\cdots 00 \oplus 00A_{i-1} \oplus 00B_{i-1} \]
\[ = 00a_2(M-i)+1a_2(M-i)+200\cdots 00 \oplus 00b_2(M-i)+1b_2(M-i)+200\cdots 00 \oplus 00T_{i-1}. \]  
\[ (6-16) \]

By our induction assumption, \( T_{i-1} \) is the concatenation of \( tempCarry_{i-1} \), \( tempRem_{i-1} \), and \( Sum_{i-1} \). Hence Equation \( 6-16 \) becomes
\[ T_i = 00a_2(M-i)+1a_2(M-i)+200\cdots 00 \oplus 00b_2(M-i)+1b_2(M-i)+200\cdots 00 \]
\[ \oplus 00tempCarry_{i-1} tempRem_{i-1} Sum_{i-1}. \]  
\[ (6-17) \]

In Step 5 of Algorithm \( addLabels \), let
\[ carrySet = \{ a_2(M-i)+1a_2(M-i)+2, b_2(M-i)+1b_2(M-i)+2, tempCarry_{i-1} \} \]
\[ = \{ aPair\{i\}, bPair\{i\}, tempCarry_{i-1} \}. \]  
\[ (6-18) \]

Now we consider the following two cases.

**Case 1:** If the first digit of \( tempRem_{i-1} \) is 0, then, in the \( i^{th} \) iteration of Step 5 of Algorithm \( addLabels \), we let \( remd_i = tempRem_{i-1} \). The \( i^{th} \) iteration of Step 5 in Algorithm \( addLabels \) computes the carry and remainder of the addition of the three elements in the set \( carrySet \). By Lemma \( 6.1.13 \), we have
\[ 00a_2(M-i)+1a_2(M-i)+2 \oplus 00b_2(M-i)+1b_2(M-i)+2 \oplus 00tempCarry_{i-1} = x_1x_2x_3x_4 \]  
\[ (6-19) \]
for some \( x_1x_2x_3x_4 \in \Lambda_4 \). In Algorithm \textit{addLabels}, obviously \( aPair\{i\} = a_{2(M-i)+1}a_{2(M-i)+2} \) and \( bPair\{i\} = b_{2(M-i)+1}b_{2(M-i)+2} \). Then Equation 6–19 becomes

\[
00aPair\{i\} \oplus 00bPair\{i\} \oplus 00\text{tempCarry}_{i-1} = x_1x_2x_3x_4. \tag{6–20}
\]

It follows that the carry and remainder of the addition of the three elements in the set \textit{carrySet} are \( x_1x_2 \) and \( x_3x_4 \), respectively. Hence \( \text{tempCarry}_i = x_1x_2 \) and \( \text{tempRem}_i = x_3x_4 \). By Equations 6–17 and 6–19, we have

\[
T_i = 00aPair\{i\}00...00 \oplus 00bPair\{i\}00...00 \oplus 00\text{tempCarry}_{i-1}\text{tempRem}_{i-1}\text{Sum}_{i-1} \\
= 00aPair\{i\}00...00 \oplus 00bPair\{i\}00...00 \\
\oplus 00\text{tempCarry}_{i-1}00...00 \oplus 0000\text{tempRem}_{i-1}\text{Sum}_{i-1} \\
= x_1x_2x_3x_400...00 \oplus 0000\text{remd}_{i}\text{Sum}_{i-1} \\
= x_1x_2x_3x_400...00 \oplus 0000\text{Sum}_{i} \\
= \text{tempCarry}_i\text{tempRem}_i00...00 \oplus 0000\text{Sum}_{i} \\
= \text{tempCarry}_i\text{tempRem}_i\text{Sum}_{i}. \tag{6–21}
\]

It follows that \( T_i \) is the concatenation of \( \text{tempCarry}_i \), \( \text{tempRem}_i \), and \( \text{Sum}_i \).

**Case 2:** The first digit of \( \text{tempRem}_{i-1} \) is nonzero, i.e., the second case in Step 5 of Algorithm \textit{addLabels}. Recall that, in Algorithm \textit{addLabels}, \( \text{carrySet}_1 \) consist of elements of \( \text{carrySet} \) whose last digit is nonzero and \( \text{carrySet}_2 \) consist of elements of \( \text{carrySet} \) whose first digit is nonzero. Also \( k_j \) denotes the size of the set \( \text{carrySet}_j \) for \( j = 1, 2 \).

If \( k_1 = 0 \), similar to Case 1, we can show that \( T_i \) is the concatenation of \( \text{tempCarry}_i \), \( \text{tempRem}_i \), and \( \text{Sum}_i \).

If \( k_1 \neq 0 \), we compute the sum

\[
S = 00a_{2(M-i)+1}a_{2(M-i)+2}00 \oplus 00b_{2(M-i)+1}b_{2(M-i)+2}00 \oplus 00\text{tempCarry}_{i-1}\text{tempRem}_{i-1}
\]
in certain procedures. By Equation 6–17, $S$ consists of the first six digits of $T_i$. By the assumption of this induction, $T_{i-1} \in \Lambda_{2i}$ is the concatenation of $\text{tempCarry}_{i-1}$, $\text{tempRem}_{i-1}$, and $\text{Sum}_{i-1}$. Hence the concatenation of $\text{tempRem}_{i-1}$ and $\text{Sum}_{i-1}$ is a legitimate Pyxis label. Since $T_i = A_i \oplus B_i \in \Lambda_{2i+2}$ and $\text{Sum}_{i-1} \in \Lambda_{2i-4}$, by Item 2 of Lemma 6.1.8, it follows that $S \in \Lambda_6$. In the $i^{th}$ iteration of Step 5 of Algorithm $\text{addLabels}$, we apply Table 6-1 to convert the expression $00a_2(M-i)+1a_2(M-i)+200 \oplus 00b_2(M-i)+1b_2(M-i)+200 \oplus 00\text{tempCarry}_{i-1}\text{tempRem}_{i-1}$ to the expression $00x00 \oplus 00y00 \oplus 00z00 \oplus 000r0$ for some $x, y, z, r \in \{0, 1, 2, 3, 4, 5, 6\}$. Let $\text{carrySet}_2$ consist of nonzero elements of the set $\{x0, y0, z0\}$ and add the elements in $\text{carrySet}_2$. By Subroutine 2, the sum of the elements in $\text{carrySet}_2$ equals $c_1c_2c_30$ for some $c_1c_2c_30 \in \Lambda_4$. It follows that $S = c_1c_2c_30r0$. Hence $\text{tempCarry}_i = c_1c_2$, $\text{tempRem}_i = c_30$, $\text{remd}_i = r0$, and $\text{Sum}_i$ is the concatenation of $\text{remd}_i = r0$ and $\text{Sum}_{i-1}$. Now Equation 6–17 becomes

\[
T_i = 00a_2(M-i)+1a_2(M-i)+200...00 \oplus 00b_2(M-i)+1b_2(M-i)+200...00 \\
\oplus 00\text{tempCarry}_{i-1}\text{tempRem}_{i-1}\text{Sum}_{i-1} \\
= 00a_2(M-i)+1a_2(M-i)+200...00 \oplus 00b_2(M-i)+1b_2(M-i)+200...00 \\
\oplus 00\text{tempCarry}_{i-1}0...0 \oplus 0000\text{tempRem}_{i-1}0...0 \oplus 000000\text{Sum}_{i-1} \\
= c_1c_2c_30r00...0 \oplus 000000\text{Sum}_{i-1}
\]

(6–22)

Hence $T_i$ is the concatenation of $\text{tempCarry}_i$, $\text{tempRem}_i$, and $\text{Sum}_i$. By induction, $T_i$ is the concatenation of $\text{tempCarry}_i$, $\text{tempRem}_i$, and $\text{Sum}_i$ for any $i$ satisfying $1 \leq i \leq M$. It follows that $T_M$ is the concatenation of $\text{tempCarry}_M$, $\text{tempRem}_M$, and $\text{Sum}_M$. 

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tempRem_M, and Sum_M. The output Sum in Algorithm addLabels is also the concatenation of tempCarry_M, tempRem_M, and Sum_M. Hence Sum = T_M. Since T_M = A_M ⊕ B_M = 00a_1a_2...a_{2M-1}a_{2M} ⊕ 00b_1b_2...b_{2M-1}b_{2M}, we have Sum = 00a_1a_2...a_{2M-1}a_{2M} ⊕ 00b_1b_2...b_{2M-1}b_{2M}.

6.2 Pyxis P(n) Does Not Tile the Underlying Lattice by Translations by a Sublattice for any n > 2

Recall from Chapter 2 that, if an array tiles its underlying lattice by translations by a sublattice, then we can compute the DFT on it. For the n^{th} level of the Pyxis structure P(n), to consider its DFT, we need to know whether or not it tiles the underlying lattice L_n by translations by a sublattice. In this section, we show some properties of the Pyxis structure and apply them to prove that P(n) does not tile the underlying lattice by translations by a sublattice for any n > 2. Hence the DFT on the Pyxis structure cannot be defined using the method in Chapter 2. For any 2 < n ∈ N, the proof that P(n) does not tile L_n when n is odd is quite different from that when n is even. Hence we deal with them in two separate subsections.

6.2.1 Pyxis P(2n−1) Does Not Tile the Underlying Lattice by Translations by a Sublattice for any n > 1

In this subsection, we first define the concept of boundary index for P(n) and use several properties of the boundary index of P(2n−1) to prove that P(2n−1) does not tile the underlying lattice L_{2n−1} by translations by a sublattice for any n ≥ 2. Hence the DFT on the Pyxis structure cannot be defined using the method in Chapter 2.

For any ∅ ≠ T ⊆ L and q ∈ T, if there exists at least one neighbor b of q such that b ∉ T, then q is called a boundary lattice point of T, the Voronoi cell of q is called a boundary hexagon of T. If q is a boundary lattice point of T and if there exist exactly d neighbors that belong to L \ T, then d is called the boundary index of q in T. The number d is also called the boundary index of the Voronoi cell of q in T. The following lemmas will be used to prove the main result of this subsection, which is Theorem 6.2.11.
Lemma 6.2.1. Let \( L \) be a hexagonal lattice, \( u \in L \), \( \emptyset \neq T \subseteq L \), and \( d \) a positive integer. Then \( q \) is a boundary lattice point of \( T \) having boundary index \( d \) if and only if \( u + q \) is a boundary lattice point of \( u + T \) having the same boundary index \( d \).

Lemma 6.2.2. For any \( a \in \beta_3 \) and \( b \in \beta_3 \), we have \( a + b \in P(3) \). Moreover there exist \( c \in \bar{\beta}_1 \) and \( d \in \beta_3 \) such that \( a + b = c + d \).

Proof. Since \( a \in P(2) \subseteq P(3) \) and \( b \in \beta_3 \subseteq P(3) \), the label of \( a \) in \( P(3) \) is either of the form \( 0i0 \) or \( i00 \), and the label of \( b \) in \( P(3) \) is of the form \( 00j \) where \( i, j \in \{0, 1, 2, ..., 6\} \).

By Table 6-1, \( 0i0 \oplus 00j = d_10d_3 \) for some \( d_10d_3 \in \Lambda_3 \). It follows that \( a + b \in P(3) \). Let \( c \in \bar{\beta}_1 \) and \( d \in \beta_3 \) such that the labels of \( c \) and \( d \) in \( P(3) \) are \( d_100 \) and \( 00d_3 \), respectively.

Since \( 0i0 \oplus 00j = d_10d_3 = d_100 \oplus 00d_3 \), we have \( a + b = c + d \). \( \square \)

Lemma 6.2.3. For any \( n \in \mathbb{N} \), \( a \in P(2n) \), and \( b \in \beta_{2n+1} \), we have \( a + b \in P(2n + 1) \).

Proof. When \( n = 1 \), it is followed directly from Lemma 6.2.2. Suppose it is true for all \( n < t \) for some integer \( t \). When \( n = t \), \( P(2n) = P(2t) = P(2t - 1) \cup (P(2t - 2) + \beta_{2t}) \).

Hence \( a \in P(2n) \) implies that either \( a \in P(2t - 1) \) or \( a \in P(2t - 2) + \beta_{2t} \). If \( a \in P(2t - 1) \), then \( a + b \in P(2t - 1) + \beta_{2t+1} = P(2t + 1) \). If \( a \in P(2t - 2) + \beta_{2t} \), then \( a = u + v \) for some \( u \in P(2t - 2) \) and \( v \in \beta_{2t} \). Since \( v \in \beta_{2t} \) and \( b \in \beta_{2t+1} \), it follows easily from Lemma 6.2.2 that \( v + b = c + d \) for some \( c \in \bar{\beta}_{2t-1} \) and \( d \in \beta_{2t+1} \). Hence \( a + b = u + v + b = u + c + d \). Since \( u \in P(2t - 2) = P(2(t - 1)) \) and either \( c \in \beta_{2t-1} \) or \( c = (0, 0) \), we have \( u + c \in P(2t - 1) \) by the assumption of this induction. Thus \( a + b = (u + c) + d \in P(2t - 1) + \beta_{2t+1} \subset P(2t + 1) \). Hence this proposition is true for \( n = t \). Therefore it is true for all \( n \in \mathbb{N} \) by induction. \( \square \)

Corollary 6.2.4. For any \( n \in \mathbb{N} \) and \( q \in P(2n) \), the lattice point \( q \) is not a boundary lattice point of \( P(2n+1) \).

Proof. By Lemma 6.2.3, all six neighbors of \( q \) in \( L_{2n+1} \) are still in \( P(2n + 1) \). Hence \( q \) is not on the boundary of \( P(2n + 1) \). \( \square \)
The following lemma will be used in the proofs of Lemma 6.2.6 and some other results.

**Lemma 6.2.5.** For any \( n \in \mathbb{N} \) and \( a \in P(n) \), if the label of \( a \) is \( a_1a_2...a_n \) with \( a_n \neq 0 \), then \( a \not\in L_{n-1} \).

**Proof.** Let \( \bar{a} \in P(n) \) and \( a_n \in \beta_n \) be lattice points of \( L_n \) such that their labels are \( a_1a_2...a_{n-2}00 \) and \( 00...0a_n \), respectively. Since the label of \( \bar{a} \) is \( a_1a_2...a_{n-2}00 \), we have \( a \in P(n - 2) \subset L_{n-2} \subset L_{n-1} \). By Item 1 of Lemma 6.1.8, we have \( a_1a_2...a_n = a_1a_2...a_{n-2}00 \oplus 00...0a_n \). It follows that \( a = \bar{a} + a_n \). Suppose \( a \in L_{n-1} \). Then \( a_n = a - \bar{a} \in L_{n-1} \cap \beta_n = \emptyset \), which is a contradiction. Thus \( a \not\in L_{n-1} \). \( \square \)

**Lemma 6.2.6.** For any \( n \in \mathbb{N} \), if three lattice points of \( P(n) \) are mutually next to each other in the lattice \( L_n \), then at least one of them is in \( P(n - 1) \).

**Proof.** Let \( a, b, c \) be three lattice points of \( P(n) \) that are mutually next to each other in the lattice \( L_n \), and let \( A = a_1a_2...a_n \in \Lambda_n \), \( B = b_1b_2...b_n \in \Lambda_n \) and \( C = c_1c_2...c_n \in \Lambda_n \) be their labels, respectively. It is enough to show that either \( a_n = 0 \) or \( b_n = 0 \) or \( c_n = 0 \). Suppose that \( a_n \neq 0, b_n \neq 0 \) and \( c_n \neq 0 \). Let \( u = b - a \) and \( v = c - a \). Since \( a \), \( b \) and \( c \) are mutually next to each other in the lattice \( L_n \), by Lemma 6.1.10, we have \( b - a, c - a, b - c \in \beta_n \). It follows that \( u, v, u - v \in \beta_n \), and \( u \) and \( v \) are next to each other in the lattice \( L_n \). Thus \( \beta_n = \{ \pm u, \pm v, \pm (u - v) \} \). Let \( \bar{a} \in P(n) \) and \( a_n \in \beta_n \) be lattice points of \( L_n \) such that their labels are \( a_1a_2...a_{n-2}00 \) and \( 00...0a_n \), respectively. By Item 1 of Lemma 6.1.8, we have \( a_1a_2...a_n = a_1a_2...a_{n-2}00 \oplus 00...0a_n \). It follows that \( a = \bar{a} + a_n \).

Now it follows from \( a_n \in \beta_n \) that either \( a_n = \pm u \) or \( a_n = \pm v \) or \( a_n = \pm (u - v) \).

If \( a_n = u \), then \( a_n \) and \( v \) are next to each other in the lattice \( L_n \). By Lemma 6.1.9, it follows that \( a_n + v \in \beta_{n-1} \subset L_{n-1} \). Hence \( a_n + c - a \in L_{n-1} \). Since the label of \( \bar{a} \) is \( a_1a_2...a_{n-2}00 \), we have \( \bar{a} \in P_{n-2} \subset L_{n-1} \). It follows that \( a_n - a = -\bar{a} \in L_{n-1} \). Hence \( c \in L_{n-1} \) which contradicts Lemma 6.2.5.

If \( a_n = v \), similar to the previous case, there is a contradiction.
If \(a_n = -u\), then \(a_n + u = 0\), i.e. \(a_n + b - a = 0\). It follows that \(b = a - a_n = a \in L_{n-1}\) which also contradicts Lemma 6.2.5.

If \(a_n = u - v\), then \(a_n\) is next to \(u\) in the lattice \(L_n\). By Lemma 6.1.9, it follows that \(a_n + u \in \beta_{n-1} \subset L_{n-1}\). Hence \(a_n + b - a \in L_{n-1}\). Since \(a_n - a = -\bar{a} \in L_{n-1}\), it follows that \(b \in L_{n-1}\) which also contradicts Lemma 6.2.5. For the remaining two cases, contradictions can be similarly shown. Therefore either \(a_n = 0\) or \(b_n = 0\) or \(c_n = 0\).

\[\square\]

**Lemma 6.2.7.** Let \(n \in \mathbb{N}\), and \(a, b, c\) be three lattice points of \(P(n)\) that are mutually next to each other in the lattice \(L_n\), and let \(t\) be the centroid of the triangle with vertices \(a, b, c\). Then \(t \in P(n + 1)\).

**Proof.** Because \(a, b, c\) are mutually next to each other in the lattice \(L_n\), there exist \(u, v \in \beta_n\) that are next to each other in the lattice \(L_n\) such that \(b = a + u\) and \(c = a + v\). By Lemma 6.2.6, we can assume that \(a \in P(n - 1)\). Since \(t\) is the centroid of the triangle with vertices \(a, b, c\), we have \(t = \frac{1}{3}(a + b + c) = a + \frac{1}{3}u + \frac{1}{3}v\). Let \(\bar{u} = \frac{1}{3}u\) and \(\bar{v} = \frac{1}{3}v\). Since \(u, v \in \beta_n\) and they are next to each other in the lattice \(L_n\), we have \(\bar{u}, \bar{v} \in \beta_{n+2}\) and are next to each other in the lattice \(L_{n+2}\). By Lemma 6.1.9, it follows that \(\bar{u} + \bar{v} \in \beta_{n+1}\). Thus \(t = a + (\bar{u} + \bar{v}) \in P(n - 1) + \beta_{n+1} \subset P(n + 1)\).

\[\square\]

**Lemma 6.2.8.** Let \(n \in \mathbb{N}\) and let \(a, b, c\) be three lattice points of \(L_{2n-1}\) that are mutually next to each other in the lattice \(L_{2n-1}\). If \(t = \frac{1}{3}(a + b + c)\) and \(a \not\in P(2n - 1)\), then \(t \not\in P(2n + 1)\).

**Proof.** Since \(a, b, c \in L_{2n-1}\), we have \(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c \in L_{2n+1}\). It follows that \(t \in L_{2n+1}\).

Because \(a, b, c \in L_{2n-1}\) and they are mutually next to each other in the lattice \(L_{2n-1}\), by Lemma 6.1.10, there exist \(u, v \in \beta_{2n-1}\) such that \(b = a + u\), \(c = a + v\), and \(u\) and \(v\) are next to each other in the lattice \(L_{2n-1}\). Since \(t = \frac{1}{3}(a + b + c)\), similar to the proof of Lemma 6.2.7, we have \(t = a + w\) with \(w = \frac{1}{3}(u + v) \in \beta_{2n}\). Suppose that \(t \in P(2n + 1)\). Let the label of \(t\) be \(T = t_1t_2...t_{2n+1} \in \Lambda_{2n+1}\). Then we consider the following two cases about \(t_{2n+1}\).
Case 1: If \( t_{2n+1} \neq 0 \), then \( t_{2n} = 0 \). Let \( \bar{t} \) and \( t_{2n+1} \) be two elements of \( P(2n + 1) \) such that their labels are \( t_1 t_2 \ldots t_{2n-1} 00 \in \lambda_{2n+1} \) and \( 00 \ldots 0 t_{2n+1} \in \Lambda_{2n+1} \), respectively. Then \( t = \bar{t} + t_{2n+1} \). Also from the previous paragraph, we have \( t = a + w \). Hence \( t_{2n+1} = a + w - \bar{t} \). Since \( a \in L_{2n-1} \subset L_{2n} \), \( w \in \beta_2 \subset L_{2n} \), and \( \bar{t} \in L_{2n-1} \subset L_{2n} \), it follows that \( t_{2n+1} \in L_{2n} \) which contradicts Lemma 6.2.5.

Case 2: If \( t_{2n+1} = 0 \), then \( t \in P(2n) \). We claim that \( t_{2n} \neq 0 \). Suppose \( t_{2n} = 0 \). It follows that \( t \in P(2n - 1) \subset L_{2n-1} \). Since \( a \in L_{2n-1} \) and \( w = t - a \), we have \( w \in L_{2n-1} \). Hence \( w \in L_{2n-1} \cap \beta_2 = \emptyset \) by Lemma 6.1.3, which is a contradiction. Thus \( t_{2n} \neq 0 \).

It follows that \( t_{2n-1} = 0 \). Let \( t_{1,2n-2} \in P(2n - 2) \) be the lattice point whose label is \( t_1 t_2 \ldots t_{2n-2} \in \Lambda_{2n-2} \) and \( t_{2n} \in P(2n) \) be the lattice point whose label is \( 00 \ldots 0 t_{2n} \in \Lambda_{2n} \). It follows that \( t = t_{1,2n-2} + t_{2n} \). Since \( t = a + w \), we have \( a = t - w = t_{1,2n-2} + t_{2n} - w \). If \( t_{2n} = w \), then \( a = t - w = t_{1,2n-2} \in P(2n - 2) \subset P(2n - 1) \) which contradicts the condition of this lemma that \( a \not\in P(2n - 1) \). Hence \( t_{2n} \neq w \). Since both \( t_{2n} \in \beta_2 \) and \( -w \in \beta_2 \), it follows from Table 6-2 that \( t_{2n} - w \in \beta_2 \cup \beta_{2n-1} \cup (\beta_{2n-2} + \beta_2) \).

If \( t_{2n} - w \in \beta_2 \), then \( t_{2n} - w = a - t_{1,2n-2} \in \beta_2 \cap L_{2n-1} = \emptyset \), which is a contradiction. If \( t_{2n} - w \in \beta_{2n-1} \), then \( a = t_{1,2n-2} + t_{2n} - w \in P(2n - 2) + \beta_{2n-1} \). By Lemma 6.2.3, \( P(2n - 2) + \beta_{2n-1} \subset P(2n - 1) \). Hence \( a \in P(2n - 1) \) which again contradicts the condition \( a \not\in P(2n - 1) \). If \( t_{2n} - w \in \beta_{2n-2} + \beta_2 \), then there are \( x \in \beta_{2n-2} \) and \( y \in \beta_2 \) such that \( t_{2n} - w = x + y \). Hence \( a = t_{1,2n-2} + t_{2n} - w = t_{1,2n-2} + x + y \). It follows that \( y = a - t_{1,2n-2} - x \in \beta_{2n} \cap L_{2n-1} = \emptyset \), which is a contradiction as well. Therefore \( t \not\in P(2n + 1) \).

Let \( \emptyset \neq T \subset L \), and let \( q \in T \) be a boundary lattice point of \( T \), and \( Q \) the Voronoi cell of \( q \). If \( S \) is a side of \( Q \) but \( S \) is not a side of any other Voronoi cell of \( T \), then \( S \) is called a \textit{B-side of} \( Q \) \textit{in} \( T \). The boundary lattice point \( q \) is called a \textit{C-boundary lattice point of} \( T \) if the union of all \textit{B}-sides of the Voronoi cell \( Q \) is a connected set of \( \mathbb{R}^2 \).

Lemma 6.2.9. For any integer \( n \geq 2 \) and any boundary lattice point \( q \) of \( P(2n - 1) \), \( q \) is a \textit{C-boundary lattice point of} \( P(2n - 1) \), and the boundary index of \( q \) in \( P(2n - 1) \) is either
1 or 3. Furthermore there exists at least one boundary lattice point of $P(2n - 1)$ whose boundary index is 1.

Proof. When $n = 2$, this lemma follows directly from the plot of the lattice points of $P(3)$ in Figure 6-2. Now assume this lemma is true for any $n \leq t$ for some integer $t$.

When $n = t + 1$, let $q$ be any boundary lattice point of $P(2t + 1)$. Since $P(2t + 1) = P(2t) \cup (P(2t - 1) + \beta_{2t+1})$, we have either $q \in P(2t)$ or $q \in P(2t - 1) + \beta_{2t+1}$. If $q \in P(2t)$, then by Corollary 6.2.4, $q$ is not a boundary lattice point of $P(2t + 1)$ which leads to a contradiction. Hence $q \in P(2t - 1) + \beta_{2t+1}$. Let $q = q_0 + r$ for some $q_0 \in P(2t - 1)$ and $r \in \beta_{2t+1}$.

Suppose $q_0$ is not a boundary lattice point of $P(2t - 1)$. Then the six neighbors of $q_0$ in the lattice $L_{2t-1}$ are also lattice points of $P(2t - 1)$. Let $\{q_i \in P(2t - 1) : i = 1, 2, ..., 6\}$ be the set consisting of the six neighbors of $q_0$ in the lattice $L_{2t-1}$ such that $q_i$ is next to $q_{i+1}$ for each $i = 1, 2, 3, 4, 5$. Also the Voronoi cell of $q_i$ is denoted $Q_i$ for each $i = 1, 2, ..., 6$. Since $q_0$, $q_1$, and $q_2$ are lattice points of $P(2t - 1)$ and mutually next to each other, by Lemma 6.2.7, the centroid of the triangle whose vertices are $q_0$, $q_1$, and $q_2$ is in $P(2t + 1)$. It follows that the hexagon $A$ in Figure 6-3 is a Voronoi cell of $P(2t + 1)$. Similarly the hexagons $B$, $C$, $D$, $E$, and $F$ are Voronoi cells of $P(2t + 1)$. Each green hexagon except $A$, $B$, $C$, $D$, $E$, and $F$ is the Voronoi cell of a lattice point in the set $\bigcup \{q_i + \beta_{2t+1} : i = 1, 2, ..., 6\}$. Since $q_i + \beta_{2t+1} \subset P(2t - 1) + \beta_{2t+1} \subset P(2t + 1)$, each green hexagon in Figure 6-3 is a Voronoi cell of $P(2t + 1)$ which contradicts that $q$ is a boundary lattice point of $P(2t + 1)$. Thus $q_0$ is a boundary lattice point of $P(2t - 1)$.

By the assumption of this induction, $q_0$ is a $C$-boundary lattice point of $P(2t - 1)$ and the boundary index of $q_0$ (in $P(2t - 1)$) is either 1 or 3. In the following, we assume that the boundary index of $q_0$ is 1 because the proof can be similarly done if the boundary index of $q_0$ is 3. It follows that the Voronoi cell $Q_0$ of $q_0$ is surrounded by five hexagons of $P(2t - 1)$ as shown in Figure 6-4 (a). Since $q_0$ and its five neighbors in the lattice $L_{2t-1}$ are lattice points of $P(2t - 1)$ and since $P(2t - 1) + \beta_{2t+1} \subset P(2t + 1)$, by Lemma 6.2.7, all
Figure 6-3. Voronoi cells of the lattice points of $P(2t+1)$ which are generated from the lattice points of $P(2t-1)$.

Figure 6-4. The boundary hexagons of $Q_0$ and $Q$. (a) The black hexagon $Q_0$ has boundary index 1. (b) The boundary hexagon $Q$ should be one of the three red hexagons.

green and red hexagons in Figure 6-4 (b) are Voronoi cells of $P(2t+1)$. Because $q = q_0 + r$ for some $r \in \beta_{2t+1}$ and because $q$ is a boundary lattice point of $P(2t + 1)$, the Voronoi cell $Q$ of $q$ should be one of the three red hexagons in Figure 6-4 (b). By Lemma 6.2.8, the hexagon $W$ in Figure 6-4 (b) is not a Voronoi cell of $P(2t + 1)$. Hence, in Figure 6-4 (b), the hexagon $C$ has boundary index 1. For a similar reason, we can show that the hexagon $A$ has boundary index 1, and the hexagon $B$ has boundary index 3 and the union of its
three $B$-sides is a connected set of $\mathbb{R}^2$. Hence this lemma is true for $n = t + 1$. Thus this lemma is true for any $n \geq 2$. 

Let $L$ be a lattice, $\emptyset \neq T \subset L$, and let $p$ and $q$ be two $C$-boundary lattice points of $T$. If the union of the $B$-sides of the Voronoi cells of $p$ and $q$ is a connected set of $\mathbb{R}^2$, then we say that $p$ and $q$ are $B$-connected in $T$.

**Lemma 6.2.10.** For any integer $n \geq 2$, there exists $w \in L_{2n-1} \setminus P(2n - 1)$ such that $w$ has at least four neighbors belonging to the set $P(2n - 1)$.

**Proof.** By Lemma 6.2.9, there exists one boundary lattice point $q_0$ of $P(2n - 3)$ whose boundary index is 1. Let $Q_0$ be the Voronoi cell of $q_0$. Let $W$ be the hexagon shown in Figure 6-4 (b). As shown in the proof of Lemma 6.2.9, $W$ is not a Voronoi cell of $P(2n - 1)$, and all green and red hexagons in Figure 6-4 (b) are Voronoi cells of $P(2n - 1)$. Let $w$ be a lattice point of $L_{2n-1}$ such that the hexagon $W$ is a Voronoi cell of $w$. Then $w \notin P(2n - 1)$ and $w$ has at least four neighbors belonging to the set $P(2n - 1)$. 

The main result in this subsection is the following theorem.

**Theorem 6.2.11.** For any integer $n \geq 2$, the $(2n - 1)^{th}$ level of the Pyxis structure does not tile its underlying lattice $L_{2n-1}$ by translations by a sublattice.

**Proof.** By Lemma 6.2.10, for any integer $n \geq 2$, there exists $w \in L_{2n-1} \setminus P(2n - 1)$ such that $w$ has at least four neighbors belonging to the set $P(2n - 1)$. Suppose that $P(2n - 1)$ tiles $L_{2n-1}$ by translations by a sublattice $L_s$. Then there is $r \in L_s$ such that $w \in r + P(2n - 1)$. Since $w \not\in P(2n - 1)$, by the definition of tiling, we have $P(2n - 1) \cap (r + P(2n - 1)) = \emptyset$. Hence the boundary index of $w$ in $r + P(2n - 1)$ is at least 4 because $w$ has at least four neighbors belonging to $P(2n - 1)$. By Lemma 6.2.1, $r + P(2n - 1)$ has the same geometry as $P(2n - 1)$. Hence by Lemma 6.2.9, the boundary index of $w$ in $r + P(2n - 1)$ is either 1 or 3. However we have shown that the boundary index of $w$ in $r + P(2n - 1)$ is at least 4, which is a contradiction. Therefore $P(2n - 1)$
Lemma 6.2.13. It follows that

Lemma 6.2.12. \( \geq P \)

By the definition of the Pyxis structure, we have

For any \( n \geq 1 \), \( \lambda_1 \lambda_2 \ldots \lambda_n \in \Lambda_{2n} \), and \( x \in \{1, 2, 3, 4, 5, 6\} \), we have

\[ \lambda_1 \lambda_2 \ldots \lambda_n 0 \oplus 00 \ldots 0 x \in \Lambda_{2n+1} \]
Figure 6-5. The 13 lattice points in the set \((q + \beta_{2n}) \cup (q + \beta_{2n-1})\) and their Voronoi cells.

**Proof.** Let \(p \in P(2n)\) such that the label of \(p\) is \(\lambda_1 \lambda_2 \ldots \lambda_{2n} \in \Lambda_{2n}\), and let \(b \in \beta_{2n+1}\) such that the label of \(b\) is \(00\ldots0x \in \Lambda_{2n+1}\). By Lemma 6.2.3, we have \(p + b \in P(2n + 1)\). It follows that \(\lambda_1 \lambda_2 \ldots \lambda_{2n} 0 \oplus 00\ldots0x \in \Lambda_{2n+1}\).

**Lemma 6.2.14.** Let \(1 < n \in \mathbb{N}\). If \(q\) is a boundary lattice point of \(P(2n)\) with label \(q_1 q_2 \ldots q_{2n-2} q_{2n-1} q_{2n} \in \Lambda_{2n}\), then either \(q_{2n-1} \neq 0\) or \(q_{2n} \neq 0\), and the lattice point labeled \(q_1 q_2 \ldots q_{2n-2}\) is a boundary lattice point of \(P(2n - 2)\).

**Proof.** If \(q_{2n-1} = q_{2n} = 0\), then \(q \in P(2n - 2)\). Obviously the six neighbors of \(q\) in \(P(2n)\) are exactly the elements of the set \(q + \beta_{2n}\). By Lemma 6.2.12, \(P(2n) = P(2n - 2) \cup (P(2n - 2) + \beta_{2n})\). It follows that \(q + \beta_{2n} \subseteq P(2n - 2) + \beta_{2n} \subseteq P(2n)\). Hence all neighbors of \(q\) in \(P(2n)\) are still in \(P(2n)\), which contradicts that \(q\) is a boundary lattice point of \(P(2n)\). Thus either \(q_{2n-1} \neq 0\) or \(q_{2n} \neq 0\).

Let \(\bar{q}\) be the lattice point of \(P(2n - 2)\) labeled \(q_1 q_2 \ldots q_{2n-2}\). To show \(\bar{q}\) is a boundary lattice point of \(P(2n - 2)\), it suffices to show that there exists \(b \in \beta_{2n-2}\) such that \(\bar{q} + b \not\in P(2n - 2)\), i.e., \(q_1 q_2 \ldots q_{2n-3} q_{2n-2} \oplus 00\ldots0j \not\in \Lambda_{2n-2}\) for some \(j \in \{1, 2, 3, 4, 5, 6\}\), where \(00\ldots0j \in \Lambda_{2n-2}\) is the label of \(b\). Since \(q\) is a boundary lattice point of \(P(2n)\) with label \(q_1 q_2 \ldots q_{2n-2} q_{2n-1} q_{2n} \in \Lambda_{2n}\), there exists \(k \in \{1, 2, 3, 4, 5, 6\}\) such that \(q_1 q_2 \ldots q_{2n-2} q_{2n-1} q_{2n} \oplus...
00...00k \not\in \Lambda_{2n}. By Table 6-2, we have 00q_{2n-1}q_{2n} \oplus 000k = 0d_2d_3d_4 \in \Lambda_4. Hence we have

\begin{align*}
q_1q_2\ldots q_{2n-3}q_{2n-2}q_{2n-1}q_{2n} & \oplus 00...000k \\
= q_1q_2\ldots q_{2n-3}q_{2n-2}000 & \oplus 00...000k \\
= q_1q_2\ldots q_{2n-3}q_{2n-2}00 & \oplus 00...0d_2d_3d_4.
\end{align*}

(6–23)

Since we have shown in the previous paragraph that either \(q_{2n-1} \neq 0\) or \(q_{2n} \neq 0\), we split the proof into two cases.

**Case 1:** If \(q_{2n-1} \neq 0\), then, by Item 1 of Corollary 6.1.7, we have \(q_{2n} = 0\).

Since \(q_{2n-1} \neq 0\) and \(q_{2n} = 0\), by Table 6-2, we have \(d_2 = 0\). Suppose \(d_2 = 0\).

Then, by Equation 6–23, we have

\[q_1q_2\ldots q_{2n-2}q_{2n-1}q_{2n} \oplus 00...00k = q_1q_2\ldots q_{2n-2}000 \oplus 00...0d_4 = q_1q_2\ldots q_{2n-2}0d_4 \in \Lambda_{2n},\]

which is a contradiction. Thus \(d_2 \neq 0\). Now suppose \(q_1q_2\ldots q_{2n-3}q_{2n-2} \oplus 00...0d_2 \in \Lambda_{2n-2}\). Let

\[y_1y_2\ldots y_{2n-3}y_{2n-2} = q_1q_2\ldots q_{2n-3}q_{2n-2} \oplus 00...0d_2.\]

Then Equation 6–23 becomes

\begin{align*}
q_1q_2\ldots q_{2n-3}q_{2n-2}q_{2n-1}q_{2n} & \oplus 00...000k \\
= q_1q_2\ldots q_{2n-3}q_{2n-2}000 & \oplus 00...0d_20d_4 \\
= q_1q_2\ldots q_{2n-3}q_{2n-2}00 & \oplus 00...0d_000 \oplus 00...000d_4 \\
= y_1y_2\ldots y_{2n-3}y_{2n-2}00 & \oplus 00...000d_4 \\
= y_1y_2\ldots y_{2n-3}y_{2n-2}0d_4 \in \Lambda_{2n},
\end{align*}

(6–24)

which contradicts that \(\mathbf{q}\) is a boundary lattice point of \(P(2n)\) with label \(q_1q_2\ldots q_{2n-2}q_{2n-1}q_{2n}\).

Thus \(q_1q_2\ldots q_{2n-3}q_{2n-2} \oplus 00...0d_2 \not\in \Lambda_{2n-2}\). Therefore the lattice point labeled \(q_1q_2\ldots q_{2n-2}\) is a boundary lattice point of \(P(2n - 2)\).
Case 2: If $q_{2n} \neq 0$, then, by Item 1 of Corollary 6.1.7, we have $q_{2n-1} = 0$. Suppose $d_2 = 0$. Then Equation 6–23 becomes

$$q_1q_2...q_{2n-3}q_{2n-2}q_{2n-1}q_{2n} \oplus 00...00k$$

$$= q_1q_2...q_{2n-3}q_{2n-2}00 \oplus 00...00d_3d_4$$

(6–25)

If $d_3 \neq 0$, then, by Item 1 of Corollary 6.1.7, we have $d_4 = 0$. Since $d_3 \neq 0$ and $d_4 = 0$, by Lemma 6.2.13, we have $q_1q_2...q_{2n-3}q_{2n-200} \oplus 00...00d_3d_4 \in \Lambda_{2n}$. By Equation 6–25, it follows that $q_1q_2...q_{2n-3}q_{2n-2}q_{2n-1}q_{2n} \oplus 00...00k \in \Lambda_{2n}$, which contradicts that $q$ is a boundary lattice point of $P(2n)$ with label $q_1q_2...q_{2n-2}q_{2n-1}q_{2n}$. If $d_3 = 0$, then $q_1q_2...q_{2n-3}q_{2n-200} \oplus 00...00d_3d_4 = q_1q_2...q_{2n-3}q_{2n-20}d_4 \in \Lambda_{2n}$, which also contradicts that $q$ is a boundary lattice point of $P(2n)$ with label $q_1q_2...q_{2n-2}q_{2n-1}q_{2n}$. Thus $d_2 \neq 0$, which implies that $d_3 = 0$. Since $q_1q_2...q_{2n-2}q_{2n-1}q_{2n} \oplus 00...00k \not\in \Lambda_{2n}$, by Equation 6–23, we have $q_1q_2...q_{2n-3}q_{2n-200} \oplus 00...00d_3d_4 \not\in \Lambda_{2n}$, i.e., $q_1q_2...q_{2n-3}q_{2n-200} \oplus 00...00d_4d_4 \not\in \Lambda_{2n}$. By Item 2 of Lemma 6.1.8, it follows that $q_1q_2...q_{2n-3}q_{2n-2}00...00d_2 \not\in \Lambda_{2n-2}$. Therefore the lattice point labeled $q_1q_2...q_{2n-2}$ is a boundary lattice point of $P(2n-2)$. \qed

The following lemma shows some invariant properties of $P(n)$ with respect to the rotation of $\mathbb{R}^2$ by an angle of $\frac{k\pi}{3}$ for some $k \in \mathbb{Z}$ about the origin.

Lemma 6.2.15. Let $n \in \mathbb{N}$, $x \in P(2n)$, and $\alpha = \frac{k\pi}{3}$ for some $k \in \mathbb{Z}$. If $y$ is the point in $\mathbb{R}^2$ obtained by rotating $x$ by an angle of $\alpha$ about the origin, then $y \in P(2n)$. Furthermore, $x$ is a boundary lattice point of $P(2n)$ with boundary index $b$ for some $b \in \{1, 2, 3, 4, 5, 6\}$ if and only if $y$ is a boundary lattice point of $P(2n)$ with boundary index $b$.

Proof. Let the label of $x$ be $x_1x_2...x_{2n-1}x_{2n} \in \Lambda_{2n}$. Then $x = \sum_{i=1}^{2n} \beta_{i,x_i}$. Let $M = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$. Since $M$ is the matrix of rotation by $\alpha = \frac{k\pi}{3}$, by the definition of $\beta_1$ and $\beta_2$, it is easy to show that, for any $j = 1, 2, ..., 6$, we have $M\beta_{1,j} \in \beta_1$ and $M\beta_{2,j} \in \beta_2$. For any $k \in \mathbb{N}$, since $\beta_{2k+1,j} = \frac{1}{3\pi} \beta_{1,j}$ and $\beta_{2k+2,j} = \frac{1}{3\pi} \beta_{2,j}$, we have $M\beta_{2k+1,j} = \frac{1}{3\pi} M\beta_{1,j} \in \beta_{2k+1}$ and $M\beta_{2k+2,j} = \frac{1}{3\pi} M\beta_{2,j} \in \beta_{2k+2}$. Obviously $M\beta_{1,0} = M0 = 0 = \beta_{1,0}$ for
any \( i \in \mathbb{N} \). Thus \( M\bar{\beta}_i \subseteq \bar{\beta}_i \) for any \( i \in \mathbb{N} \). It follows that \( M\beta_{i,x_i} \in \bar{\beta}_i \) where \( \bar{\beta}_i = \beta_i \cup \{0\} \), and there exists \( y_i \in \{0,1,\ldots,6\} \) such that \( \beta_{i,y_i} = M\beta_{i,x_i} \) for each \( i \) satisfying \( 1 \leq i \leq 2n \).

Since \( \|M\beta_{i,x_i}\| = \|\beta_{i,x_i}\|, x_i \neq 0 \) if and only if \( y_i \neq 0 \). Since \( x_1x_2\ldots x_{2n-1}x_{2n} \in \Lambda_{2n} \), it follows that \( y_1y_2\ldots y_{2n-1}y_{2n} \in \Lambda_{2n} \). Because \( y \) is the point in \( \mathbb{R}^2 \) obtained by rotating \( x \) by an angle of \( \alpha \) about the origin, we have

\[
y = Mx = \sum_{i=1}^{2n} M\beta_{i,x_i} = \sum_{i=1}^{2n} \beta_{i,y_i}.
\]

(6–26)

Since either \( y_i = 0 \) or \( y_{i+1} = 0 \) for each \( i \) satisfying \( 1 \leq i \leq 2n - 1 \), by Lemma 6.1.5, it follows that \( y \in P(2n) \).

Furthermore, if \( x \) is a boundary lattice point of \( P(2n) \) with boundary index \( b \) for some \( b \in \{1,2,3,4,5,6\} \), then there exists \( B \subset \{1,2,3,4,5,6\} \) such that \( |B| = b \) and, for each \( j \in \{1,2,3,4,5,6\} \), \( x + \beta_{2n,j} \notin P(2n) \) if and only if \( j \in B \). By the result of the previous paragraph, we have \( x + \beta_{2n,j} \in P(2n) \) if and only if \( Mx + M\beta_{2n,j} \in P(2n) \).

Hence \( y \) is a boundary lattice point of \( P(2n) \) with boundary index \( b \). Conversely, if \( y \) is a boundary lattice point of \( P(2n) \) with boundary index \( b \) for some \( b \in \{1,2,3,4,5,6\} \), similarly we can show that \( x \) is also a boundary lattice point of \( P(2n) \) with boundary index \( b \) since \( x = M^{-1}y = \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{pmatrix} y \).

The following lemma shows that each boundary lattice point of \( P(2n) \) is a \( C \)-boundary lattice point, and its boundary index is 1, 2, or 4.

**Lemma 6.2.16.** Let \( q \) be a boundary lattice point of \( P(2n) \). If the label of \( q \) is \( \lambda 0j \) for some \( \lambda \in \Lambda_{2n-2} \) and \( j \in \{1,2,3,4,5,6\} \), then the boundary index of \( q \) is 1. If the label of \( q \) is \( \lambda j0 \) for some \( \lambda \in \Lambda_{2n-2} \) and \( j \in \{1,2,3,4,5,6\} \), then the boundary index of \( q \) is either 2 or 4, and \( q \) is a \( C \)-boundary lattice point of \( P(2n) \).
Proof. If the label of \( q \) is \( \lambda 0j \) for some \( \lambda = \lambda_1 \lambda_2 \ldots \lambda_{2n-2} \in \Lambda_{2n-2} \) and \( j \in \{1, 2, 3, 4, 5, 6\} \), then the labels of the six neighbors of \( q \) in the lattice \( L_{2n} \) constitute the set

\[
\{ \lambda_1 \lambda_2 \ldots \lambda_{2n-2} 0j \oplus 00 \ldots 00k : k = 1, 2, \ldots, 6 \}.
\]

For any \( k \in \{1, 2, 3, 4, 5, 6\} \) with \( k \neq j \), by Table 6-2, we have \( 000j \oplus 000k = 00x_1x_2 \in \Lambda_4 \) where \( x_1x_2 \in \Lambda_2 \). By applying Item 4 of Lemma 6.1.8 twice, it follows that \( 0j \oplus 0k = x_1x_2 \in \Lambda_2 \). Hence we have

\[
\lambda_1 \lambda_2 \ldots \lambda_{2n-2} 0j \oplus 00 \ldots 00k = \lambda_1 \lambda_2 \ldots \lambda_{2n-2} 00 \oplus 00 \ldots 0x_1x_2.
\]  (6–27)

When \( x_1 = 0 \), Equation 6–27 becomes

\[
\lambda_1 \lambda_2 \ldots \lambda_{2n-2} 0j \oplus 00 \ldots 00k = \lambda_1 \lambda_2 \ldots \lambda_{2n-2} 00 \oplus 00 \ldots 0x_2
\]

\[
= \lambda_1 \lambda_2 \ldots \lambda_{2n-2} 0x_2 \in \Lambda_{2n}.
\]  (6–28)

When \( x_2 = 0 \), Equation 6–27 becomes

\[
\lambda_1 \lambda_2 \ldots \lambda_{2n-2} 0j \oplus 00 \ldots 00k = \lambda_1 \lambda_2 \ldots \lambda_{2n-2} 00 \oplus 00 \ldots 0x_10.
\]  (6–29)

By Lemma 6.2.13, \( \lambda_1 \lambda_2 \ldots \lambda_{2n-2} 0 \oplus 00 \ldots 0x_1 \in \Lambda_{2n-1} \). It follows that \( \lambda_1 \lambda_2 \ldots \lambda_{2n-2} 00 \oplus 00 \ldots 0x_10 \in \Lambda_{2n} \). Hence by Equation 6–29, we have \( \lambda_1 \lambda_2 \ldots \lambda_{2n-2} 0j \oplus 00 \ldots 00k \in \Lambda_{2n} \). Thus at least five neighbors of \( q \) in the lattice \( L_{2n} \) are still in \( P(2n) \). Therefore the boundary index of \( q \) is 1.

If the label of \( q \) is \( \lambda j 0 \) for some \( \lambda = \lambda_1 \lambda_2 \ldots \lambda_{2n-2} \in \Lambda_{2n-2} \) and \( j \in \{1, 2, 3, 4, 5, 6\} \), then the labels of the six neighbors of \( q \) in the lattice \( L_{2n} \) constitute the set

\[
\{ \lambda_1 \lambda_2 \ldots \lambda_{2n-2} j 0 \oplus 00 \ldots 00k : k = 1, 2, \ldots, 6 \}.
\]

For \( k = 1, 2, \ldots, 6 \), denote \( n_k \) the neighbor of \( q \) whose label is \( \lambda_1 \lambda_2 \ldots \lambda_{2n-2} j 0 \oplus 00 \ldots 00k \). Without loss of generality we assume that \( j = 1 \) because, by Lemma 6.2.15, the boundary index would be the same if \( j \in \{2, 3, 4, 5, 6\} \). If \( k = 3 \), by Table 6-2, we have \( 00j0 \oplus 000k = \)
0010 \oplus 0003 = 0001. Then we have
\[
\lambda_1 \lambda_2 \ldots \lambda_{2n-2} j0 \oplus 00\ldots00k = \lambda_1 \lambda_2 \ldots \lambda_{2n-2} 00 \oplus 00\ldots001
\]
\[
= \lambda_1 \lambda_2 \ldots \lambda_{2n-2} 01 \in \Lambda_{2n}.
\]

It follows that \( n_3 \in P(2n) \). Similarly \( 00 j0 \oplus 0004 = 0006 \) implies that \( \lambda_1 \lambda_2 \ldots \lambda_{2n-2} j0 \oplus 00\ldots004 \in \Lambda_{2n} \). Thus \( n_4 \in P(2n) \). Now consider \( n_1 \) and \( n_2 \). By Table 6-2, \( 00 j0 \oplus 0001 = 0010 \oplus 0001 = 0105 \) and \( 00 j0 \oplus 0002 = 0010 \oplus 0002 = 0104 \). It follows that \( n_1 \not\in P(2n) \) if and only if \( \lambda_1 \lambda_2 \ldots \lambda_{2n-2} j0 \oplus 00\ldots001 \not\in \Lambda_{2n} \) if and only if \( \lambda_1 \lambda_2 \ldots \lambda_{2n-3} \lambda_{2n-2} 00 \oplus 00\ldots0105 \not\in \Lambda_{2n} \) if and only if \( \lambda_1 \lambda_2 \ldots \lambda_{2n-3} \lambda_{2n-2} 00 \oplus 00\ldots0104 \not\in \Lambda_{2n} \) if and only if \( n_2 \not\in P(2n) \). Similarly we can show that \( n_5 \not\in P(2n) \) if and only if \( n_6 \not\in P(2n) \). Thus the boundary index of \( q \) is either 2 or 4. Obviously \( n_5 \) is next to \( n_6 \), \( n_6 \) is next to \( n_1 \), and \( n_1 \) is next to \( n_2 \). It follows that \( q \) is a \( \mathcal{C} \)-boundary lattice point of \( P(2n) \).

**Lemma 6.2.17.** Let \( 1 \leq n \in \mathbb{N} \) and let \( q \) be a boundary lattice point of \( P(2n) \). If the boundary index of \( q \) is 1, then there exist two boundary lattice points \( c_1 \) and \( c_2 \) of \( P(2n) \) such that, for each \( j = 1, 2, \) \( c_j \) is \( \mathcal{B} \)-connected to \( q \) in \( P(2n) \) and the boundary index of \( c_j \) is either 2 or 4. If the boundary index of \( q \) is 2 or 4, then there exist two boundary lattice points \( c_1 \) and \( c_2 \) of \( P(2n) \) such that, for each \( j = 1, 2, \) \( c_j \) is \( \mathcal{B} \)-connected to \( q \) in \( P(2n) \) and the boundary index of \( c_j \) is 1.

**Proof.** We prove it by induction. If \( n = 1 \), by the plot of the Voronoi cells of \( P(2) \) in Figure 6-2, this lemma is obviously true. Assume this lemma is true for all integers less than \( n \). Now let \( q \) be a boundary lattice point of \( P(2n) \) and let \( q_1 q_2 \ldots q_{2n-2} q_{2n-1} q_{2n} \in \Lambda_{2n} \) be the label of \( q \). By Lemma 6.2.14, we have \( q_{2n-1} q_{2n} = 0i \) or \( q_{2n-1} q_{2n} = i0 \) for some \( i \in \{1, 2, 3, 4, 5, 6\} \).

Let \( \lambda = q_1 q_2 \ldots q_{2n-2} \in \Lambda_{2n-2} \) and let \( \bar{q} \) be a lattice point of \( P(2n-2) \) such that the label of \( \bar{q} \) is \( \lambda \). Then the label of \( q \) is \( \lambda q_{2n-1} q_{2n} \). Since \( q \) is a boundary lattice point of \( P(2n) \), by Lemma 6.2.14, \( \bar{q} \) is a boundary lattice point of \( P(2n - 2) \). Furthermore, by
Lemma 6.2.12, we have \( P(2n) = P(2n - 2) \cup (P(2n - 2) + \beta_{2n-1}) \cup (P(2n - 2) + \beta_{2n}) \).

It follows that \( q + \beta_{2n} \subset P(2n) \), and \( q + \beta_{2n-1} \subset P(2n) \). For any \( i \in \{1, 2, 3, 4, 5, 6\} \), by Lemma 6.2.13, we have

\[
q_1q_2...q_{2n-2}0 \oplus 00...0i \in \Lambda_{2n-1}.
\]

(6–31)

Let \( \mu_i = q_1q_2...q_{2n-2}0 \oplus 00...0i \). When \( q \) is taken as an element of \( P(2n - 1) \), the label of \( q \) is \( q_1q_2...q_{2n-2}0 \). By Equation 6–31, it follows that \( q + \beta_{2n-1,i} \in P(2n - 1) \) and the label of \( q + \beta_{2n-1,i} \) is \( \mu_i \in \Lambda_{2n-1} \). Hence, if \( q + \beta_{2n-1,i} \) is taken as an element of \( P(2n) \), its label is \( \mu_i 0 \in \Lambda_{2n} \). When \( q \) is taken as an element of \( P(2n) \), the label of \( q \) is \( q_1q_2...q_{2n-2}00 \). It follows that the label of \( q + \beta_{2n,i} \) is \( q_1q_2...q_{2n-2}00 \oplus 00...0i \). By Item 2 of Lemma 6.1.8, we have \( q_1q_2...q_{2n-2}00 \oplus 00...0i = q_1q_2...q_{2n-2}0i \in \Lambda_{2n} \). Let \( \lambda = q_1q_2...q_{2n-2} \). Then the label of \( q + \beta_{2n,i} \) is \( \lambda 0i \). By Lemma 6.2.16, the boundary index of \( q \) is 1, 2, or 4, and \( q \) is a \( C \)-boundary lattice point of \( P(2n - 2) \). Without loss of generality, we assume that the boundary index of \( q \) is 4 since the proofs for the other two cases can be similarly done.

Then the Voronoi cell of \( q \) in the lattice \( L_{2n-2} \) and the Voronoi cells of lattice points of \( L_{2n} \) in the set \( (q + \beta_{2n}) \cup (q + \beta_{2n-1}) \) are shown in Figure 6-6, where the Voronoi cells of lattice points in the set \( q + \beta_{2n} \) are labeled \( \mu_i \) for \( i = 0, 1, 2, ..., 6 \), and the Voronoi cells of lattice points in the set \( q + \beta_{2n-1} \) are labeled \( \lambda 0i \) for \( i = 1, 2, ..., 6 \). By Lemma 6.2.16, the boundary index of \( q \) is 1, 2, or 4. Hence we consider the following cases.

If the boundary index of \( q \) is 1, then, by Lemma 6.2.16, we have \( q_{2n-1}q_{2n} = 0i \) for some \( i \in \{1, 2, ..., 6\} \). Hence the label of \( q \) is \( \lambda 0i \). As shown in Figure 6-6, there exist two boundary lattice points \( c_1 \) and \( c_2 \) of \( P(2n) \) such that, for each \( j = 1, 2 \), \( c_j \) is \( B \)-connected to \( q \) in \( P(2n) \) and the boundary index of \( c_j \) is either 2 or 4.

Similarly if the boundary index of \( q \) is 2 or 4, then by Lemma 6.2.16, we have \( q_{2n-1}q_{2n} = i0 \) for some \( i \in \{1, 2, ..., 6\} \). Since \( q_{2n-1} = i \neq 0 \), by Item 1 of Corollary 6.1.7, we have \( q_{2n-2} = 0 \). By Item 2 of Lemma 6.1.8, it follows that \( q_1q_2...q_{2n-2}q_{2n-1}q_{2n} = q_1q_2...q_{2n-2}00 \oplus 00...0i0 = \mu_i 0 \) where \( \mu_i \) is defined in the second paragraph of this proof. Hence the label of \( q \) is \( \mu_i 0 \) where \( i \in \{1, 2, 3, 4, 5, 6\} \). As shown in Figure 6-6, there
exist two boundary lattice points \( c_1 \) and \( c_2 \) of \( P(2n) \) such that, for each \( j = 1, 2 \), \( c_j \) is \( B \)-connected to \( q \) in \( P(2n) \) and the boundary index of \( c_j \) is 1. By induction, this lemma is true for any \( n \in \mathbb{N} \).

**Lemma 6.2.18.** For any \( n \) satisfying \( 2 \leq n \in \mathbb{N} \), there exists at least one \( C \)-boundary lattice point of \( P(2n) \) with boundary index 2.

**Proof.** For any \( i \in \mathbb{N} \), let \( \Gamma_i = 1010...10 \in \Lambda_{2i} \) and, for any \( j \in \{1, 2, 3, 4, 5, 6\} \), let \( \Theta_{i,j} = 00...0j \in \Lambda_i \) which is the label of \( \beta_{i,j} \) in \( P(i) \). We claim that \( \Gamma_i \oplus \Theta_{2i,1} \not\in \Lambda_{2i} \) for any \( i \in \mathbb{N} \). If \( i = 1 \), by Table 6-2, we have \( 0010 \oplus 0001 = 0105 \). By applying Item 4 of Lemma 6.1.8 twice, it follows that \( 10 \oplus 01 \not\in \Lambda_2 \). Hence \( \Gamma_1 \oplus \Theta_{2,1} \not\in \Lambda_2 \) and the claim is true for \( i = 1 \). Assume the claim is true for any \( i \leq N_0 \) for some \( N_0 \in \mathbb{N} \). In the following, we show that the claim is also true if \( i = N_0 + 1 \). Obviously

\[
\Gamma_{N_0+1} \oplus \Theta_{2N_0+2,1} = 1010...1010 \oplus 0000...00001
\]
\[
= 1010...1000 \oplus 0000...0010 \oplus 0000...0001
\]
\[
= \Gamma_{N_0} \oplus 0000...0010 \oplus 0000...0001.
\]
Since \(0010 \oplus 0001 = 0105\), by applying Item 4 of Lemma 6.1.8 for \(2N_0 - 2\) times, we have \(0000...0010 \oplus 0000...0001 = 0000...0105 \in \Lambda_{2N_0+2}\). Hence Equation 6–32 becomes

\[\Gamma_{N_0+1} \oplus \Theta_{2N_0+2,1} = \Gamma_{N_0} \oplus 0000...0105. \quad (6–33)\]

By the assumption, we have \(\Gamma_{N_0} \oplus \Theta_{2N_0,1} \not\in \Lambda_{2N_0}\), i.e., \(\Gamma_{N_0} \oplus 0000...01 \not\in \Lambda_{2N_0}\). By Equation 6–33 and Item 2 of Lemma 6.1.8, it follows that \(\Gamma_{N_0+1} \oplus \Theta_{2N_0+2,1} \not\in \Lambda_{2N_0+2}\). Thus the claim is true for any \(i \in \mathbb{N}\).

Now we prove Lemma 6.2.18. If \(n = 2\), by the plot of Voronoi cells of \(P(4)\) as shown in Figure 6-2, this lemma is obviously true. If \(n > 2\), let \(Q = 1010...1030 \in \Lambda_{2n}\) and let \(q\) be a lattice point of \(P(2n)\) whose label is \(Q\). By the definition of the lattice \(L_{2n}\) and the definition of \(\beta_{2n}\), the six neighbors of \(q\) in the lattice \(L_{2n}\) are exactly the elements of the set \(q + \beta_{2n}\). By the definition of labels, we have \(q + \beta_{2n,k} \in P(2n)\) if and only if \(Q \oplus \Theta_{2n,k} \in \Lambda_{2n}\). By Item 1 of Lemma 6.1.8, we have \(Q = 1010...100000 \oplus 0000...001030 = \Gamma_{n-2}0000 \oplus 0000...001030\). Hence

\[Q \oplus \Theta_{2n,k} = \Gamma_{n-2}0000 \oplus 0000...01030 \oplus 0000...0000k \quad (6–34)\]

In the following, we show that \(Q \oplus \Theta_{2n,k} \not\in \Lambda_{2n}\) for \(k = 1, 2\), and \(Q \oplus \Theta_{2n,k} \in \Lambda_{2n}\) for \(k = 3, 4, 5, 6\).

By Table 6-2, we have \(0030 \oplus 0001 = 0205\). Hence \(000030 \oplus 000001 = 000205\). Since \(001030 = 001000 \oplus 000030\), it follows that

\[001030 \oplus 000001 = 001000 \oplus 000030 \oplus 000001 = 001000 \oplus 000205 \quad (6–35)\]

\[= 001000 \oplus 000200 \oplus 000005.\]

By Table 6-2, we have \(0010 \oplus 0002 = 0104\). It follows that \(001000 \oplus 000200 = 010400\). Then Equation 6–35 becomes \(001030 \oplus 000001 = 010400 \oplus 000005 = 010405\). By Equation 6–34,
we have

\[ Q \oplus \Theta_{2n,1} = \Gamma_{n-2}0000 \oplus 0000...010405 \]
\[ = \Gamma_{n-2}0000 \oplus 0000...010000 \oplus 0000...000405 \]
\[ = \Gamma_{n-2}0000 \oplus \Theta_{2n-4,1}0000 \oplus 0000...000405. \quad (6-36) \]

By the claim in the first paragraph, we have \( \Gamma_{n-2} \oplus \Theta_{2n-4,1} \not\in \Lambda_{2(n-2)}. \) By Item 2 of Lemma 6.1.8 and Equation 6–36, it follows that \( Q \oplus \Theta_{2n,1} \not\in \Lambda_{2n}. \) Similarly we can show that \( Q \oplus \Theta_{2n,2} \not\in \Lambda_{2n}. \)

By Table 6-2, we have \( 0030 \oplus 0003 = 0301. \) Since \( 1030 \oplus 0003 = 1000 \oplus 0301 \oplus 0003, \) it follows that \( 1030 \oplus 0003 = 1000 \oplus 0101 \in \Lambda_4. \) By Equation 6–34, it follows that \( Q \oplus \Theta_{2n,3} = \Gamma_{n-2}0000 \oplus 0000...001030 \oplus 0000...000003 = \Gamma_{n-2}0301 \in \Lambda_{2n}. \) Similarly we can show that \( Q \oplus \Theta_{2n,k} \in \Lambda_{2n} \) for \( k = 4, 5, 6. \)

We have shown that \( Q \oplus \Theta_{2n,k} \not\in \Lambda_{2n} \) for \( k = 1, 2, \) and \( Q \oplus \Theta_{2n,k} \in \Lambda_{2n} \) for \( k = 3, 4, 5, 6. \)

By the definition of the sum of two labels, it follows that \( q + \beta_{2n,k} \not\in P(2n) \) for \( k = 1, 2, \) and \( q + \beta_{2n,k} \in P(2n) \) for \( k = 3, 4, 5, 6. \) Thus \( q \) is a boundary lattice point of \( P(2n) \) and the boundary index of \( P(2n) \) is 2. Furthermore, since \( q + \beta_{2n,1} \) is next to \( q + \beta_{2n,2} \) in the lattice \( L_{2n}, \) \( q \) is a \( C \)-boundary lattice point of \( P(2n). \)

In the following, we use the previous results to prove the main result of this subsection.

**Theorem 6.2.19.** For any integer \( n \geq 2, \) the \((2n)^{th}\) level of the Pyxis structure \( P(2n) \) does not tile its underlying lattice \( L_{2n} \) by translations by a sublattice.

**Proof.** By Lemma 6.2.18, there exists a \( C \)-boundary lattice point, denoted \( a, \) of \( P(2n) \) with boundary index 2. By Lemma 6.2.17, there exist two boundary lattice points \( b_j \) of \( P(2n) \) for \( j = 1, 2 \) such that \( a \) is \( B \)-connected to \( b_j \) in \( P(2n) \) and \( b_j \) has boundary index 1 for each \( j = 1, 2. \) Let \( b_3 = a \) and let \( B_j \) be the Voronoi cell of \( b_j \) for \( j = 1, 2, 3. \) Because \( b_3 \) has boundary index 2 and is \( B \)-connected to \( b_1 \) and \( b_2 \) in \( P(2n), \) we can assume that
the Voronoi cells $B_j$, $B_j$, and $B_j$ are shown in Figure 6-7 (a). Also, for $k = 1, 2$, let $u_k \in L_{2n} \setminus P(2n)$ be a lattice point such that $u_k$ is next to $b_k$ in the lattice $L_{2n}$, and let $U_k$ be the Voronoi cell of $u_k$. Since $u_k \in L_{2n} \setminus P(2n)$ and is next to $b_k$ for $k = 1, 2$, the location of $U_k$ is as shown in Figure 6-7 (a) for $k = 1, 2$.

Suppose that $P(2n)$ tiles $L_{2n}$ by translations by a sublattice $L_s$. Then

$$\bigcup \{u + P(2n) : u \in L_s\} = L_{2n}$$

and $u + P(2n) = v + P(2n)$ whenever $(u + P(2n)) \cap (v + P(2n)) \neq \emptyset$ for $u, v \in L_s$.

Since $u_k \in L_{2n}$, it follows that there exists $r_k \in L_s$ such that $u_k \in r_k + P(2n)$ for each $k = 1, 2$, and either $r_1 + P(2n) = r_2 + P(2n)$ or $(r_1 + P(2n)) \cap (r_2 + P(2n)) = \emptyset$. Since $u_k \in r_k + P(2n)$ and $u_k \not\in P(2n)$, we have $r_k + P(2n) \neq 0 + P(2n) = P(2n)$. Hence $P(2n) \cap (r_k + P(2n)) = \emptyset$ for $k = 1, 2$. Because either $r_1 + P(2n) = r_2 + P(2n)$ or $(r_1 + P(2n)) \cap (r_2 + P(2n)) = \emptyset$, we consider the following two cases.

**Case 1:** $r_1 + P(2n) = r_2 + P(2n)$. In this case, we have $u_k \in r_1 + P(2n)$ for each $k = 1, 2$. It follows that $u_1$ and $u_2$ are two boundary lattice points of $r_1 + P(2n)$ and next to each other in the lattice $L_{2n}$. Since there are two lattice points $b_1$ and $b_3$ which are next to $u_1$ in the lattice $L_{2n}$ but not in $r_1 + P(2n)$, the boundary index of $u_1$ in $r_1 + P(2n)$ is at least 2 as shown in Figure 6-7 (a). For a similar reason, the boundary index of $u_2$
in \( r_1 + P(2n) \) is at least 2 as shown in Figure 6-7 (a). However, by Lemma 6.2.17, \( u_1 \) is \( B \)-connected to two boundary boundary lattice points in \( r_1 + P(2n) \) each having boundary index 1. It follows that the boundary index of \( u_2 \) in \( r_1 + P(2n) \) should be 1, which contradicts that the boundary index of \( u_2 \) in \( r_1 + P(2n) \) is at least 2.

**Case 2:** \( (r_1 + P(2n)) \cap (r_2 + P(2n)) = \emptyset \). Let \( w_3 \) be a lattice point of \( L_{2n} \) which is different from \( b_3 \) such that its Voronoi cell \( W_3 \) is next to both \( U_1 \) and \( U_2 \) as shown in Figure 6-7 (b). Since \( (r_1 + P(2n)) \cap (r_2 + P(2n)) = \emptyset \) in this case, either \( w_3 \not\in r_1 + P(2n) \) or \( w_3 \not\in r_2 + P(2n) \). Because \( U_1 \) and \( U_2 \) in Figure 6-7 (b) are symmetric about \( W \), without loss of generality, we assume that \( w_3 \not\in r_2 + P(2n) \). We claim that the boundary index of \( u_2 \) (in \( r_2 + P(2n) \)) is 5. Since the boundary index of \( b_2 \) (in \( P(2n) \)) is 1, the only neighbor of \( b_2 \) which does not belong to \( P(2n) \) is \( u_2 \). It follows that the black hexagon sitting on the top of \( B_2 \) must be a Voronoi cell of \( P(2n) \). Hence \( u_2 \) has three neighbors in \( P(2n) \) which are \( b_3, b_2 \), and the one sitting on the top of \( b_2 \). Since \( (r_2 + P(2n)) \cap P(2n) = \emptyset \), these three neighbors are not in \( P(2n) \). Furthermore, because \( w_3 \not\in r_2 + P(2n) \) and \( u_1 \in (r_1 + P(2n)) \setminus (r_2 + P(2n)) \), \( w_3 \) and \( u_1 \) are also neighbors of \( u_2 \) which are not elements of \( r_2 + P(2n) \). Thus the boundary index of \( u_2 \) (in \( r_2 + P(2n) \)) is at least 5. However by Lemma 6.2.16, the boundary index of \( u_2 \) is less than or equal to 4, which leads to a contradiction. Thus \( P(2n) \) does not tile its underlying lattice \( L_{2n} \) by translations by a sublattice.
CHAPTER 7
FRACIAL DIMENSION OF THE BOUNDARY OF THE PYXIS STRUCTURE

7.1 The Limit of the Pyxis Structure

Let \((\mathbb{R}^2, d)\) be the 2-dimensional Euclidean space, where \(d(x, y)\) denotes the distance between \(x\) and \(y\) for any \(x, y \in \mathbb{R}^2\), and let \(\mathcal{H}(\mathbb{R}^2)\) be the set consisting of all nonempty compact subsets of \((\mathbb{R}^2, d)\). The next three definitions are from Barnsley [4]. For any \(x \in \mathbb{R}^2\) and \(B \in \mathcal{H}(\mathbb{R}^2)\), let \(d(x, B) = \min\{d(x, y) : y \in B\}\). The number \(d(x, B)\) is called the distance from the point \(x\) to the set \(B\). For any \(A, B \in \mathcal{H}(\mathbb{R}^2)\), define \(d(A, B) = \max\{d(x, B) : x \in A\}\). The number \(d(A, B)\) is called the distance from the set \(A \in \mathcal{H}(\mathbb{R}^2)\) to the set \(B \in \mathcal{H}(\mathbb{R}^2)\). Also let \(h(A, B) = \max\{d(A, B), d(B, A)\}\). The number \(h(A, B)\) is called the Hausdorff distance between \(A\) and \(B\).

By Theorem 1 on page 37 of Barnsley [4], \((\mathcal{H}(\mathbb{R}^2), h)\) is a complete metric space. Recall from Chapter 6 that

\[
\begin{align*}
v_1^A &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & v_2^A &= \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, & v_1^B &= \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, & v_2^B &= \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix},
\end{align*}
\]

\(L^A = \{n_1(v_1^A) + n_2(v_2^A) : n_1, n_2 \in \mathbb{Z}\}\), \(L^B = \{n_1(v_1^B) + n_2(v_2^B) : n_1, n_2 \in \mathbb{Z}\}\), \(\rho_1\) is a fixed positive number, and \(\rho_n = (\frac{1}{\sqrt{3}})^{n-1}\rho_1\) for any \(n\) satisfying \(1 < n \in \mathbb{N}\). Also \(v_{2n-1,k} = \rho_{2n-1}v_k^A\) and \(v_{2n,k} = \rho_{2n}v_k^B\) for any \(n\) satisfying \(1 < n \in \mathbb{N}\) and \(k \in \{1, 2\}\). We have defined a sequence of lattices by \(L_n = \{n_1v_{n,1} + n_2v_{n,2} : n_1, n_2 \in \mathbb{Z}\}\), \(n \in \mathbb{N}\). It is shown in Chapter 6 that \(P(n) \subset L_n\) and \(P(n) \subset P(n + 1)\). Now let \(s_n\) denote the length of one side of a Voronoi cell of the lattice \(L_n\) and let \(\tilde{P}_n\) be the union of all Voronoi cells of the lattice points of \(P(n)\). Since \(L_n\) is a hexagonal lattice with generators \(v_{n,1}\) and \(v_{n,2}\) each having length \(\rho_n\), we have \(s_n = \frac{\rho_n}{\sqrt{3}}\). The following lemma will be used to show that each of the two sequences, \(\{P(n) : n \in \mathbb{N}\}\) and \(\{\tilde{P}_n : n \in \mathbb{N}\}\), is a Cauchy sequence in the space \((\mathcal{H}(\mathbb{R}^2), h)\).

**Lemma 7.1.1.** For any \(n \in \mathbb{N}\), we have

1. \(h(P(n), P(n + 1)) \leq \rho_{n+1}\) and \(h(P(n), P(n + 2)) \leq \rho_{n+1}\);
2. \( h(\tilde{P}_n, \tilde{P}_{n+1}) \leq (2\sqrt{3}+1)\rho_{n+2}; \)

3. \( h(\tilde{P}_n, P(n)) \leq s_n = \frac{1}{\sqrt{3}}\rho_n. \)

**Proof.** 1. Because \( P(n) \subset P(n+1) \), the distance from \( P(n) \) to \( P(n+1) \) is 0, i.e.,
\( d(P(n), P(n+1)) = 0 \). For any \( x \in P(n+1) \), since \( P(n+1) = P(n) \cup (P(n-1) + \beta_{n+1}) \) and \( P(n-1) \subset P(n) \), there exist \( y \in P(n) \) and \( z \in \beta_{n+1} \) such that \( x = y + z \). It follows that \( d(x, P(n)) \leq d(x, y) = \|z\| = \rho_{n+1} \), where \( \|z\| \) denotes the norm of \( z \). Hence
\[
d(P(n+1), P(n)) = \max \{ d(x, P(n)) : x \in P(n+1) \} \leq \rho_{n+1}.
\]
Thus
\[
h(P(n), P(n+1)) = \max \{ d(P(n), P(n+1)), d(P(n+1), P(n)) \} \leq \rho_{n+1}.
\]
Similarly \( d(P(n), P(n+2)) = 0 \). For any \( x \in P(n+2) \), since \( P(n+2) = P(n+1) \cup (P(n) + \beta_{n+2}) \), we have either \( x \in P(n+1) \) or \( x \in P(n) + \beta_{n+2} \). If \( x \in P(n+1) \), then \( d(x, P(n)) \leq d(P(n+1), P(n)) \leq h(P(n+1), P(n)) \leq \rho_{n+1} \). If \( x \in P(n) + \beta_{n+2} \), then there exist \( y \in P(n) \) and \( z \in \beta_{n+2} \) such that \( x = y + z \). It follows that \( d(x, P(n)) \leq \|z\| = \rho_{n+2} \leq \rho_{n+1} \). Thus \( h(P(n), P(n+2)) \leq \rho_{n+1} \).

2. For any \( x \in \tilde{P}_n \), there exists \( y \in P(n) \) such that \( x \) lies in the Voronoi cell of \( y \). Since \( L_n \) is a hexagonal lattice and \( s_n \) is the length of one side of a Voronoi cell of the lattice \( L_n \), we have \( d(x, y) \leq s_n \). It follows that \( d(x, P(n)) \leq s_n \). Hence \( d(\tilde{P}_n, P(n)) \leq s_n \). Because \( P(n) \subset \tilde{P}_n \), we have \( d(P(n), \tilde{P}_n) = 0 \). Thus \( h(\tilde{P}_n, P(n)) \leq s_n \).

3. By Item 2, we have \( h(\tilde{P}_n, P(n)) \leq s_n \) for any \( n \in \mathbb{N} \). By the triangle inequality, we have
\[
h(\tilde{P}_n, \tilde{P}_{n+1}) \leq h(\tilde{P}_n, P(n)) + h(P(n), P(n+1)) + h(P(n+1), \tilde{P}_{n+1}) \leq s_n + \rho_{n+1} + s_{n+1}.
\]
(7–1)

Since \( s_n = \frac{1}{\sqrt{3}}\rho_n \) for any \( n \in \mathbb{N} \), we have
\[
s_n + \rho_{n+1} + s_{n+1} = \frac{1}{\sqrt{3}}\rho_n + \rho_{n+1} + \frac{1}{\sqrt{3}}\rho_{n+1} = 2\rho_{n+1} + \frac{1}{\sqrt{3}}\rho_{n+1} = (2\sqrt{3} + 1)\rho_{n+2}.
\]
\[\square\]
Proposition 7.1.2. Each of the two sequences, \( \{P(n) : n \in \mathbb{N}\} \) and \( \{\tilde{P}_n : n \in \mathbb{N}\} \), converges in \((\mathcal{H}(\mathbb{R}^2), h)\), and these two sequences have the same limit.

Proof. By Item 1 of Lemma 7.1.1, we have \( h(P(n), P(n + 1)) \leq \rho_{n+1} \) for any \( n \in \mathbb{N} \). Since \( \rho_n = (\frac{1}{\sqrt{3}})^n \rho_1 \) where \( \rho_1 \) is a fixed positive number, the sequence \( \{P(n) : n \in \mathbb{N}\} \) is a Cauchy sequence in the space \((\mathcal{H}(\mathbb{R}^2), h)\). Because \((\mathcal{H}(\mathbb{R}^2), h)\) is a complete metric space, it follows that the sequence \( \{P(n) : n \in \mathbb{N}\} \) converges in \( \mathcal{H}(\mathbb{R}^2) \).

Similarly, by Item 2 of Lemma 7.1.1, the sequence \( \{\tilde{P}_n : n \in \mathbb{N}\} \) is a Cauchy sequence in the space \((\mathcal{H}(\mathbb{R}^2), h)\). Therefore the sequence \( \{\tilde{P}_n : n \in \mathbb{N}\} \) also converges in \( \mathcal{H}(\mathbb{R}^2) \).

Let \( \mathcal{P} \) be the limit of the sequence \( \{P(n) : n \in \mathbb{N}\} \). For any \( \epsilon > 0 \), there exists \( n_1 \in \mathbb{N} \) such that \( h(P(n), \mathcal{P}) < \frac{1}{2} \epsilon \) for any \( n > n_1 \). By Item 3 of Lemma 7.1.1, we have \( h(\tilde{P}_n, P(n)) \leq \frac{1}{\sqrt{3}} \rho_n = (\frac{1}{\sqrt{3}})^n \rho_1 \) for any \( n \in \mathbb{N} \). It follows that there exists \( n_2 \in \mathbb{N} \) such that \( h(\tilde{P}_n, P(n)) < \frac{1}{2} \epsilon \) for any \( n > n_2 \). Let \( n_3 = \max\{n_1, n_2\} \). Then, for any \( n > n_3 \), we have \( h(\tilde{P}_n, \mathcal{P}) < h(\tilde{P}_n, P(n)) + h(P(n), \mathcal{P}) < \frac{1}{2} \epsilon + \frac{1}{2} \epsilon = \epsilon \). Thus \( \mathcal{P} \) is also the limit of the sequence \( \{\tilde{P}_n : n \in \mathbb{N}\} \). \( \blacksquare \)

Let \( \mathcal{P} \) be the limit of the sequence \( \{P(n) : n \in \mathbb{N}\} \) in \( \mathcal{H}(\mathbb{R}^2) \). We call \( \mathcal{P} \) the limit of the Pyxis structure. The boundary of \( \mathcal{P} \) is denoted \( \mathcal{P}_B \) and is called the boundary of the Pyxis structure.

7.2 Fractal Dimension of the Boundary of the Pyxis Structure

To make this research more self-contained, we review one definition and one theorem from Page 174 and Page 176 of Barnsley [4], respectively. Let \( A \in \mathcal{H}(\mathbb{R}^2) \). For each \( \epsilon > 0 \), let \( \mathcal{N}(A, \epsilon) \) denote the smallest number of closed balls of radius \( \epsilon > 0 \) needed to cover \( A \).

If \( D = \lim_{\epsilon \to 0} \frac{\ln(\mathcal{N}(A, \epsilon))}{\ln(1/\epsilon)} \) exists, then \( D \) is called the fractal dimension of \( A \).

Theorem 7.2.1. Let \( A \in \mathcal{H}(\mathbb{R}^2) \) and \( C > 0 \). If there exists some \( r \) with \( 0 < r < 1 \) such that \( \epsilon_n = Cr^n \) for any \( n \in \mathbb{N} \), and \( D = \lim_{n \to \infty} \frac{\ln(\mathcal{N}(A, \epsilon_n))}{\ln(1/\epsilon_n)} \) exists, then \( A \) has fractal dimension \( D \).

To determine the fractal dimension of \( \mathcal{P}_B \), in the following, we will show that \( 4^n + 2 \leq \mathcal{N}(\mathcal{P}_B, \epsilon_n) \leq 4^{n+1} - 4 \) for any \( n \in \mathbb{N} \), where \( \epsilon_n = \frac{3}{2} \rho_{2n+1} = \frac{3\rho_1}{2}(\frac{1}{3})^n \). For any \( n \in \mathbb{N} \)
and \( j = 1, 2, 4 \), let \( B_{n,j} \) be the set consisting of all boundary lattice points of \( P(2n) \) whose boundary index is \( j \) and let \( b_{n,j} \) be the size of the set \( B_{n,j} \). The following lemma shows one relation among these sizes.

**Lemma 7.2.2.** For any \( n \in \mathbb{N} \), we have \( b_{n,1} = b_{n,2} + b_{n,4} \).

**Proof.** Define a bipartite graph \( G \) with parts \( V_1 = B_{n,1} \) and \( V_2 = B_{n,2} \cup B_{n,4} \) such that \( e \) is an edge of this graph if and only if the two vertices which are end points of \( e \) are \( B \)-connected in \( P(2n) \). Let \( E(G) \) denote the edge set of the graph \( G \). By Lemma 6.2.17, each vertex in the set \( V_1 \) is \( B \)-connected to exactly two vertices in the set \( V_2 \). It follows that \( |E(G)| = 2|V_1| \). Also, by Lemma 6.2.17, each vertex in the set \( V_2 \) is \( B \)-connected to exactly two vertices in the set \( V_1 \). It follows that \( |E(G)| = 2|V_2| \). Thus \( |V_1| = |V_2| \) and hence \( b_{n,1} = b_{n,2} + b_{n,4} \). \( \square \)

![Figure 7-1. The boundary hexagons of \( P(2n) \) and \( P(2n+2) \). (a) The black hexagons \( Q, R, \) and \( T \) of \( P(2n) \) such that \( Q \) is \( B \)-connected to \( R \) and \( T \), the boundary index of \( Q \) is 4, and the boundary index of \( R \) and \( T \) is 1. (b) The boundary hexagons of \( P(2n+2) \) which have boundary index 2 or 4. Their Voronoi cells overlap the interior of the boundary hexagon \( Q \) of \( P(2n) \), where \( A_4, B_4, \) and \( C_4 \) have boundary index 4, and \( D_2 \) and \( E_2 \) have boundary index 2.](image)

**Lemma 7.2.3.** For any \( n \in \mathbb{N} \), the \((2n)^{th}\) level of the Pyxis structure has \( 4^n + 1 - 4 \) boundary hexagons. Furthermore, \( 4^n + 2 \) of them have boundary index 4, \( 4^n - 4 \) of them have boundary index 2, and \( 2(4^n) - 2 \) of them have boundary index 1.
Proof. By Lemma 7.2.2, we have the following Equation.

\[ b_{n,1} = b_{n,2} + b_{n,4}. \]  (7–2)

Hence it suffices to compute \( b_{n,2} \) and \( b_{n,4} \). Let \( Q \) be a boundary hexagon of \( P(2n) \). By Lemma 6.2.16, the boundary index of \( Q \) is 1, 2, or 4. If the boundary index of \( Q \) is 4, then it follows from Lemma 6.2.17 that, as shown in Figure 7-1 (a), there are exactly two boundary hexagons \( R \) and \( T \) of \( P(2n) \) such that each of \( R \) and \( T \) is \( B \)-connected to \( Q \) and has boundary index 1. By Lemma 6.2.12, it follows that, as shown in Figure 7-1 (b), there exist three boundary hexagons \( A_4 \), \( B_4 \), and \( C_4 \) of \( P(2n + 2) \) with boundary index 4 and there exist two boundary hexagons \( D_2 \) and \( E_2 \) of \( P(2n + 2) \) with boundary index 2 such that each of them has interior overlapping the interior of \( Q \). Similarly if the boundary index of \( Q \) is 2, then as shown in Figure 7-2 (a), there are exactly two boundary hexagons \( R \) and \( T \) of \( P(2n) \) such that each of \( R \) and \( T \) is \( B \)-connected to \( Q \) and has boundary index 1. By Lemma 6.2.12, it follows that there exists one boundary hexagon \( A_4 \) with boundary index 4 and there exist two boundary hexagons \( B_2 \) and \( C_2 \) of \( P(2n + 2) \) with boundary index 2.

Figure 7-2. The boundary hexagons of \( P(2n) \) and \( P(2n+2) \). (a) The black hexagons \( Q, R, \) and \( T \) of \( P(2n) \) such that \( Q \) is \( B \)-connected to \( R \) and \( T \), the boundary index of \( Q \) is 2, and the boundary index of \( R \) and \( T \) is 1. (b) The boundary hexagons of \( P(2n+2) \) which have boundary index 2 or 4, and whose Voronoi cells overlap the interior of the boundary hexagon \( Q \) of \( P(2n) \), where \( A_4 \) has boundary index 4, and \( B_2 \) and \( C_2 \) have boundary index 2.
index 2 such that each of them has interior overlapping the interior of $Q$ as shown in Figure 7-2 (b). If the boundary index of $Q$ is 1, then it follows from Lemma 6.2.17 that, as shown in Figure 7-3 (a), there are exactly two boundary hexagons $R$ and $T$ of $P(2n)$ such that each of $R$ and $T$ is $B$-connected to $Q$ and has boundary index 2 or 4. By Lemma 6.2.12, it follows that there exist two boundary hexagons $A$ and $B$ of $P(2n + 2)$ with boundary index 2 such that each of them has interior overlapping the interior of $Q$ as shown in Figure 7-3 (b). But boundary hexagons $A$ and $B$ are already counted from hexagons $R$ and $T$ which have boundary index 2 or 4. Therefore we have the following system of equations.

$$b_{n+1,4} = 3b_{n,4} + b_{n,2}$$
$$b_{n+1,2} = 2b_{n,4} + 2b_{n,2}. \quad (7-3)$$

Let $X_j = \begin{pmatrix} b_{j,4} \\ b_{j,2} \end{pmatrix}$ for any $j \in \mathbb{N}$, and let $M = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$. By System 7-3, $X_{n+1} = MX_n$ for any $n \in \mathbb{N}$. Therefore we have

$$X_{n+1} = M^n X_1. \quad (7-4)$$
Since there exist exactly six boundary hexagons of \( P(2) \) and each of them has boundary index 4, we have \( b_{1,4} = 6 \) and \( b_{1,2} = 0 \). Hence \( \mathbf{X}_1 = \begin{pmatrix} b_{1,4} \\ b_{1,2} \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \). The eigenvalues of the matrix \( \mathbf{M} \) are 1 and 4, and \( V_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \) and \( V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) are their associated eigenvectors. Let \( \mathbf{V} = (V_1 \ V_2) \) and \( \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \). Since \( \mathbf{M} \mathbf{V} = \mathbf{V} \mathbf{A} \), we have \( \mathbf{M} = \mathbf{V} \mathbf{A} \mathbf{V}^{-1} \). Thus \( \mathbf{M}^n = \mathbf{V} \mathbf{A}^n \mathbf{V}^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4^n \end{pmatrix} \left( \frac{1}{3} \right) \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} = \left( \frac{1}{3} \right) \begin{pmatrix} 1 + 2(4^n) & 4^n - 1 \\ 2(4^n) - 2 & 4^n + 2 \end{pmatrix} \). Then by Equation 7–4 we have

\[
\mathbf{X}_{n+1} = \left( \frac{1}{3} \right) \begin{pmatrix} 1 + 2(4^n) & 4^n - 1 \\ 2(4^n) - 2 & 4^n + 2 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 4^{n+1} + 2 \\ 4^{n+1} - 4 \end{pmatrix} \tag{7–5}
\]

It follows that \( b_{2n+1,4} = 4^{n+1} + 2 \) and \( b_{2n,2,2} = 4^{n+1} - 4 \). Hence \( b_{2n+2,1} = b_{2n,2,1} + b_{2n+2,4} = 2(4^{n+1}) - 2 \). Therefore \( b_{2n,4} = 4^{n} + 2 \), \( b_{2n,2} = 4^{n} - 4 \), and \( b_{2n,1} = 2(4^{n}) - 2 \). \( \square \)

The following lemma will be used in the proof of Lemma 7.2.7.

**Lemma 7.2.4.** \( \tilde{P}_{2n} \subseteq \tilde{P}_{2n+2} \) for any \( n \in \mathbb{N} \). Hence \( \{ \tilde{P}_{2n} : n \in \mathbb{N} \} \) is an increasing sequence of \( \mathcal{H}(\mathbb{R}^2) \) with respect to set inclusion.

**Proof.** By Lemma 6.2.12, we have

\[
P(2n + 2) = P(2n) \bigcup (P(2n) + \beta_{2n+1}) \bigcup (P(2n) + \beta_{2n+2})
\]

for any \( n \in \mathbb{N} \). It follows that, for any \( \mathbf{q} \in P(2n) \), we have \( \mathbf{q} + \beta_{2n+1} \bigcup \mathbf{q} + \beta_{2n+2} \subseteq P(2n + 2) \). As shown in Figure 6-5 of Chapter 6, the Voronoi cell of \( \mathbf{q} \in P(2n) \) is contained in the
union of the Voronoi cells of the lattice points of $L_{2n+2}$ in the set $q + \beta_{2n+1} \bigcup q + \bar{\beta}_{2n+2}$.

Thus $\tilde{P}_{2n} \subseteq \tilde{P}_{2n+2}$. 

To prove Lemma 7.2.7, we need the following lemma.

**Lemma 7.2.5.** For any $k \in \mathbb{N}$, we have $\tilde{P}_{2k} \subseteq P$.

*Proof.* By Proposition 7.1.2, $P$ is the limit of the sequence $\left\{ \tilde{P}_{2n} : n \in \mathbb{N} \right\}$ in the space $(\mathcal{H}(\mathbb{R}^2), h)$. By Theorem 1 on Page 37 through page 38 from Barnsley [4], we have $P = \left\{ x \in \mathbb{R}^2 : \text{there is a Cauchy sequence} \left\{ x_n \in \tilde{P}_{2n} \right\} \text{that converges to} \, x \right\}$. For any $x \in \tilde{P}_{2k}$, let $x_n = x$ for any $n \geq k$, and let $x_n = 0 \in \tilde{P}_{2n}$ for any $n < k$. By Lemma 7.2.4, $\left\{ \tilde{P}_{2n} : n \in \mathbb{N} \right\}$ is an increasing sequence of $\mathcal{H}(\mathbb{R}^2)$ with respect to set inclusion. Since $x \in \tilde{P}_{2k}$, it follows that $x_n \in \tilde{P}_{2n}$ for any $n \geq k$. Obviously the sequence $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence which converges to $x$. Hence $x \in P$. Therefore $\tilde{P}_{2k} \subseteq P$. \qed

**Lemma 7.2.6.** Let $L$ be a hexagonal lattice, $x \in \mathbb{R}^2$, $y \in L$, and let $N_y$ be the set consisting of the six neighbors of $y$ in the lattice $L$. If $x$ is not contained in the Voronoi cell of $y$, then $d(x, y) > d(x, N_y)$.

*Proof.* We consider the following two cases.

**Case 1:** If there exists $z \in N_y$ such that $x$ is contained in the Voronoi cell of $z$, then $d(x, z) < d(x, y)$ since $x$ is not contained in the Voronoi cell of $y$. Because $z \in N_y$, we have $d(x, z) \geq d(x, N_y)$. Hence $d(x, y) > d(x, z) \geq d(x, N_y)$.

**Case 2:** If there does not exist $z \in N_y$ such that $x$ is contained in the Voronoi cell of $z$, then let $N_y = \{a, b, c, d, e, f\}$ and let $H_N$ be the hexagon with vertices in the set $N_y$. Since $x$ is not contained in the Voronoi cell of any element in the set $N_y \cup \{y\}$, the lattice point $x$ lies outside the hexagon $H_N$. It follows that there is exactly one intersection point between the line segment connecting $x$ and $y$ and the sides of $H_N$. Let $p$ denote that intersection point.

If $p \in N_y$, then $d(x, y) > d(x, p)$ and $d(x, p) \geq d(x, N_y)$. Hence $d(x, y) > d(x, N_y)$. 

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Figure 7-4. The distance from a point $x$ to a given lattice point $y$ and the distance from $x$ to the six neighbors of $y$ in the lattice.

If $p \notin N_y$, then $p$ lies on exactly one side, say the side connecting the two vertices $a$ and $b$, of the hexagon $H_N$. By symmetry, without loss of generality, we assume that $d(x, a) \leq d(x, b)$ as shown in Figure 7-4. Consider the triangle with vertices $a$, $x$, and $y$. The three angles of that triangle with vertices $a$, $x$, and $y$ are denoted $A$, $X$, and $Y$, respectively. Since $d(x, a) \leq d(x, b)$, we have $Y \leq \frac{\pi}{6}$. However the angle $A$ is bigger than the angle formed by the ray from $a$ to $y$ and the ray from $a$ to $b$, which is $\frac{\pi}{3}$. Hence the angle $A$ is bigger than the angle $Y$. It follows that $d(x, y) > d(x, a)$. Since $d(x, a) \geq d(x, N_y)$, we have $d(x, y) > d(x, N_y)$. \qed

**Lemma 7.2.7.** For any $x \in P_B$, there exists a boundary lattice point $y$ of $P(2n)$ such that $d(x, y) \leq \frac{3}{2} \rho_{2n+1}$.

**Proof.** By Item 1 of Lemma 7.1.1, we have $h(P(i), P(i + 2)) \leq \rho_{i+1}$ for any $i \in \mathbb{N}$. Since $\mathcal{P}$ is the limit of the sequence $\{P(2i) : i \in \mathbb{N}\}$, we have $h(P(2n), \mathcal{P}) \leq \sum_{i=n}^{\infty} h(P(2i), P(2i + 2))$. Thus $h(P(2n), \mathcal{P}) \leq \sum_{i=n}^{\infty} \rho_{2i+1}$. If we let $j = i - n$, then it follows that $h(P(2n), \mathcal{P}) \leq \sum_{j=0}^{\infty} \rho_{2j+2n+1} = \sum_{j=0}^{\infty} \frac{1}{3} \rho_{2n+1} = \frac{3}{2} \rho_{2n+1}$. Hence $d(\mathcal{P}, P(2n)) \leq \frac{3}{2} \rho_{2n+1}$. Since $\mathcal{P}_B \subset \mathcal{P}$, it follows that $d(\mathcal{P}_B, P(2n)) \leq \frac{3}{2} \rho_{2n+1}$. For any $x \in \mathcal{P}_B$, we have $d(x, P(2n)) \leq d(\mathcal{P}_B, P(2n)) \leq \frac{3}{2} \rho_{2n+1}$. Because $P(2n)$ is a finite set, there exists $y \in P(2n)$ such that $d(x, P(2n)) = d(x, y)$. We claim that $y$ is a boundary lattice point of $P(2n)$. Let $N_y$
be the set consisting of the six neighbors of \( y \) in the lattice \( L_{2n} \). Suppose that \( y \) is not a boundary lattice point of \( P(2n) \). Since \( x \in P_B \) and since \( \tilde{P}_{2n} \subseteq P \) by Lemma 7.2.5, it follows that \( x \) lies outside the Voronoi cell of \( y \). Hence by Lemma 7.2.6, \( d(x, y) > d(x, N_y) \geq d(x, P(2n)) \), which contradicts that \( d(x, P(2n)) = d(x, y) \). Thus \( y \) is a boundary point of \( \tilde{P}_{2n} \) and \( d(x, y) = d(x, P(2n)) \leq \frac{3}{2} \rho_{2n+1} \). 

To get the fractal dimension of \( P_B \), we still need the following lemmas. For any \( n \in \mathbb{N} \) and \( x \in L_n \), let \( V_n(x) \) denote the Voronoi cell of \( x \).

**Lemma 7.2.8.** For any \( n \in \mathbb{N} \), we have \( \tilde{P}_{2n} \subseteq \tilde{P}_{2n-1} \).

**Proof.** For any \( x \in \tilde{P}_{2n} \), by the definition of \( \tilde{P}_{2n} \), there exists \( y \in P(2n) \) such that \( x \in V_{2n}(y) \). Since \( P(2n) = P(2n - 1) \cup (P(2n - 2) + \beta_{2n}) \), either \( y \in P(2n - 1) \) or \( y \in P(2n - 2) + \beta_{2n} \). In the following, we consider these two cases separately.

**Case 1:** If \( y \in P(2n - 1) \), then \( V_{2n}(y) \subset V_{2n-1}(y) \) as shown in Figure 7-5 (a) where the Voronoi cell \( V_{2n}(y) \) has a horizontal side and the Voronoi cell \( V_{2n-1}(y) \) has a vertical side. Since \( x \in V_{2n}(y) \), it follows that \( x \in V_{2n-1}(y) \subseteq \tilde{P}_{2n-1} \).

![Figure 7-5](image.png)

Figure 7-5. The containment relation among Voronoi cells of \( P(2n) \) and \( P(2n+1) \). (a) shows that the (blue) Voronoi cell \( V_{2n}(y) \) is contained in the (black) Voronoi cell \( V_{2n-1}(y) \), where \( V_{2n}(y) \) has a horizontal side and \( V_{2n-1}(y) \) has a vertical side. (b) shows that \( y \in P(2n) \) is the centroid of the triangle with vertices \( q \in P(2n - 1) \), \( r \in P(2n - 1) \), and \( t \in P(2n - 1) \). It also shows that the (blue) Voronoi cell of \( y \in L_{2n} \) is contained in the union of the (black) Voronoi cells of \( q \in L_{2n-1} \), \( r \in L_{2n-1} \), and \( t \in L_{2n-1} \).
Case 2: If \( y \in P(2n-2) + \beta_{2n} \), then there exists \( y_0 \in P(2n-2) \) and \( i \in \{1, 2, 3, 4, 5, 6\} \) such that \( y = y_0 + \beta_{2n,i} \).

If \( i \in \{1, 3, 5\} \), then let \( \tilde{q} = \beta_{2n,i} + \beta_{2n,2} \), \( \tilde{r} = \beta_{2n,i} + \beta_{2n,4} \), and \( \tilde{t} = \beta_{2n,i} + \beta_{2n,6} \). Assume \( i = 1 \) because the proof for \( i = 3 \) or \( i = 5 \) is completely similar to that for \( i = 1 \).

By Table 6-2, \( 0001 \oplus 0002 = 0020 \in \Lambda_4 \). By applying Item 4 of Lemma 6.1.8 for \( 2n - 4 \) times, it follows that

\[
00...0001 \oplus 00...0002 = 00...0020 \in \Lambda_{2n}. \quad (7-6)
\]

By the definition of \( \beta_{2n-1,2} \), the label of \( \beta_{2n-1,2} \) is \( 00...002 \in \Lambda_{2n-1} \). By Item 3 of Lemma 6.1.8, the label of \( \beta_{2n-1,2} \) in \( P(2n) \) is \( 00...0020 \in \Lambda_{2n} \). By Equation 7-6, it follows that \( \beta_{2n,1} + \beta_{2n,2} = \beta_{2n-1,2} \), i.e., \( \tilde{q} = \beta_{2n-1,2} \). Similarly, by Table 6-2, we have \( 0001 \oplus 0004 = 0000 \in \Lambda_4 \) and \( 0001 \oplus 0006 = 0010 \in \Lambda_4 \). Then we can show that \( \beta_{2n,1} + \beta_{2n,4} = 0 \) and \( \beta_{2n,1} + \beta_{2n,6} = \beta_{2n-1,1} \), i.e., \( \tilde{r} = 0 \) and \( \tilde{t} = \beta_{2n-1,1} \). Hence \( \tilde{q}, \tilde{r}, \tilde{t} \in \beta_{2n-1,1} \) and they are mutually next to each other in the lattice \( L_{2n-1} \). Since

\[
\tilde{q} + \tilde{r} + \tilde{t} = 3 \beta_{2n,i} + \beta_{2n,2} + \beta_{2n,4} + \beta_{2n,6}
\]

and since \( \beta_{2n,2} + \beta_{2n,4} + \beta_{2n,6} = 0 \), we have \( \beta_{2n,i} = \frac{1}{3} (\tilde{q} + \tilde{r} + \tilde{t}) \). Let \( q = y_0 + \tilde{q}, r = y_0 + \tilde{r}, \) and \( t = y_0 + \tilde{t} \). Then

\[
\frac{1}{3} (q + r + t) = y_0 + \frac{1}{3} (\tilde{q} + \tilde{r} + \tilde{t}) = y_0 + \beta_{2n,i} = y.
\]

It follows that \( y \) is the centroid of the triangle with vertices \( q, r, \) and \( t \). Since \( y_0 \in P(2n - 2) \) and \( \tilde{q}, \tilde{r}, \tilde{t} \in \beta_{2n-1,1} \), by Lemma 6.2.3, we have \( q, r, t \in P(2n - 1) \). Because \( \tilde{q}, \tilde{r}, \) and \( \tilde{t} \) are mutually next to each other in the lattice \( L_{2n-1} \), the three lattice points \( q, r, \) and \( t \) are also mutually next to each other in the lattice \( L_{2n-1} \). Hence \( y \) is the centroid of the triangle with vertices \( q \in L_{2n-1}, r \in L_{2n-1}, \) and \( t \in L_{2n-1} \) which are mutually next to each other in the lattice \( L_{2n-1} \). As shown in Figure 7-5 (b), it follows that \( V_{2n}(y) \) is contained in the union of \( V_{2n-1}(q), V_{2n-1}(r), \) and \( V_{2n-1}(t) \), where the Voronoi cells of the lattice \( L_{2n-1} \) have a vertical side and the Voronoi cells of the lattice \( L_{2n} \) have a horizontal side. Hence \( V_{2n}(y) \subseteq \tilde{P}_{2n-1} \). Since \( x \in V_{2n}(y) \), it follows that \( x \in \tilde{P}_{2n-1} \).
If $i \in \{2, 4, 6\}$, then let $\bar{q} = \beta_{2n,i} + \beta_{2n,1}$, $\bar{r} = \beta_{2n,i} + \beta_{2n,3}$, and $\bar{t} = \beta_{2n,i} + \beta_{2n,5}$. Similar to the proof in the previous paragraph, we can show that $x \in \bar{P}_{2n-1}$. Thus $\bar{P}_{2n} \subset \bar{P}_{2n-1}$.

\begin{lemma}
\label{lem:7.2.9}
For any $n \in \mathbb{N}$, we have $\bar{P}_{2n+1} \subset \bar{P}_{2n-1}$.
\end{lemma}

\begin{proof}
For any $x \in \bar{P}_{2n+1}$, there exists $y \in \bar{P}(2n + 1)$ such that $x \in V_{2n+1}(y)$. Since $P(2n + 1) = P(2n) \cup (P(2n - 1) + \beta_{2n+1})$ and $P(2n) = P(2n - 1) \cup (P(2n - 2) + \beta_{2n})$, we have $P(2n + 1) = P(2n - 1) \cup (P(2n - 2) + \beta_{2n}) \cup (P(2n - 1) + \beta_{2n+1})$. Hence either $y \in P(2n - 1)$ or $y \in P(2n - 2) + \beta_{2n}$ or $y \in P(2n - 1) + \beta_{2n+1}$.

\textbf{Case 1:} If $y \in P(2n - 1)$, then $V_{2n+1}(y) \subset V_{2n-1}(y) \subset \bar{P}_{2n-1}$. It follows that $x \in \bar{P}_{2n-1}$.

\textbf{Case 2:} If $y \in P(2n - 2) + \beta_{2n}$, then by the proof in Case 2 of Lemma 7.2.9, there exist three lattice points $q \in L_{2n-1}$, $r \in L_{2n-1}$, and $t \in L_{2n-1}$ which are mutually next to each other in the lattice $L_{2n-1}$ such that $y$ is the centroid of the triangle with vertices $q$, $r$, and $t$. It follows that $V_{2n}(y) \subset \bar{P}_{2n-1}$ as shown in Figure 7-5 (b). Since $V_{2n}(y)$ and $V_{2n+1}(y)$ have the same centroid and since the length of one side of the Voronoi cell $V_{2n}(y)$ is $\sqrt{3}$ times that of the Voronoi cell $V_{2n+1}(y)$, we have $V_{2n+1}(y) \subset V_{2n}(y)$. Hence $V_{2n+1}(y) \subset \bar{P}_{2n-1}$ and $x \in \bar{P}_{2n-1}$.

\textbf{Case 3:} If $y \in P(2n - 1) + \beta_{2n+1}$, then there exists $y_0 \in P(2n - 1)$ and $i \in \{1, 2, 3, 4, 5, 6\}$ such that $y = y_0 + \beta_{2n+1,i}$. Since the length of the vector $\beta_{2n+1,i}$ is $\rho_{2n+1} = \frac{1}{3}\rho_{2n-1}$, as shown in Figure 7-6, we have $V_{2n+1}(y) \subset V_{2n-1}(y_0)$. Hence $x \in V_{2n-1}(y_0) \subset \bar{P}_{2n-1}$. Therefore $\bar{P}_{2n+1} \subset \bar{P}_{2n-1}$.
\end{proof}

\begin{corollary}
For any $k \in \mathbb{N}$, we have $\mathcal{P} \subseteq \bar{P}_{2k-1}$.
\end{corollary}

\begin{proof}
Since $\mathcal{P}$ is the limit of the sequence $\{ \bar{P}_{2n} : k \leq n \in \mathbb{N} \}$ in the space $(\mathcal{H}(\mathbb{R}^2), h)$.

By Theorem 1 on Page 37 through page 38 from [4], we have

$$\mathcal{P} = \left\{ x \in \mathbb{R}^2 : \text{there is a Cauchy sequence} \left\{ x_n \in \bar{P}_{2n} : n \geq k \right\} \text{that converges to} x \right\}.$$

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By Lemma 7.2.8, we have $\tilde{P}_n \subset \tilde{P}_{n-1}$ for any $n \geq k$. By Lemma 7.2.9, for any $n \geq k$, we have $\tilde{P}_{n-1} \subset \tilde{P}_{2k-1}$. Hence $\tilde{P}_n \subset \tilde{P}_{2k-1}$ for any $n \geq k$. For any $n \geq k$, since $x_n \in \tilde{P}_n$, it follows that $x_n \in \tilde{P}_{2k-1}$. Because $\tilde{P}_{2k-1}$ is a closed set of $\mathbb{R}^2$ and $x = \lim_{n \to \infty} x_n$, it follows that $x \in \tilde{P}_{2k-1}$. Therefore $P \subseteq \tilde{P}_{2k-1}$.

For any $n \in \mathbb{N}$ and $i \in \{1, 2, 3, 4, 5, 6\}$, recall from Chapter 6 that $\Theta_{n,i} = 0000...0i \in \Lambda_n$ is the label of $\beta_{n,i}$ where $\beta_{n,i} \in \beta_n \subseteq P(n)$.

**Lemma 7.2.11.** Let $n \in \mathbb{N}$, $q$ be a boundary lattice point of $P(2n)$ which is labeled $q_1q_2...q_{2n-2}i0 \in \Lambda_{2n}$ with $i \in \{1, 2, 3, 4, 5, 6\}$, and $\bar{q}$ be a lattice point of $P(2n+2)$ which is labeled $q_1q_2...q_{2n-2}i0i0 \in \Lambda_{2n+2}$. If the boundary index of $q$ is 4, then $\bar{q}$ is a boundary lattice point of $P(2n+2)$ with boundary index 4.

**Proof.** By Lemma 6.2.15, we only need to consider the case $i = 1$ because any of the other cases can be proven by rotating the vectors $q$ and $\bar{q}$ by an angle of $\frac{k\pi}{3}$ for some $k \in \mathbb{N}$. By Table 6-2, we have $0010 \oplus 0003 = 0001 \in \Lambda_4$. By applying Item 4 of Lemma 6.1.8 twice, it follows that $10 \oplus 03 = 01 \in \Lambda_2$, i.e., $i0 \oplus 03 = 01 \in \Lambda_2$. By applying Item 4 of Lemma 6.1.8 for $2n-2$ times, i.e., by attaching $2n-2$ zeros to the left side on each term of the previous equation, we have $00...0i0 \oplus 00...003 = 00...001 \in \Lambda_{2n}$. 

Figure 7-6. The green Voronoi cell $V_{2n+1}(y)$ is contained in the black Voronoi cell $V_{2n-1}(y_0)$.
By Item 1 of Lemma 6.1.8, we have

\[ q_1 q_2 \ldots q_{2n-2} i 0 = q_1 q_2 \ldots q_{2n-2} 00 \oplus 00 \ldots 0i 0. \]

Hence

\[ q_1 q_2 \ldots q_{2n-2} i 0 \oplus \Theta_{2n,3} = q_1 q_2 \ldots q_{2n-2} 00 \oplus 00 \ldots 0i 0 \oplus 00 \ldots 003 \]
\[ = q_1 q_2 \ldots q_{2n-2} 00 \oplus 00 \ldots 001 \] (7–7)
\[ = q_1 q_2 \ldots q_{2n-2} 01 \in \Lambda_{2n}. \]

Similarly it follows from \( 0010 \oplus 0004 = 0006 \in \Lambda_4 \) that \( q_1 q_2 \ldots q_{2n-2} 10 \oplus \Theta_{2n,4} = q_1 q_2 \ldots q_{2n-2} 06 \in \Lambda_{2n} \). Since the boundary index of \( q \) is 4 and since \( q_1 q_2 \ldots q_{2n-2} 10 \oplus 00 \ldots 00i \in \Lambda_{2n} \) for \( i = 3, 4 \), we have \( q_1 q_2 \ldots q_{2n-2} 10 \oplus 00 \ldots 000i \not\in \Lambda_{2n} \) for each \( i \in \{1, 2, 5, 6\} \).

Similar to the previous computation, we can show that

\[ q_1 q_2 \ldots q_{2n-2} 1010 \oplus \Theta_{2n+2,3} = q_1 q_2 \ldots q_{2n-2} 1001 \in \Lambda_{2n+2} \]

and

\[ q_1 q_2 \ldots q_{2n-2} 1010 \oplus \Theta_{2n+2,4} = q_1 q_2 \ldots q_{2n-2} 1006 \in \Lambda_{2n+2}. \]

In the following, we show that \( q_1 q_2 \ldots q_{2n-2} 1010 \oplus \Theta_{2n+2,j} \not\in \Lambda_{2n+2} \) for each \( j \in \{1, 2, 5, 6\} \).

By Table 6-2, we have \( 0010 \oplus 0001 = 0105 \in \Lambda_4 \). By Item 4 of Lemma 6.1.8 for \( 2n - 2 \) times, it follows that \( 00 \ldots 00i 0 \oplus 00 \ldots 000i = 00 \ldots 0105 \in \Lambda_{2n+2} \). Hence

\[ q_1 q_2 \ldots q_{2n-2} i 0 i 0 \oplus \Theta_{2n+2,1} = q_1 q_2 \ldots q_{2n-2} 1010 \oplus 00 \ldots 00001 \]
\[ = q_1 q_2 \ldots q_{2n-2} 1000 \oplus 00 \ldots 00010 \oplus 00 \ldots 00001 \] (7–8)
\[ = q_1 q_2 \ldots q_{2n-2} 1000 \oplus 00 \ldots 00105. \]

Since \( q_1 q_2 \ldots q_{2n-2} 10 \oplus 00 \ldots 001 = q_1 q_2 \ldots q_{2n-2} 10 \oplus \Theta_{2n,1} \not\in \Lambda_{2n} \), by Item 4 and Item 2 of Lemma 6.1.8, we have \( q_1 q_2 \ldots q_{2n-2} 1000 \oplus 00 \ldots 00105 \not\in \Lambda_{2n+2} \). By Equation 7–8, it follows that \( q_1 q_2 \ldots q_{2n-2} i 0 i 0 \oplus \Theta_{2n+2,1} \not\in \Lambda_{2n+2} \).
By Table 6-2, we have \(0010 \oplus 0002 = 0104 \in \Lambda_4\). Hence we can show that 
\(q_1q_2...q_{2n-2}i0i0 \oplus \Theta_{2n+2,2} \not\in \Lambda_{2n+2}\) using a similar proof as in the previous paragraph.

Now consider \(q_1q_2...q_{2n-2}i0i0 \oplus \Theta_{2n+2,5}\). By Table 6-2, we have \(0010 \oplus 0005 = 0603 \in \Lambda_4\). By applying Item 4 of Lemma 6.1.8 for \(2n - 2\) times, it follows that \(00...0010 \oplus 00...0005 = 00...0603 \in \Lambda_{2n+2}\). Hence

\[
q_1q_2...q_{2n-2}i0i0 \oplus \Theta_{2n+2,5} = q_1q_2...q_{2n-2}1010 \oplus 00...00005
\]

\[
= q_1q_2...q_{2n-2}1000 \oplus 00...0010 \oplus 00...0005
\]

\[
= q_1q_2...q_{2n-2}1000 \oplus 00...00603.
\]

Since \(q_1q_2...q_{2n-2}10 \oplus 00...06 = q_1q_2...q_{2n-2}10 \oplus \Theta_{2n,6} \not\in \Lambda_{2n}\), by Item 4 and Item 2 of Lemma 6.1.8, we have \(q_1q_2...q_{2n-2}1000 \oplus 00...00603 \not\in \Lambda_{2n+2}\). By Equation 7–9, it follows that \(q_1q_2...q_{2n-2}i0i0 \oplus \Theta_{2n+2,5} \not\in \Lambda_{2n+2}\). Similarly we can show that \(q_1q_2...q_{2n-2}i0i0 \oplus \Theta_{2n+2,6} \not\in \Lambda_{2n+2}\).

We have shown that \(q_1q_2...q_{2n-2}i0i0 \oplus \Theta_{2n+2,j} \in \Lambda_{2n+2}\) for \(j = 3, 4\), and \(q_1q_2...q_{2n-2}i0i0 \oplus \Theta_{2n+2,j} \not\in \Lambda_{2n+2}\) for any \(j \in \{1, 2, 5, 6\}\). Since \(q_1q_2...q_{2n-2}i0i0\) is the label of \(\tilde{q}\), it follows that the boundary index of \(\tilde{q}\) is 4.

\[\square\]

**Lemma 7.2.12.** If \(q\) is a boundary lattice point of \(P(2n)\) and \(q \in L_{2n-1}\), then \(q\) is also a boundary lattice point of \(P(2n - 1)\).

**Proof.** Let \(q_1q_2...q_{2n} \in \Lambda_{2n}\) be the label of \(q\). We claim that \(q_{2n} = 0\). Suppose \(q_{2n} \neq 0\).

Then \(q_{2n} = i\) for some \(i \in \{1, 2, 3, 4, 5, 6\}\). Recall that the lattice point labeled \(00...0i\) \(\in \Lambda_{2n}\) is \(\beta_{2n,i}\) which is an element of \(\beta_{2n}\). Let \(r\) be a lattice point of \(P(2n - 1)\) which is labeled \(q_1q_2...q_{2n-1}\). By Item 1 of Lemma 6.1.8, we have \(q_1q_2...q_{2n-1}q_{2n} = q_1q_2...q_{2n-1}0 \oplus 00...0q_{2n}\). It follows that \(q = r + \beta_{2n,i}\). Hence \(\beta_{2n,i} = q - r\). Because \(q \in L_{2n-1}\) and \(r \in P(2n - 1) \subset L_{2n-1}\), it follows that \(q - r \in L_{2n-1}\), i.e., \(\beta_{2n,i} \in L_{2n-1}\). Since \(\beta_{2n,j} \in \beta_{2n}\), it follows that \(\beta_{2n,i} \in L_{2n-1} \cap \beta_{2n} = \emptyset\) by Lemma 6.1.3, which is a contradiction. Thus \(q_{2n} = 0\). It follows that \(q \in P(2n - 1)\).
Now we show that $q$ is also a boundary lattice point of $P(2n - 1)$. Suppose $q$ is not a boundary lattice point of $P(2n - 1)$. Then $q + \beta_{2n-1} \subseteq P(2n - 1)$. Let $\beta_{2n-1,7} = \beta_{2n-1,1}$, and let $u_j = q + \beta_{2n-1,j}$ for $j = 1, 2, ..., 7$. Also let $t_j = \frac{1}{3}(q + u_j + u_{j+1})$ for each $j \in \{1, 2, 3, 4, 5, 6\}$. Obviously
\[
t_j = q + \frac{1}{3}(\beta_{2n-1,j} + \beta_{2n-1,j+1}) = q + \frac{1}{3}\beta_{2n-2,j} = q + \beta_{2n,j}.
\]
For each $j \in \{1, 2, 3, 4, 5, 6\}$, since $q$, $u_j$, and $u_{j+1}$ are three lattice points of $P(2n - 1)$ which are next to each other in the lattice $L_{2n-1}$, by Lemma 6.2.7, we have $t_j \in P(2n)$, i.e., $q + \beta_{2n,j} \in P(2n)$. Thus $q + \beta_{2n} \subseteq P(2n)$. It follows that $q$ is not a boundary lattice point of $P(2n)$, which contradicts the given condition of this Lemma. Thus $q$ is also a boundary lattice point of $P(2n - 1)$. □

**Lemma 7.2.13.** Let $1 < n \in \mathbb{N}$ and let $q$ be a lattice point of $P(2n)$ whose label is $q_1q_2...q_{2n} \in \Lambda_{2n}$ and $\bar{q}$ be a boundary lattice point of $P(2n - 2)$ whose label is $q_1q_2...q_{2n-2} \in \Lambda_{2n-2}$. If the boundary index of $\bar{q}$ is 1 and if $q$ is a boundary lattice point of $P(2n)$, then the boundary index of $q$ is 1.

**Proof.** Since $\bar{q}$ is a boundary lattice point of $P(2n)$ and the label of $\bar{q}$ is $q_1q_2...q_{2n-2}$, by Lemma 6.2.14, either $q_{2n-3} \neq 0$ or $q_{2n-2} \neq 0$. Because the boundary index of $\bar{q}$ is 1, by Lemma 6.2.16, $q_{2n-3} = 0$ and $q_{2n-2} \neq 0$. Because $q_1q_2...q_{2n} \in \Lambda_{2n}$ and $q_{2n-2} \neq 0$, by Item 1 of Corollary 6.1.7, we have $q_{2n-1} = 0$. Since $q$ is a boundary lattice point of $P(2n)$, by Lemma 6.2.14, it follows that $q_{2n} \neq 0$. Hence, by Lemma 6.2.16, the boundary index of $q$ is 1. □

**Lemma 7.2.14.** Let $n \in \mathbb{N}$, $q \in P(2n)$ and let $q_1q_2...q_{2n}$ be the label of $q$. If $q$ is a boundary lattice point of $P(2n)$ with boundary index 2 or 4, then $q_{2i} = 0$ and $q_{2i-1} \neq 0$ for any $i$ satisfying $1 \leq i \leq n$.

**Proof.** If $n = 1$, then, by the plot of the lattice points of $P(2)$, this lemma is obviously true. Now assume this lemma is true for any $n < k$ for some $k \in \mathbb{N}$. When $n = k$, let
\( \tilde{q} \) be the lattice points of \( P(2k - 2) \) whose label is \( q_1q_2...q_{2k-2} \). Since \( q \) is a boundary lattice point of \( P(2k) \), by Lemma 6.2.14, either \( q_{2k-1} \neq 0 \) or \( q_{2k} \neq 0 \), and \( \tilde{q} \) is a boundary lattice point of \( P(2k - 2) \). If \( q_{2k} \neq 0 \), then by Lemma 6.2.16, the boundary index of \( q \) is 1 which contradicts the given condition that the boundary index of \( q \) is either 2 or 4. Thus \( q_{2k-1} \neq 0 \) and \( q_{2k} = 0 \). By Lemma 7.2.13, if the boundary index of \( \tilde{q} \) is 1, then the boundary index of \( q \) is also 1, which again contradicts the given condition that the boundary index of \( q \) is either 2 or 4. Thus the boundary index of \( \tilde{q} \) is either 2 or 4. By the assumption of this induction, we have \( q_{2i} = 0 \) and \( q_{2i-1} \neq 0 \) for any \( i \) satisfying \( 1 \leq i \leq k - 1 \). Thus this lemma is true when \( n = k \). By induction, it is true for any \( n \in \mathbb{N} \).

\[
\Psi_n(q) = q + \frac{1}{2}b_{2n-1,a}.
\]

The following lemma shows that \( \Psi_n \) is a map from \( B_{2n,4} \) to \( \mathcal{P}_B \), where \( \mathcal{P}_B \) is the boundary of \( \mathcal{P} \) and defined at the beginning of this chapter.

**Lemma 7.2.15.** For any \( n \in \mathbb{N} \) and \( q \in B_{2n,4} \), we have \( \Psi_n(q) \in \mathcal{P}_B \).

**Proof.** Since \( q \in B_{2n,4} \subset P(2n) \), let the label of \( q \) be \( q_1q_2...q_{2n} \in \Lambda_{2n} \). Then \( \Psi_n(q) = q + \frac{1}{2}b_{2n-1,a} \) where \( a = q_{2n-1} \). Since \( q \in B_{2n,4} \), the boundary index of \( q \) is 4. By Lemma 7.2.14, we have \( q_{2i} = 0 \) and \( q_{2i-1} \neq 0 \) for each \( i \) satisfying \( 1 \leq i \leq n \). Because \( q_{2n} = 0 \), we have \( q \in P(2n - 1) \). Since \( P(2n - 1) + \beta_{2n+1} \subset P(2n + 1) \), it follows that \( q + \beta_{2n+1,a} \in P(2n + 1) \). By induction, it is easy to show that \( q + \sum_{i=0}^{k} \beta_{2n+2i+1,a} \in P(2n + 2k + 1) \) for each \( k \geq 0 \). Since \( P(2n + 2k + 1) \subset P(2n + 2k + 2) \subset P_{2n+2k+2} \), we have \( q + \sum_{i=0}^{k} \beta_{2n+2i+1,a} \in P_{2n+2k+2} \) for each \( k \in \mathbb{N} \). By Lemma 7.2.5, we have \( P_{2n+2k+2} \subset \mathcal{P} \). It follows that \( q + \sum_{i=0}^{k} \beta_{2n+2i+1,a} \in \mathcal{P} \) for each \( k \in \mathbb{N} \).
Since $\beta_{2n+2i+1,a} = \frac{1}{3^i} \beta_{2n+1,a}$ for each $i \geq 0$, the sequence \( \left\{ \sum_{i=0}^{k} \beta_{2n+2i+1,a} : k \in \mathbb{N} \right\} \) is a Cauchy sequence in the space \( (\mathbb{R}^2, d) \). Hence the limit of this sequence exists. As usual, the limit of the sequence \( \left\{ \sum_{i=0}^{k} \beta_{2n+2i+1,a} : k \in \mathbb{N} \right\} \) is denoted \( \sum_{i=0}^{\infty} \beta_{2n+2i+1,a} \). Obviously we have
\[
\sum_{i=0}^{\infty} \beta_{2n+2i+1,a} = \sum_{i=0}^{\infty} \frac{1}{3^i} \beta_{2n+1,a} = \frac{3}{2} \beta_{2n+1,a} = \frac{1}{2} \beta_{2n-1,a}.
\]
It follows that \( q + \sum_{i=0}^{\infty} \beta_{2n+2i+1,a} = q + \frac{1}{2} \beta_{2n-1,a} \). Hence \( q + \frac{1}{2} \beta_{2n-1,a} \) is a limit point of the sequence \( \left\{ q + \sum_{i=0}^{k} \beta_{2n+2i+1,a} : k \in \mathbb{N} \right\} \). Since \( q + \sum_{i=0}^{k} \beta_{2n+2i+1,a} \in \mathcal{P} \) for each \( k \in \mathbb{N} \) as shown in the previous paragraph and since \( \mathcal{P} \) is a closed subset of \( \mathbb{R}^2 \), it follows that \( q + \frac{1}{2} \beta_{2n-1,a} \in \mathcal{P} \), i.e., \( \Psi_n(q) \in \mathcal{P} \).

We claim that \( \Psi_n(q) \in \mathcal{P}_B \). Suppose \( \Psi_n(q) \notin \mathcal{P}_B \). Then there exists an open \( \delta \)-neighborhood \( \mathcal{N}_\delta(\Psi_n(q)) = \{ y \in \mathbb{R}^2 : d(\Psi_n(q), y) < \delta \} \) for some \( \delta > 0 \) such that \( \mathcal{N}_\delta(\Psi_n(q)) \subseteq \mathcal{P} \). Because \( \delta > 0 \) and \( \lim_{n \to \infty} \rho_n = 0 \), there exists \( c \in \mathbb{N} \) such that \( \rho_{2c+1} < \frac{\delta}{2} \).

Since \( q \in B_{2n,4} \) and \( a = q_{2n-1} \), it follows from Lemma 7.2.11 that \( q + \sum_{i=0}^{k} \beta_{2n+2i+1,a} \) is a boundary lattice point of \( P(2n + 2k + 2) \). To make the following expressions simpler, let \( s_k = q + \sum_{i=0}^{k} \beta_{2n+2i+1,a} \). Because \( s_k = q + \sum_{i=0}^{k} \beta_{2n+2i+1,a} \in P(2n + 2k + 1) \) as shown in the previous paragraph and because \( s_k \) is a boundary lattice point of \( P(2n + 2k + 2) \), it follows from Lemma 7.2.12 that \( s_k \) is also a boundary lattice point of \( P(2n + 2k + 1) \).

Hence there exists \( l \in \{1, 2, 3, 4, 5, 6\} \) such that \( s_k + \beta_{2n+2k+1,l} \notin P(2n + 2k + 1) \). Since \( s_k + \beta_{2n+2k+1,l} \in L_{2n+2k+1} \), it follows that \( s_k + \beta_{2n+2k+1,l} \notin \tilde{P}_{2n+2k+1} \). By Corollary 7.2.10, we have \( \mathcal{P} \subseteq \tilde{P}_{2n+2k+1} \). It follows that \( s_k + \beta_{2n+2k+1,l} \notin \mathcal{P} \). Since \( \Psi_n(q) = q + \frac{1}{2} \beta_{2n-1,a} \), we have
\[
\Psi_n(q) - s_c = (q + \frac{1}{2} \beta_{2n-1,a}) - (q + \sum_{i=0}^{c} \beta_{2n+2i+1,a}) = \frac{1}{2} \beta_{2n-1,a} - \sum_{i=0}^{c} \frac{1}{3^i} \beta_{2n+1,a}.
\]
It follows that \( d(s_c, \Psi_n(q)) = ||\Psi_n(q) - s_c|| = ||\frac{1}{2} \beta_{2n-1,a} - \sum_{i=0}^{c} \frac{1}{3^i} \beta_{2n+1,a}|| = ||\frac{1}{2} \beta_{2n-1,a} - \frac{3}{2} (1 - \frac{1}{3^{c+1}}) \beta_{2n+1,a}|| = ||\frac{1}{2} \beta_{2n-1,a} - \frac{1}{2} (1 - \frac{1}{3^{c+1}}) \beta_{2n-1,a}|| = ||\frac{1}{2(3^{c+1})} \beta_{2n-1,a}|| = \frac{1}{2(3^{c+1})} \rho_{2n-1} = \frac{1}{2} \rho_{2n+2c+1} \).
By the triangle inequality, we have
\[
d(s_c + \beta_{2n+2c+1}, \Psi_n(q)) \leq d(s_c + \beta_{2n+2c+1}, t, s_c) + d(s_c, \Psi_n(q)).
\]
Since \(d(s_c + \beta_{2n+2c+1}, s_c) = \|\beta_{2n+2c+1}\| = \rho_{2n+2c+1}\) and since \(d(s_c, \Psi_n(q)) = \frac{1}{2}\rho_{2n+2c+1}\), it follows that \(d(s_c + \beta_{2n+2c+1}, \Psi_n(q)) \leq \rho_{2n+2c+1} + \frac{1}{2}\rho_{2n+2c+1} < 2\rho_{2n+2c+1} = \frac{2}{\beta}\rho_{2c+1} \leq 2\rho_{2c+1} < \delta.\) Hence \(s_c + \beta_{2n+2c+1} \in N_\delta(\Psi_n(q)) \subseteq P.\) This contradicts that \(s_k + \beta_{2n+2k+1} \notin P\) for any \(k \in \mathbb{N}.\) Therefore \(\Psi_n(q) \in P_B.\)

The next three lemmas will be applied in the proof of Lemma 7.2.20.

**Lemma 7.2.16.** Let \(n \in \mathbb{N}\) and \(x, y \in P(2n).\) If both \(x\) and \(y\) are boundary lattice points of \(P(2n)\) each having boundary index 2 or 4, then \(d(x, y) \neq \rho_{2n}\) and \(d(x, y) \neq 2\rho_{2n}.\)

**Proof.** Let \(x_1x_2...x_{2n} \in \Lambda_{2n}\) and \(y_1y_2...y_{2n} \in \Lambda_{2n}\) be the labels of \(x\) and \(y,\) respectively. Since the boundary index of \(x\) is either 2 or 4, by Lemma 7.2.14, we have \(x_{2n} = 0.\) It follows that \(x \in P(2n-1) \subset L_{2n-1}.\) Similarly we have \(y \in L_{2n-1}.\)

If \(d(x, y) = \rho_{2n},\) then \(x\) is next to \(y\) in the lattice \(L_{2n}.\) It follows that \(x = y + z\) for some \(z \in \beta_{2n}.\) Since \(z = x - y\) and \(x, y \in L_{2n-1},\) we have \(z \in L_{2n-1} \cap \beta_{2n}.\) However, by Lemma 6.1.3, we have \(L_{2n-1} \cap \beta_{2n} = \emptyset.\) It follows that \(z \in \emptyset\) which is a contradiction. Thus \(d(x, y) \neq \rho_{2n}.\)

If \(d(x, y) = 2\rho_{2n},\) then \(\|x - y\| = 2\rho_{2n}.\) Since \(L_{2n}\) is a hexagonal lattice, it follows that \(x - y = 2\beta_{2n,i}\) for some \(i \in \{1, 2, 3, 4, 5, 6\}.\) Hence \(x = y + 2\beta_{2n,i}.\) By Table 6-2, we have \(000i \oplus 000i = 0j0k \in \Lambda_4\) and \(k \neq 0.\) By applying Item 4 of Lemma 6.1.8 for \(2n-4\) times, it follows that \(00...000i \oplus 00...000i = 00...0j0k \in \Lambda_{2n}.\) Hence \(\beta_{2n,i} + \beta_{2n,i} = \beta_{2n-2,j} + \beta_{2n,k},\) i.e., \(2\beta_{2n,i} = \beta_{2n-2,j} + \beta_{2n,k}.\) Since \(x = y + 2\beta_{2n,i},\) it follows that \(x = y + \beta_{2n-2,j} + \beta_{2n,k}.\) Hence \(\beta_{2n,k} = x - y - \beta_{2n-2,j}.\) Because \(x, y \in L_{2n-1}\) and \(\beta_{2n-2,j} \in L_{2n-2} \subset L_{2n-1},\) it follows that \(\beta_{2n,k} = x - y - \beta_{2n-2,j} \in L_{2n-1}.\) Since \(\beta_{2n,k} \in \beta_{2n},\) it follows that \(\beta_{2n,k} \in L_{2n-1} \cap \beta_{2n} = \emptyset,\) which is a contradiction. Thus \(d(x, y) \neq 2\rho_{2n}.\)

\(\square\)
Lemma 7.2.17. Let $L$ be a hexagonal lattice with generators $a$ and $b$ such that $\|a\| = \|b\| = r$ for some $0 < r \in \mathbb{R}$, and the angle between $a$ and $b$ is $\frac{2\pi}{3}$. If $x \in L$ and $0 < \|x\| < r\sqrt{7}$, then either $\|x\| = r$ or $\|x\| = r\sqrt{3}$ or $\|x\| = 2r$.

Proof. Since $x \in L$ and $L$ is a hexagonal lattice with generators $a$ and $b$, there exist $m, n \in \mathbb{Z}$ such that $x = ma + nb$. Because $\|a\| = \|b\| = r$ and the angle between $a$ and $b$ is $\frac{2\pi}{3}$, we have

$$\|x\| = \|ma + nb\| = \sqrt{m^2\|a\|^2 + n^2\|b\|^2 + 2mn\|a\| \cdot \|b\| \cos \left(\frac{2\pi}{3}\right)}$$

(7-11)

If $m < 0$ or $n < 0$, we can rotate $x$ by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ to obtain $y = m_2a + n_2b$ so that $m_2 \geq 0$ and $n_2 \geq 0$. Without loss of generality, we assume that $m, n \geq 0$. By the symmetry of $m$ and $n$ in Equation 7–11, we also assume that $m \geq n$.

If $n = 0$ or $n = m$, then $\|x\| = rm$. Since $0 < \|x\| < r\sqrt{7}$, it follows that $0 < m < \sqrt{7}$. Because $m \in \mathbb{Z}$, it follows that $m = 1$ or $m = 2$. Hence $\|x\| = r$ or $\|x\| = 2r$.

Otherwise we have $n \geq 1$ and $m - n \geq 1$. Then we consider the following two cases.

Case 1: If $m - n = 1$, then $m = n + 1$ and $\|x\| = r\sqrt{n^2 + n + 1}$. Since $0 < \|x\| < r\sqrt{7}$, it follows that $n^2 + n + 1 < 7$. Hence $(n + 3)(n - 2) < 0$. Because $n > 0$, it follows that $n - 2 < 0$. Hence $1 \leq n < 2$. It follows that $n = 1$. Thus $\|x\| = r\sqrt{3}$.

Case 2: If $m - n \geq 2$, then $m \geq n + 2$ and $\|x\| = r\sqrt{m(m-n)+n^2} \geq r\sqrt{(n+2)^2 + n^2} = r\sqrt{n^2 + 2n + 4}$. Since $n \geq 1$, it follows that $\|x\| = r\sqrt{7}$ which contradicts the condition that $0 < \|x\| < r\sqrt{7}$.

Lemma 7.2.18. For any $n \in \mathbb{N}$ and $x, y \in B_{2n,2} \cup B_{2n,4}$ with $d(x, y) = \sqrt{3}\rho_{2n}$, there exists $z \in P(2n)$ such that $z$ is next to both $x$ and $y$ in the lattice $L_{2n}$.

Proof. If $n = 1$, this lemma is true by the plot of lattice points of $P(2)$ (Figure 6-2).

Assume this lemma is true for any $n \leq k$ for some $k \in \mathbb{N}$. Now we show that it is also
true when $n = k + 1$. Let the labels of $x$ and $y$ be $x_1x_2\ldots x_{2k}x_{2k+1}x_{2k+2} \in \Lambda_{2k+2}$ and $y_1y_2\ldots y_{2k}y_{2k+1}y_{2k+2} \in \Lambda_{2k+2}$, respectively. Since $x, y \in B_{2k+2,2} \cup B_{2k+2,4}$, by Lemma 7.2.14, we have $x_{2i} = y_{2i} = 0$, $x_{2i-1} \neq 0$, and $y_{2i-1} \neq 0$ for any $i$ satisfying $1 \leq i \leq k + 1$.

Let $q$ and $r$ be lattice points of $P(2k)$ whose labels are $x_1x_2\ldots x_{2k}x_{2k}x_{2k+1}x_{2k+2} \in \Lambda_{2k}$ and $y_1y_2\ldots y_{2k}y_{2k-1}y_{2k} \in \Lambda_{2k}$, respectively. Also let $a = x_{2k+1}$ and $b = y_{2k+1}$. By Item 1 of Lemma 6.1.8, we have $x_1x_2\ldots x_{2k-1}x_{2k+1}x_{2k+2} = x_1x_2\ldots x_{2k-1}x_{2k}000\ldots 00x_{2k+1}x_{2k+2}$. Since $x_{2k+1}x_{2k+2} = ab$, it follows that $x = q + \beta_{2k+1,a}$. Similarly we have $y = r + \beta_{2k+1,b}$.

Now we consider the following two cases.

**Case 1:** If $q = r$, then $y = q + \beta_{2k+1,b}$. Hence $x, y \in q + \beta_{2k+1}$. By Lemma 6.2.12, we have $q + \beta_{2k+1} \cup q + \beta_{2k+2} \in P(2k + 2)$. In Figure 6-5 of Chapter 6, the Voronoi cells of lattice points in $q + \beta_{2k+1}$ and $q + \beta_{2k+2}$ are in blue and in green, respectively. Since $d(x, y) = \sqrt{3}\rho_{2n} = \sqrt{3}\rho_{2k+2}$, as shown in that figure, there exists $z \in q + \beta_{2k+2}$ such that $z$ is next to both $x$ and $y$ in the lattice $L_{2k+2}$.

**Case 2:** If $q \neq r$, then we claim that $d(q, r) = \sqrt{3}\rho_{2k}$. By the triangle inequality, we have

$$d(q, r) \leq d(q, q + \beta_{2k+1,a}) + d(q + \beta_{2k+1,a}, r + \beta_{2k+1,b}) + d(r + \beta_{2k+1,b}, r)$$

$$= \| (q + \beta_{2k+1,a}) - q \| + d(x, y) + \| r - (r + \beta_{2k+1,b}) \|$$

$$= \| \beta_{2k+1,a} \| + d(x, y) + \| \beta_{2k+1,b} \|$$

$$= \rho_{2k+1} + d(x, y) + \rho_{2k+1}$$

$$= d(x, y) + 2\rho_{2k+1}.$$  

Hence $d(x, y) \geq d(q, r) - 2\rho_{2k+1}$. Suppose $d(q, r) \geq 2\rho_{2k}$. Then $d(x, y) \geq 2\rho_{2k} - 2\rho_{2k+1} = 6\rho_{2k+2} - 2\sqrt{3}\rho_{2k+2} = 2\sqrt{3}(\sqrt{3} - 1)\rho_{2k+2} > \sqrt{3}\rho_{2k+2}$ which contradicts the condition that $d(x, y) = \sqrt{3}\rho_{2n}$. Thus $d(q, r) < 2\rho_{2k}$. Since $q \neq r$ and since $d(q, r) < 2\rho_{2k} < \sqrt{7}\rho_{2k}$, by Lemma 7.2.17, we have either $d(q, r) = \rho_{2k}$ or $d(q, r) = \sqrt{3}\rho_{2k}$. Because $x, y \in B_{2k+2,2} \cup B_{2k+2,4}$, by Lemma 6.2.14, $q$ and $r$ are boundary lattice points of $P(2k)$. Since
$x_{2k} = 0$, by Lemma 6.2.16, the boundary index of $q$ is either 2 or 4. So is the boundary index of $r$. Hence, by Lemma 7.2.16, $d(q, r) \neq \rho_{2k}$. Thus $d(q, r) = \sqrt{3} \rho_{2k}$.

![Image of hexagons and Voronoi cells](image)

**Figure 7-7.** For any two hexagons of $P(2k+2)$ each having boundary index 2 or 4 such that the distance between them is $\rho_{2k+2}$, there exists a hexagon of $P(2k+2)$ that is next to both of them, where $Q$, $R$ and $S$ are hexagons of $P(2k)$. (a) The hexagon $S$ is next to $Q$ and $R$, and its centroid lies below the line connecting the centroids of $Q$ and $R$. (b) shows that, for any two blue hexagons of $P(2k+2)$ (each having boundary index 2 or 4) such that the distance between them is $\rho_{2k+2}$, there exists a hexagon of $P(2k+2)$ that is next to these two blue hexagons.

Since $d(q, r) = \sqrt{3} \rho_{2k}$, by the induction assumption, there exists $s \in P(2k)$ such that $s$ is next to both $q$ and $r$ in the lattice $L_{2k}$. Because $\|r - q\| = d(q, r) = \sqrt{3} \rho_{2k}$ and $r - q \in L_{2k}$, we have $r - q = \beta_{2k-1,j}$ for some $j \in \{1, 2, 3, 4, 5, 6\}$. By Lemma 6.2.15, we can rotate the vectors $q$, $r$, and $s$ by an angle of $-\frac{\pi}{3}$ or $-\frac{\pi}{3} + \pi$ so that the line connecting $q$ and $r$ is horizontal. Hence we assume that the line connecting $q$ and $r$ is horizontal and the Voronoi cells $Q$, $R$, and $S$ of $q$, $r$, and $s$ are as shown in Figure 7-7 (a), where the Voronoi cells $Q$ and $R$ are in black and the Voronoi cell $S$ is dashed. Since $q, r, s \in P(2k)$, by Lemma 6.2.12, it follows that all blue, green and red hexagons are Voronoi cells of $P(2k+2)$. Since $x = q + \beta_{2k+1,a}$ and $y = r + \beta_{2k+1,b}$, the Voronoi cell of $x \in P(2k+2)$ is one of the blue hexagons overlapping $Q$ and the Voronoi cell of $y \in P(2k+2)$ is one of the blue hexagons overlapping $R$. Because $d(x, y) = \sqrt{3} \rho_{2k+2}$, the Voronoi cells $X$ and $Y$ of $x \in P(2k+2)$ and $y \in P(2k+2)$ must be as shown in Figure 7-7 (b). Let $z$ be a lattice.
point of \( P(2k + 2) \) whose Voronoi cell \( Z \) is shown in Figure 7-7 (b). Then \( z \) is next to both \( x \) and \( y \). 

To simplify the proof of Lemma 7.2.20, we need the following lemma.

**Lemma 7.2.19.** Let \( n \in \mathbb{N} \) and \( x, y \in B_{2n, 4} \). If \( R \) is the rotation of \( \mathbb{R}^2 \) by an angle of \( \frac{k\pi}{3} \) about the origin for some \( k \in \mathbb{Z} \), then \( R(x), R(y) \in B_{2n, 4} \) and \( d(\Psi_n(R(x)), \Psi_n(R(y))) = d(\Psi_n(x), \Psi_n(y)) \).

**Proof.** Since \( x, y \in B_{2n, 4} \), by Lemma 6.2.15, we have \( R(x), R(y) \in B_{2n, 4} \).

Let \( x_1x_2...x_{2n-1}x_{2n} \in \Lambda_{2n}, y_1y_2...y_{2n-1}y_{2n} \in \Lambda_{2n}, \bar{x}_1\bar{x}_2...\bar{x}_{2n-1}\bar{x}_{2n} \in \Lambda_{2n} \) and \( \bar{y}_1\bar{y}_2...\bar{y}_{2n-1}\bar{y}_{2n} \in \Lambda_{2n} \) be the labels of \( x, y, R(x) \) and \( R(y) \), respectively. Also let \( a = x_{2n-1}, b = y_{2n-1}, c = \bar{x}_{2n-1}, \) and \( d = \bar{y}_{2n-1} \). By the definition of \( \Psi_n \), we have \( \Psi_n(x) = x + \frac{1}{2}\beta_{2n-1,a}, \Psi_n(y) = y + \frac{1}{2}\beta_{2n-1,b}, \Psi_n(R(x)) = R(x) + \frac{1}{2}\beta_{2n-1,c}, \Psi_n(R(y)) = R(y) + \frac{1}{2}\beta_{2n-1,d}. \)

Since \( x_1x_2...x_{2n-1}x_{2n} \) is the label of \( x \), we have \( x = \sum_{i=1}^{2n} \beta_{i,x_i}. \) Hence

\[
R(x) = \sum_{i=1}^{2n} R(\beta_{i,x_i}). \tag{7-13}
\]

For any \( i \) satisfying \( 1 \leq i < 2n \), by Item 1 of Corollary 6.1.7, we have either \( x_i = 0 \) or \( x_{i+1} = 0 \). Hence either \( \beta_{i,x_i} = 0 \) or \( \beta_{i+1,x_{i+1}} = 0 \). Since \( R \) is the rotation of \( \mathbb{R}^2 \) about the origin, it follows that either \( R(\beta_{i,x_i}) = 0 \) or \( R(\beta_{i+1,x_{i+1}}) = 0 \). For each \( i \) satisfying \( 1 \leq i \leq 2n \), if \( \beta_{i,x_i} \neq 0 \), then \( \beta_{i,x_i} \in \beta_i \). Because \( R \) is the rotation of \( \mathbb{R}^2 \) by an angle of \( \frac{k\pi}{3} \) about the origin for some integer \( k \), it follows that \( R(\beta_{i,x_i}) \in \beta_i \). Thus \( R(\beta_{i,x_i}) \in \bar{\beta}_i \) for each \( i \) satisfying \( 1 \leq i \leq 2n \). Hence Equation 7–13 is the standard expression of \( R(x) \) in \( P(2n) \). Since \( \bar{x}_1\bar{x}_2...\bar{x}_{2n-1}\bar{x}_{2n} \in \Lambda_{2n} \) is the label of \( R(x) \), the Equation

\[
R(x) = \sum_{i=1}^{2n} \beta_{i,\bar{x}_i}. \tag{7-14}
\]

is also the standard expression of \( R(x) \) in \( P(2n) \). By Theorem 6.1.6 for the unique standard expression, it follows from Equation 7–13 and Equation 7–14 that \( R(\beta_{i,x_i}) = \beta_{i,\bar{x}_i} \) for each \( i \) satisfying \( 1 \leq i \leq 2n \). Hence \( R(\beta_{2n-1,a}) = \beta_{2n-1,c} \). Since \( \Psi_n(R(x)) = \)
it follows that

\[ \Psi_n(R(x)) = R(x) + \frac{1}{2}R(\beta_{2n-1,a}) = R(x + \frac{1}{2}\beta_{2n-1,a}) = R(\Psi_n(x)). \]

For a similar reason, we have

\[ \Psi_n(R(y)) = R(y) + \frac{1}{2}R(\beta_{2n-1,b}) = R(y + \frac{1}{2}\beta_{2n-1,b}) = R(\Psi_n(y)). \]

Therefore

\[ d(\Psi_n(R(x)), \Psi_n(R(y))) = d(R(\Psi_n(x)), R(\Psi_n(y))) = d(\Psi_n(x), \Psi_n(y)). \]

\[ \square \]

The next lemma will be used to show that \( \Psi_n \) is a one to one map and to obtain the fractal dimension of \( \calP_B \).

**Lemma 7.2.20.** For any \( n \in \mathbb{N} \) and \( x, y \in B_{2n,4} \) with \( x \neq y \), we have \( d(\Psi_n(x), \Psi_n(y)) > 3\rho_{2n+1} \).

**Proof.** Let \( x_1x_2...x_{2n-1}x_{2n} \in \Lambda_{2n} \) and \( y_1y_2...y_{2n-1}y_{2n} \in \Lambda_{2n} \) be the labels of \( x \) and \( y \), respectively. Since \( x, y \in B_{2n,4} \), by Lemma 7.2.14, we have \( x_{2i-1} \neq 0, y_{2i-1} \neq 0 \), and \( x_{2i} = y_{2i} = 0 \) for each \( i = 1, 2, ..., n \). By the definition of \( \Psi_n \), we have \( \Psi_n(x) = x + \frac{1}{2}\beta_{2n-1,a} \) with \( a = x_{2n-1} \), and \( \Psi_n(y) = y + \frac{1}{2}\beta_{2n-1,b} \) with \( b = y_{2n-1} \). If \( n = 1 \), let \( q = r = 0 \). If \( n > 1 \), let \( q \) and \( r \) be the lattice points of \( P(2n-2) \) which are labeled \( x_1x_2...x_{2n-2} \) and \( y_1y_2...y_{2n-2} \), respectively. Since \( x_{2n-1} = a \) and \( y_{2n-1} = b \), we have \( x = q + \beta_{2n-1,a} \) and \( y = r + \beta_{2n-1,b} \). Hence we have

\[ \Psi_n(x) = x + \frac{1}{2}\beta_{2n-1,a} = q + \beta_{2n-1,a} + \frac{1}{2}\beta_{2n-1,a} = q + \frac{3}{2}\beta_{2n-1,a} \quad (7-15) \]

and

\[ \Psi_n(y) = y + \frac{1}{2}\beta_{2n-1,b} = r + \beta_{2n-1,b} + \frac{1}{2}\beta_{2n-1,b} = r + \frac{3}{2}\beta_{2n-1,b}. \quad (7-16) \]
Now we consider the following two cases.

**Case 1:** If \( q = r \), since \( x \neq y \), we have \( x_{2n-1} \neq y_{2n-1} \), i.e., \( a \neq b \). It follows that \( \beta_{2n-1,a} \) and \( \beta_{2n-1,b} \) are two different lattice points of the lattice \( L_{2n-1} \). Hence
\[
d(\beta_{2n-1,a}, \beta_{2n-1,b}) \geq \rho_{2n-1}.
\]
Since \( q = r \), it follows that
\[
d(\Psi_n(x), \Psi_n(y)) = d(q + \frac{3}{2} \beta_{2n-1,a}, r + \frac{3}{2} \beta_{2n-1,b}) = d(\frac{3}{2} \beta_{2n-1,a}, \beta_{2n-1,b}) = \frac{3}{2} d(\beta_{2n-1,a}, \beta_{2n-1,b}) \geq \frac{3}{2} \rho_{2n-1} = \frac{9}{2} \rho_{2n+1} > 3 \rho_{2n+1}.
\]

**Case 2:** If \( d(q, r) \geq \sqrt{7} \rho_{2n-2} \), then by Equations 7–15 and 7–16, we have
\[
d(\Psi_n(x), \Psi_n(y)) = d(q + \frac{3}{2} \beta_{2n-1,a}, r + \frac{3}{2} \beta_{2n-1,b})
\]  
(7–17)
By the triangle inequality and Equation 7–17, we have
\[
d(q, r) \leq d(q, q + \frac{3}{2} \beta_{2n-1,a}) + d(q + \frac{3}{2} \beta_{2n-1,a}, r + \frac{3}{2} \beta_{2n-1,b}) + d(r + \frac{3}{2} \beta_{2n-1,b}, r)
\]
\[
= \| (q + \frac{3}{2} \beta_{2n-1,a}) - q \| + d(\Psi_n(x), \Psi_n(y)) + \| r - (r + \frac{3}{2} \beta_{2n-1,b}) \|
\]
\[
= \frac{3}{2} \| \beta_{2n-1,a} \| + d(\Psi_n(x), \Psi_n(y)) + \frac{3}{2} \| \beta_{2n-1,b} \|
\]  
(7–18)
\[
= \frac{3}{2} \rho_{2n-1} + d(\Psi_n(x), \Psi_n(y)) + \frac{3}{2} \rho_{2n-1}
\]
\[
= d(\Psi_n(x), \Psi_n(y)) + 3 \rho_{2n-1}.
\]
It follows from Equation 7–18 that \( d(\Psi_n(x), \Psi_n(y)) \geq d(q, r) - 3 \rho_{2n-1} \). Since \( d(q, r) \geq \sqrt{7} \rho_{2n-2} \), it follows that
\[
d(\Psi_n(x), \Psi_n(y)) \geq \sqrt{7} \rho_{2n-2} - 3 \rho_{2n-1} = 3 \sqrt{7} \rho_{2n} - 3 \sqrt{3} \rho_{2n}. \]
Since \( \rho_{2n} = \sqrt{3} \rho_{2n+1} \), it follows that
\[
d(\Psi_n(x), \Psi_n(y)) \geq 3 \sqrt{21} \rho_{2n+1} - 9 \rho_{2n+1} = 3(\sqrt{21} - 3) \rho_{2n+1} > 3 \rho_{2n+1}.
\]

**Case 3:** If \( q \neq r \) and \( d(q, r) < \sqrt{7} \rho_{2n-2} \), then \( 0 < \| q - r \| < \sqrt{7} \rho_{2n-2} \). By Lemma 7.2.17, it follows that either \( \| q - r \| = \rho_{2n-2} \) or \( \| q - r \| = \sqrt{3} \rho_{2n-2} \) or \( \| q - r \| = 2 \rho_{2n-2} \). Hence either \( d(q, r) = \rho_{2n-2} \) or \( d(q, r) = \sqrt{3} \rho_{2n-2} \) or \( d(q, r) = 2 \rho_{2n-2} \).

Since \( x \) and \( y \) are boundary lattice points of \( P(2n) \), by Lemma 6.2.14, \( q \) and \( r \) are boundary lattice points of \( P(2n - 2) \). Because the label of \( q \) is \( x_1 x_3 0 \ldots x_{2n-3} 0 \in \Lambda_{2n-2} \) and \( x_{2n-3} \neq 0 \), by Lemma 6.2.16, the boundary index of \( q \) is either 2 or 4. For a similar reason, the boundary index of \( r \) is either 2 or 4 as well. Hence, by Lemma 7.2.16, we have \( d(q, r) \neq \rho_{2n-2} \) and \( d(q, r) \neq 2 \rho_{2n-2} \). Therefore \( d(q, r) = \sqrt{3} \rho_{2n-2} \). Since \( q, r \in \)
\( B_{2n-2,2} \cup B_{2n-2,4} \), by Lemma 7.2.18, it follows that there exists \( s \in P(2n - 2) \) such that \( s \) is next to both \( q \) and \( r \) in the lattice \( L_{2n-2} \). Because \( q, r \in P(2n - 2) \subseteq L_{2n-2} \), we have \( r - q \in L_{2n-2} \). Since \( \|r - q\| = d(q, r) = \sqrt{3}\rho_{2n-2} \), it follows that the angle between the vector \( r - q \) and the positive \( x \)-axis is \( \frac{j\pi}{3} \) for some \( j \in \mathbb{Z} \). After the lattice points \( x, y, q \) and \( r \) are rotated by \( -\frac{j\pi}{3} \) about the origin, the vector \( r - q \) is horizontal. If \( s \) lies above the line connecting \( q \) and \( r \), then we continue to rotate those lattice points by \( \pi \) about the origin so that \( s \) lies below the line connecting \( q \) and \( r \). By Lemma 7.2.19, for any rotation \( R \) of \( \mathbb{R}^2 \) by \( \frac{k\pi}{3} \) for some \( k \in \mathbb{Z} \), we have 

\[
d(\Psi_n(R(x)), \Psi_n(R(y))) = d(\Psi_n(x), \Psi_n(y)).
\]

Hence, without loss of generality, we assume that the vector \( r - q \) is horizontal and \( s \) lies below the line connecting \( q \) and \( r \) as shown in Figure 7-8. By Equations 7–15 and 7–16, we have

\[
d(\Psi_n(x), \Psi_n(y)) = d(q + \frac{3}{2}b_{2n-1,a}, r + \frac{3}{2}b_{2n-1,b}) \\
= \|r - q + \frac{3}{2}(b_{2n-1,b} - b_{2n-1,a})\|. \tag{7–19}
\]

By the symmetry of \( q \) and \( r \) in Equation 7–19, without loss of generality, we also assume that \( q \) lies to the left side of \( r \) as shown in Figure 7-8. Then the two vectors \( r - q \) and \( b_{2n-1,6} \) have the same direction. Because \( \|r - q\| = d(q, r) = \sqrt{3}\rho_{2n-2} = 3\rho_{2n-1} = 3\|b_{2n-1,6}\| \), it follows that

\[
r - q = 3b_{2n-1,6}. \tag{7–20}
\]

Since \( q, r, s \in P(2n - 2) \), by Lemma 6.2.12, it follows that all blue, green and red hexagons in Figure 7-8 are Voronoi cells of \( P(2n) \). Because \( x \in B_{2n,4} \), the boundary index of \( x \) is 4. Referring to Figure 7-8, since \( x = q + b_{2n-1,a} \), it follows that \( a \in \{1, 2, 3, 4\} \). Similarly, because \( y \in B_{2n,4} \), we have \( b \in \{1, 2, 5, 6\} \). By combining Equations 7–19 and 7–20, we
have

\[ d(\Psi_n(x), \Psi_n(y)) = \|3\beta_{2n-1,6} + \frac{3}{2}(\beta_{2n-1,b} - \beta_{2n-1,a})\| = \frac{3}{2}\|2\beta_{2n-1,6} + \beta_{2n-1,b} - \beta_{2n-1,a}\|. \]

(7–21)

Figure 7-8. The lattice points \( q, r, s \in P(2n - 2) \) and the lattice points in \( q + \beta_{2n}, r + \beta_{2n}, \) and \( s + \beta_{2n}, \) where \( s \) is next to both \( q \) and \( r \) in the lattice \( L_{2n-2}. \)

If \( a = b, \) then Equations 7–21 becomes

\[ d(\Psi_n(x), \Psi_n(y)) = \frac{3}{2}\|2\beta_{2n-1,6}\| = 3\|\beta_{2n-1,6}\| = 3\rho_{2n-1} = 9\rho_{2n+1} > 3\rho_{2n+1}. \]

If \( \beta_{2n-1,b} = -\beta_{2n-1,a}, \) then Equations 7–21 becomes

\[ d(\Psi_n(x), \Psi_n(y)) = \frac{3}{2}\|2\beta_{2n-1,6} - 2\beta_{2n-1,a}\| = 3\|\beta_{2n-1,6} - \beta_{2n-1,a}\|. \]

(7–22)

Since \( \beta_{2n-1,6}, \beta_{2n-1,a} \in L_{2n-1}, \) we have \( \beta_{2n-1,6} - \beta_{2n-1,a} \in L_{2n-1}. \) Because \( a \neq 6, \beta_{2n-1,6} - \beta_{2n-1,a} \neq 0. \) Hence \( 0 \neq \beta_{2n-1,6} - \beta_{2n-1,a} \in L_{2n-1}. \) It follows that \( \|\beta_{2n-1,6} - \beta_{2n-1,a}\| \geq \rho_{2n-1}. \)

Then by Equation 7–22 we have \( d(\Psi_n(x), \Psi_n(y)) \geq 3\rho_{2n-1} = 9\rho_{2n+1} > 3\rho_{2n+1}. \)

If \( a \neq b \) and \( \beta_{2n-1,b} \neq -\beta_{2n-1,a}, \) then the two vectors \( \beta_{2n-1,a} \) and \( \beta_{2n-1,b} \) are not parallel. Since \( \|\beta_{2n-1,a}\| = \|\beta_{2n-1,b}\| = \rho_{2n-1}, \) it follows that \( \|\beta_{2n-1,a} - \beta_{2n-1,b}\| < 2\rho_{2n-1}. \) However \( \|2\beta_{2n-1,6}\| = 2\rho_{2n-1}. \) Hence \( 2\beta_{2n-1,6} \neq \beta_{2n-1,a} - \beta_{2n-1,b}, \) i.e., \( 2\beta_{2n-1,6} + \beta_{2n-1,b} - \)
\(\beta_{2n-1,a} \neq 0\). Because \(\beta_{2n-1,a}, \beta_{2n-1,b}, \beta_{2n-1,6} \in L_{2n-1}\) and \(L_{2n-1}\) is a lattice, we have 
\[2\beta_{2n-1,6} + \beta_{2n-1,b} - \beta_{2n-1,a} \in L_{2n-1}.\] 
Thus \(0 \neq 2\beta_{2n-1,6} + \beta_{2n-1,b} - \beta_{2n-1,a} \in L_{2n-1}.\) It follows that 
\[\|2\beta_{2n-1,6} + \beta_{2n-1,b} - \beta_{2n-1,a}\| \geq \rho_{2n-1}.\]

Therefore, by Equation 7–21, we have 
\[d(\Psi_n(x), \Psi_n(y)) \geq \frac{3}{2} \rho_{2n-1} = \frac{9}{2} \rho_{2n+1} > 3 \rho_{2n+1}.\] 
\[\square\]

The next corollary follows directly from Lemma 7.2.20.

**Corollary 7.2.21.** For any \(n \in \mathbb{N}\), \(\Psi_n\) is a one to one map from \(\mathcal{B}_{2n,4}\) to \(\mathcal{P}_B\).

**Theorem 7.2.22.** The fractal dimension of \(\mathcal{P}_B\) is \(\frac{\ln 4}{\ln 3}\), where \(\mathcal{P}_B\) is the boundary of the limit of the Pyxis structure.

**Proof.** For any \(n \in \mathbb{N}\), let \(\epsilon_n = \frac{3}{2} \rho_{2n+1}\). Since \(\rho_{2n+1} = \frac{1}{3^n} \rho_1\), we have \(\epsilon_n = C r^n\) where \(C = \frac{3^n}{2}\) and \(r = \frac{1}{3}\). By Lemma 7.2.7, \(\mathcal{P}_B\) can be covered by those closed balls with radius \(\epsilon_n\) and centered at boundary lattice points of \(P(2n)\). By Lemma 7.2.3, it follows that 
\[\mathcal{N}(\mathcal{P}_B, \epsilon_n) \leq 4^n + 4,\] 
where \(\mathcal{N}(\mathcal{P}_B, \epsilon_n)\) is the smallest number of closed balls of radius \(\epsilon_n\) needed to cover \(\mathcal{P}_B\).

Recall that \(\mathcal{B}_{2n,4}\) is the set consisting of boundary lattice points of \(P(2n)\) each having boundary index 4. Let \(\tilde{\mathcal{B}}_{2n,4} = \{ \Psi_n(q) : q \in \mathcal{B}_{2n,4}\}\). By Corollary 7.2.21, the map \(\Psi_n\) is one to one. It follows that \(|\tilde{\mathcal{B}}_{2n,4}| = |\mathcal{B}_{2n,4}|\). By Lemma 7.2.3, we have \(|\mathcal{B}_{2n,4}| = 4^n + 2\).

Hence \(|\tilde{\mathcal{B}}_{2n,4}| = 4^n + 2\). By Lemma 7.2.20, for any \(x, y \in \mathcal{B}_{2n,4}\) with \(x \neq y\), we have 
\[d(\Psi_n(x), \Psi_n(y)) > 3 \rho_{2n+1} = 2 \epsilon_n.\] 
Since \(|\tilde{\mathcal{B}}_{2n,4}| = 4^n + 2\), it follows that 
\[\mathcal{N}(\tilde{\mathcal{B}}_{2n,4}, \epsilon_n) \geq 4^n + 2,\] 
where \(\mathcal{N}(\tilde{\mathcal{B}}_{2n,4}, \epsilon_n)\) is the smallest number of closed balls of radius \(\epsilon_n\) needed to cover \(\tilde{\mathcal{B}}_{2n,4}\).

By Lemma 7.2.15, we have \(\tilde{\mathcal{B}}_{2n,4} \subseteq \mathcal{P}_B\). It follows that 
\[\mathcal{N}(\tilde{\mathcal{B}}_{2n,4}, \epsilon_n) \leq \mathcal{N}(\mathcal{P}_B, \epsilon_n).\] 
Hence 
\[\mathcal{N}(\mathcal{P}_B, \epsilon_n) \geq 4^n + 2.\]

We have shown that \(4^n + 2 \leq \mathcal{N}(\mathcal{P}_B, \epsilon_n) \leq 4^{n+1} - 4\) for any \(n \in \mathbb{N}\). It follows that 
\[
\frac{\ln(4^n + 2)}{\ln(1/\epsilon_n)} \leq \frac{\ln(\mathcal{N}(\mathcal{P}_B, \epsilon_n))}{\ln(1/\epsilon_n)} \leq \frac{\ln(4^{n+1} - 4)}{\ln(1/\epsilon_n)}. \tag{7–23}
\]
Since \( \epsilon_n = C r^n = \frac{C}{d^n} \), we have

\[
\lim_{n \to \infty} \frac{\ln(4^n + 2)}{\ln(1/\epsilon_n)} = \lim_{n \to \infty} \frac{\ln(4^n + 2)}{\ln(3^n/C)} = \lim_{n \to \infty} \frac{n \ln 4 + \ln(1 + 2(4^{-n}))}{n \ln 3 - \ln C} = \frac{\ln 4}{\ln 3}.
\]

Similarly we can show that \( \lim_{n \to \infty} \frac{\ln(4^n + 1 - 4)}{\ln(1/\epsilon_n)} = \frac{\ln 4}{\ln 3} \). Hence, by Inequality 7–23, we have

\[
\lim_{n \to \infty} \frac{\ln(\mathcal{N}(\mathcal{P}_B, \epsilon_n))}{\ln(1/\epsilon_n)} = \frac{\ln 4}{\ln 3}.
\]

By Theorem 7.2.1, it follows that the fractal dimension of \( \mathcal{P}_B \) is \( \frac{\ln 4}{\ln 3} \). \( \square \)
CHAPTER 8
SUMMARY OF THIS RESEARCH

We have provided a systematic mathematical theory for computing the DFT on a lattice and studied the DFT on some important hexagonal array structures such as the regular hexagonal structure and the Pyxis structure. It is shown that each array of the type $A$ (type $B$) RHS is a set of coset representatives of the quotient group of two hexagonal lattices. Hence the DFT on such an array is amenable to the standard DFT. We developed an efficient method for computing the DFT on these arrays and have shown the relation between these arrays and some previously studied hexagonal arrays.

The Pyxis structure consists of a sequence of hexagonal arrays with a scheme for labeling the lattice points of the arrays. It originated with PYXIS Innovation Inc. and is applied in a digital earth reference model (Zheng et al. [50]). Developing algebra and DFT for the Pyxis structure, computing the fractal dimension of the Pyxis structure, and developing algorithms for the addition of Pyxis labels are research topics proposed by the Pyxis Innovation Inc. We have provided a recursive definition of the Pyxis structure and an algebraic method to label the lattice points of the Pyxis structure. Based on the recursive definition and labeling of the Pyxis structure, we have developed an algorithm with computational complexity of order $n$ to add any two labels of length $n$. Also we have shown that $P(n)$, the $n^{th}$ level of the Pyxis structure, does not tile the underlying hexagonal lattice by translations by a sublattice for any $n > 2$. Hence the DFT on $P(n)$ can not be evaluated using the method in Chapter 2. Furthermore, we have shown that the limit of the Pyxis structure in the space of non empty compact subsets of the two dimensional Euclidean space with Hausdorff metric exists, and the fractal dimension of its boundary is $\frac{\ln 4}{\ln 3}$.

The methods for computing the DFT on hexagonal arrays developed in this research can be applied to many general periodically sampled images. As shown in Wikipedia [45], the DFT is widely employed in signal processing and related fields to analyze the
frequencies contained in a sampled signal, to solve partial differential equations, and to perform other operations such as convolutions. Since hexagonal grids have some advantages over the usual square grids in several application domains, our research has promising applications.
REFERENCES


BIOGRAPHICAL SKETCH

Xiqiang Zheng was born in Jiangxi Province of China. He received his Master of Science in mathematics from Jiangxi University in January of 1991. From 1991 to 1999, he taught at Nanchang Institute of Aeronautical Technology, China. Before he came to University of Florida for his PhD degree, he studied at Bowling Green State University for one year. He will receive his PhD degree on applied mathematics in December of 2007 and his research interests include applied mathematics, image processing, and pattern recognition.