DISCRETE GROUPS FROM A COARSE PERSPECTIVE

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To the memory of Dawn Rogers Smith
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We first use the functorial properties of coarse structures and coarse maps as well as algebraic properties to deduce that the asymptotic dimension of the Grigorchuk group is infinity. Next, by thinking in terms of coarse invariants rather than quasi-isometry invariants, we are able to extend the notion of a word metric to all countable groups. We describe nice relations between the asymptotic dimension of the countable group and the asymptotic dimensions of its finitely generated subgroups. Finally, using the so-called sublinear coarse structure, we are able to show that, for a large class of proper metric spaces including finitely generated groups, the asymptotic Assouad-Nagata dimension can be realized as the dimension of the sublinear Higson corona. Using this, we prove that crossing a space with the real numbers lifts the asymptotic Assouad-Nagata dimension by one.
Introduction. An early example of neglecting local properties in favor of large-scale geometry is Legendre’s attempted proof of the parallel axiom from Euclidean geometry. Attempting to show a contradiction in assuming the existence of a triangle whose angles sum to less than $\pi$, he constructed larger and larger triangles whose angle sums decreased at a steady rate. Since this process could not go on indefinitely, he concluded that such triangles could not exist, and consequently that the parallel axiom must hold. The argument is sketched in more detail in [23] and [16]. Though the argument is ultimately incorrect, it invites one to further investigate the large-scale properties of spaces.

One of the first invariants of coarse topology was the notion of the ends of a space, introduced by Freudenthal in 1931. Hopf investigated the ends of finitely generated groups, showing among other things that such groups have 0, 1, 2, or infinitely many ends. Stallings later gave an alternate characterization of groups with infinitely many ends.

In this paper we will also investigate coarse and quasi-isometry invariants of finitely generated (and countable) groups. The quasi-isometry invariants we will look at are the asymptotic dimension and the asymptotic Assouad-Nagata dimension.

The notion of asymptotic dimension was introduced by Gromov in order to study large-scale invariants of discrete groups [19]. A striking application of asymptotic dimension came when G. Yu proved that, for certain finitely generated groups with finite asymptotic dimension, the Higher Novikov Signature Conjecture holds [26]. Other conjectures were proved for spaces with finite asymptotic dimension (under various other conditions), including the coarse Baum-Connes conjecture [26], the K-theoretic integral Novikov conjecture [9, 10], and the integral Novikov conjecture [11]. Based on these results, it is important to have techniques for determining if a space has finite asymptotic dimension. One technique for obtaining results about finite dimensionality is to try to prove theorems analogous to those found in classical dimension theory. For instance, it has been shown that the product and (finite) union of spaces with finite
asymptotic dimension also have finite asymptotic dimension. Included among spaces of finite asymptotic dimension are free groups, free abelian groups of finite rank, hyperbolic groups, and Coxeter groups.

There are two recurrent themes in this treatise. One is the use of coarse structures in the investigation of large-scale dimensions on discrete groups. The other is the search for techniques to find lower bounds on the asymptotic dimension of the space. The Grigorchuk group example is an easy instance of this, while the product formula given towards the end is a more difficult one. A specific problem which motivated some of the work in this paper, particularly the work with the sublinear coarse structure, is the following: for a finitely generated group $G$, is it true that $\text{asdim} G \times \mathbb{Z} = \text{asdim} G + 1$ holds? It is known that this formula does not hold for all proper metric spaces, i.e. there is a proper metric space $X$ with $\text{asdim} X \times \mathbb{Z} = \text{asdim} X$ [12]. Though this question is yet to be resolved, we do present a partial result. We prove that $\text{AN-asdim} G \times \mathbb{Z} = \text{AN-asdim} G + 1$ for finitely generated groups $G$. One can indeed think of this as a partial result, since Dydak et al. [6] showed that, given a finitely generated group $G$, there is a left-invariant proper metric $d'$ on $G$ such that $\text{asdim} G = \text{AN-asdim}(G, d')$.

**Large-scale dimensions.** We start by defining three dimensions, each defined for metric spaces, which will occupy our attention throughout this treatise. The difference between these definitions is the restriction placed on the obtained covers. We say that a family $\mathcal{V}$ of subsets of $X$ is $R$–disjoint if $d(U, V) \geq R$ for all $U, V \in \mathcal{V}$ with $U \neq V$. $\mathcal{V}$ is said to be $S$–bounded if $\text{diam} V \leq S$ for all $V \in \mathcal{V}$, and is said to be uniformly bounded if it is $S$–bounded for some $S$.

**Definition 1.1.** [19] A metric space $(X, d)$ is said to have asymptotic dimension $\leq n$, denoted $\text{asdim}(X, d) \leq n$, if for each $r > 0$, there is an $S > 0$ and $r$–disjoint, $S$–bounded families $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ of subsets of $X$ such that $\mathcal{U} := \cup_i \mathcal{U}_i$ is a cover of $X$.

Note that, as will be the case for the following dimensions, we say that $\text{asdim} X = n$ if $\text{asdim} X \leq n$ yet $\text{asdim} X \leq n - 1$ does not hold. If $\text{asdim} X \leq n$ fails to be true for each
n \geq 0$, then we say \( \text{asdim } X = \infty \). Also, we simply write \( \text{asdim } X \) rather than \( \text{asdim}(X, d) \) if the metric is understood.

**Definition 1.2.** A metric space \((X, d)\) is said to have asymptotic Assouad-Nagata dimension \( \leq n \), denoted \( \text{AN-asdim } X \leq n \), if there exists a \( C > 0 \) and a \( D \geq 0 \) such that, for each \( r > 0 \), there are \( r \)-disjoint, \((Cr + D)\)-bounded families \( \mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n \) of subsets of \( X \) such that \( \bigcup_i \mathcal{U}_i \) is a cover of \( X \).

**Definition 1.3.** A metric space \((X, d)\) is said to have Assouad-Nagata dimension \( \leq n \), denoted \( \text{dim } AN X \leq n \), if there exists a \( C > 0 \) such that, for each \( r > 0 \), there are \( r \)-disjoint, \( Cr \)-bounded families \( \mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n \) of subsets of \( X \) such that \( \bigcup_i \mathcal{U}_i \) is a cover of \( X \).

This construction was used by Assouad in [1] to resolve a question of Nagata. Many properties of \( \text{asdim} \) have been verified for this asymptotic Assouad-Nagata dimension and the Assouad-Nagata dimension. For instance, it was shown in [6] that \( \text{AN-asdim } X \times Y \leq \text{AN-asdim } X + \text{AN-asdim } Y \) for metric spaces \( X \) and \( Y \). They also proved a similar result for \( \text{dim } AN \), which was also proved in [22]. Also, we note that \( \text{asdim } X \leq \text{AN-asdim } X \leq \text{dim } AN X \). For more details, see the papers of Lang and Schlichenmaier [22], Buyalo and Lebedeva [7], [8], and Dydak, Brodskiy, Levin, Mitra, and Higes [5],[6].

These definitions can be formulated in many ways. In the case of the asymptotic Assouad-Nagata dimension, the following proposition gives some alternative characterizations.

**Proposition 1.4.** Let \((X, d)\) be a metric space. The following are equivalent.

1. \( \text{AN-asdim } X \leq n \);

2. there exists a \( C > 0 \) and an \( r_0 > 0 \) such that, for each \( r \geq r_0 \), there are \( r \)-disjoint, \( C r \)-bounded families \( \mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n \) of subsets of \( X \) such that \( \bigcup_i \mathcal{U}_i \) is a cover of \( X \);

3. there is a \( C > 0 \) and an \( \epsilon_0 > 0 \) such that for all \( \epsilon \leq \epsilon_0 \) (\( \epsilon > 0 \)), there is an \( \epsilon \)-Lipschitz, \( C/\epsilon \)-cobounded map \( p : X \to P \) to an \( n \)-dimensional simplicial complex \( P \).
4. There is a $C > 0$ and an $r_0 > 0$ such that, for all $r \geq r_0$, there is a cover $U$ of $X$ such that $\text{mesh} U \leq Cr$, $\text{mult} U \leq n + 1$, and $L_U > r$;

5. There is a $C > 0$ and an $r_0 > 0$ such that for all $r \geq r_0$, there is a cover $U$ of $X$ such that $\text{mesh} U \leq Cr$ and $B_r(x)$ meets at most $n + 1$ elements of $U$ for each $x \in X$.

By $\text{mult} U \leq m$ we mean that every intersection of $m + 1$ distinct elements of $U$ is empty. Also, $\text{mesh} U = \sup_{U \in \mathcal{U}} \text{diam}(U)$. Here $L_U$ denotes the Lebesgue number of the cover, $L_U = \inf \{ \sup_{U \in \mathcal{U}} d(x, X \setminus U) \mid x \in X \}$. The relevant definitions (for the third statement, in particular) can be found in either [2], [3], [4], or [15]. This proposition can be modified to obtain statements for the asymptotic dimension as well as the Assouad-Nagata dimension.

**Word Metrics.** We will focus mostly on proper metric spaces in this treatise.

Recall that a proper metric space is a metric space for which closed, bounded sets are compact. Also, a metric $d$ on a group $G$ is said to be left-invariant if $d(fg, fh) = d(g, h)$ for all $f, g, h \in G$. More generally, a group acts on a metric space by isometries if $d(fx, fy) = d(x, y)$ for all $f \in G, x, y \in X$.

Let $G$ be a group. We recall that a map $\| \cdot \| : G \to [0, \infty)$ is said to be a norm on $G$ if $\|x^{-1}\| = \|x\|$ for all $x \in G$, $\|x\| = 0$ if and only if $x = e$, and $\|xy\| \leq \|x\| + \|y\|$ for all $x, y \in G$. Norms on a group $G$ yield left-invariant metrics: given a norm $\| \cdot \|$ on $G$, define a metric by $d(x, y) = \|x^{-1}y\|$. We call a norm proper if it is proper as a map (where $G$ has the topology induced by the associated metric). Note that proper norms correspond to proper metrics.

Let $G$ be a finitely generated group with finite generating set $S$. We define $\|x\| = \inf \{ n \mid x = \gamma_1 \gamma_2 \cdots \gamma_n, \gamma_i \in S \cup S^{-1} \}$. $\| \cdot \|$ is a proper norm. If we define $d_S(x, y) = \|x^{-1}y\|$, then $d_S$ is a left-invariant, proper metric on $G$, called the word metric on $G$ associated with $S$. It should perhaps be explicitly mentioned here that finitely generated groups with word metric are discrete, and hence proper here means that bounded sets are finite. On finitely generated groups, word metrics form an important source of left-invariant, proper metrics.
Definitions for Metric Spaces. Most of the definitions to follow in this chapter can be found in [23]. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. We say that a map \(f : X \to Y\) is \((metrically)\) proper if the preimage of each bounded set is bounded. It is bornologous if, for each \(R > 0\), there is an \(S > 0\) such that \(d_Y(f(x), f(y)) < S\) whenever \(d_X(x, y) < R\). A map is called coarse if it is both proper and bornologous. If \(S\) is a set, then \(f, g : S \to X\) are said to be close if \(\sup_{s \in S} d(f(s), g(s)) < \infty\).

A coarse map \(f : X \to Y\) is said to be a coarse equivalence if there is a coarse map \(g : Y \to X\) such that \(g \circ f\) is close to \(\text{id}_X\) and \(f \circ g\) is close to \(\text{id}_Y\). A coarse map \(f : X \to Y\) is said to be a coarse embedding if \(f : (X, d_X) \to (f(X), d_Y|_{f(X)})\) is a coarse equivalence. In particular, an isometric embedding is a coarse embedding. Also, if \(f_i : (X_i, d_{X_i}) \to (Y_i, d_{Y_i})\) \((i = 1, 2)\) are coarse equivalences, then so is \(f_1 \times f_2 : (X_1 \times X_2, \delta_1) \to (Y_1 \times Y_2, \delta_2)\), where \(\delta_1\) and \(\delta_2\) are the corresponding sum metrics.

**Proposition 1.5.** Asymptotic dimension is a coarse invariant, i.e. coarsely equivalent spaces have the same asymptotic dimension.

**Proof.** See Chapter 9 of [23].

We recall that a mapping \(f : X \to Y\) between metric spaces is said to be a quasi-isometry if there are numbers \(\lambda > 0, C \geq 0,\) and \(D \geq 0\) such that

\[
\frac{1}{\lambda} d(x, y) - C \leq d(f(x), f(y)) \leq \lambda d(x, y) + C
\]

and every point of \(Y\) is within distance \(D\) of \(\phi(X)\). It is not difficult to see that a quasi-isometry is a coarse equivalence. Also, for a finitely generated group \(G\) with generating set \(S\), the isometric embedding \(G \hookrightarrow C(G, S)\) of the group with word metric into its Cayley graph (here equipped with a geodesic metric such that each edge has length one) is a quasi-isometry.

Since the word metrics associated with any two finite generating sets of a finitely generated group are coarsely equivalent (even quasi-isometric), we regard asymptotic
dimension as a group invariant (for finitely generated groups). In section §3 we extend these notions to all countable groups.

**Coarse Structures.** Let $X$ be a set. For $E, F \subset X \times X$, we define the product of $E$ and $F$, denoted $E \circ F$, by $E \circ F = \{(x, z) \in X \times X \mid \exists y \text{ s.t. } (x, y) \in E, (y, z) \in F\}$. For $E \subset X \times X$, we define the inverse of $E$ to be $E^{-1} = \{(y, x) \mid (x, y) \in E\}$. For $E \subset X \times X$ and $K \subset X$, we define $E[K] = \{x \in X \mid (x, y) \in E \text{ and } y \in K\}$. Also, for $x \in X$, define $E_x = E[\{x\}]$ and $E^x = E^{-1}[\{x\}]$; these are known as $E$–balls in $X$ (see the example below for the reason for the terminology).

**Definition 1.6.** For a set $X$, a coarse structure $\mathcal{E}$ on $X$ is a collection of subsets of $X \times X$ such that the diagonal $\Delta$ belongs to $\mathcal{E}$ and $\mathcal{E}$ is closed under the formation of subsets, finite unions, products, and inverses. The elements of $\mathcal{E}$ are called controlled sets. Also, $(X, \mathcal{E})$ (that is, a set $X$ equipped with a coarse structure $\mathcal{E}$) is called a coarse space.

**Example 1.7.** For a metric space $(X, d)$, let $\mathcal{E}$ consist of all those subsets $E$ of $X \times X$ for which $d(E) \subset \mathbb{R}$ is bounded (here $\mathbb{R}$ has the usual metric, and we have applied the distance map $d : X \times X \to \mathbb{R}$ to the set $E \subset X \times X$); this is a coarse structure, known as the bounded coarse structure. In particular, for a finitely generated group $G$, since any two word metrics on $G$ are equivalent, we have that the bounded coarse structure does not depend on the choice of the word metric. Also, if $r > 0$, and we define $E = \{(x, y) \mid d(x, y) < r\}$, then we have $E_x = E^x = B_r(x)$; hence the terminology $E$–ball. Indeed, $E[K] = N_r(K)$, the $r$–neighborhood of the set $K$.

For a coarse space $(X, \mathcal{E})$, we say that $B \subset X$ is bounded if $B \times B \in \mathcal{E}$. We say the coarse space is coarsely connected if $\{x, y\}$ is bounded for any pair of points $x, y \in X$. We say that a map $f : (X, \mathcal{E}) \to (Y, \mathcal{F})$ is proper if the preimage of a bounded set is bounded; it is bornologous if $f \times f$ takes controlled sets to controlled sets; a map is called coarse if it is both proper and bornologous. If $S$ is a set and $X$ is a coarse space (with coarse structure $\mathcal{E}$), then $f, g : S \to X$ are said to be close if $\{(f(x), g(x)) : x \in X\}$ is a controlled set. A coarse map $f : (X, \mathcal{E}) \to (Y, \mathcal{F})$ is said to be a coarse equivalence if there is a coarse...
map $g : (Y, \mathcal{F}) \to (X, \mathcal{E})$ such that $g \circ f$ is close to $\text{id}_X$ and $f \circ g$ is close to $\text{id}_Y$. A coarse map $f : (X, \mathcal{E}) \to (Y, \mathcal{F})$ is said to be a coarse embedding if $f : (X, \mathcal{E}) \to (f(X), \mathcal{F}_Y)$ is a coarse equivalence (here $\mathcal{F}_Y = \{E \in \mathcal{E} : E \subset Y \times Y\}$). When dealing with metric spaces and the bounded coarse structure, our definitions for coarse structures revert to the definitions for metric spaces given above.

For a coarse structure $\mathcal{E}$ on $X$ and a coarse structure $\mathcal{F}$ on $Y$, define a product structure on $X \times Y$ by $\mathcal{E} \times \mathcal{F} = \{E \subset (X \times Y) \times (X \times Y) : \pi_X(E) \in \mathcal{E}, \pi_Y(E) \in \mathcal{F}\}$, where $\pi_X : (X \times Y)^2 \to X^2$ is the projection (similarly for $Y$). Also, if $f_i : (X_i, \mathcal{E}_i) \to (Y_i, \mathcal{F}_i)$ ($i = 1, 2$) are coarse equivalences, then so is $f_1 \times f_2 : (X_1 \times X_2, \mathcal{E}_1 \times \mathcal{E}_2) \to (Y_1 \times Y_2, \mathcal{F}_1 \times \mathcal{F}_2)$.

Suppose that $X$ is a topological space. $E \subset X \times X$ is said to be proper if, for each relatively compact set $K$, both $E[K]$ and $E^{-1}[K]$ are relatively compact. We say that a coarse space $(X, \mathcal{E})$ is consistent with the topology on $X$ if it has the property that $B \subset X$ is (coarsely) bounded if and only if $B$ is relatively compact (i.e., bounded sets coincide with relatively compact sets). One can easily show a consistent coarse space $(X, \mathcal{E})$ is coarsely connected and each $E \in \mathcal{E}$ is proper.

**Definition 1.8.** Let $X$ be a topological space, and suppose that $\mathcal{E}$ is a coarse structure which is consistent with the topology on $X$. We say that $f : X \to \mathbb{C}$ is a Higson function, denoted $f \in C_h(X, \mathcal{E})$, if for every $E \in \mathcal{E}$ and every $\epsilon > 0$, there is a compact set $K$ such that $|f(x) - f(y)| < \epsilon$ whenever $(y, x) \in E \setminus K \times K$. The Higson functions form a $C^*$-algebra, and so by the Gelfand-Naimark-Segal (GNS) theorem there is a compactification $h_\mathcal{E}X$ of $X$ called the Higson compactification such that the algebra of Higson functions $C_h(X, \mathcal{E})$ is isomorphic to $C(h_\mathcal{E}X)$. The Higson corona is defined by $\nu_\mathcal{E}X = h_\mathcal{E}X \setminus X$.

The following theorem can be inferred from [23].

**Proposition 1.9.** Let $X$ and $Y$ be locally compact, Hausdorff spaces equipped with coarse structures $\mathcal{E}$ and $\mathcal{F}$, respectively, which are consistent with the topologies. If $f : X \to Y$ is
a coarse, continuous map, then \( f \) extends to a continuous map \( \overline{f} : h_\mathcal{E}X \to h_\mathcal{F}Y \) such that \( \overline{f}(\nu_\mathcal{E}X) \subset \nu_\mathcal{F}Y \).

For a locally compact, Hausdorff space \( X \) equipped with a coarse structure \( \mathcal{E} \) that is consistent with the topology, then we say that \((X, \mathcal{E})\) is a proper coarse space if there is a controlled neighborhood of the diagonal of \( X \times X \). The following is Proposition 4.1 of \([23]\).

**Proposition 1.10.** Let \( X \) and \( Y \) be proper coarse spaces. A coarse map \( \phi : X \to Y \) extends to a continuous map \( \nu\phi : \nu X \to \nu Y \). If \( \phi, \psi : X \to Y \) are close then \( \nu\phi = \nu\psi \).

If additionally \( f \) is a coarse equivalence, then \( \nu(f) \) is a homeomorphism.

All of the coarse structures in this treatise will be proper coarse structures, and will be founded upon a proper metric space.

**Compactifications.** Let \( \bar{X} \) be a compactification of a locally compact space \( X \), and let \( V \) be an open subset of \( X \). Then there is a unique maximal open set \( \tilde{V} \) in \( \bar{X} \) such that \( \tilde{V} \cap X = V \). In fact, \( \tilde{V} = \bar{X} \setminus X \setminus V \). One can show that \( \tilde{V} \subset V \).

The following propositions are not difficult. Further details can be found in \([15]\).

**Proposition 1.11.** Let \( X \) be a compactification of a locally compact space \( X \), and let \( \nu X = \bar{X} \setminus X \). Then \( \{ \tilde{V} \cap \nu X : V \text{ is open in } X \} \) forms a basis for \( \nu X \).

**Proposition 1.12.** Let \( \bar{X} \) be a compactification of a locally compact space \( X \), and let \( \nu X = \bar{X} \setminus X \). Suppose \( U \) is an open subset of \( \nu X \) and \( x \in U \). Then there is a set \( V \) which is open in \( X \), \( x \in \tilde{V} \cap \nu X \), and \( \tilde{V} \cap \nu X \subset U \).
CHAPTER 2
THE GRIGORCHUK GROUP

The first Grigorchuk group is described in [17], [18], and [21]. This Grigorchuk group, which we will denote by \( \Gamma \), has many interesting properties. It is a finitely generated 2–group with intermediate growth, whose word problem is solvable, and which does not admit a finite dimensional linear representation that is faithful. Also, \( \Gamma \) and \( \Gamma \times \Gamma \) are commensurable, which means that \( \Gamma \) and \( \Gamma \times \Gamma \) have subgroups of finite index which are isomorphic. A detailed exposition can be found in [21].

We prove that \( \Gamma \) has asymptotic dimension infinity, asdim\( \Gamma = \infty \). If one excludes Gromov’s “random groups” [20], all previously known examples of groups \( G \) with asdim\( G = \infty \) are based on the fact that \( G \) has a free abelian subgroup of arbitrary large rank. The Grigorchuk group is of different nature: since \( \Gamma \) is a 2–group, it does not have a (nontrivial) free abelian subgroup.

When \( G \) and \( H \) are groups equipped with word metrics \( d_G \) and \( d_H \), then the sum metric \( d_G + d_H \) on \( G \times H \) is also a word metric (for the natural generating set). If \( H \leq G \) is a subgroup of finite index of (the finitely generated group) \( G \), then the inclusion map \( H \to G \) is a coarse equivalence. Also, an isomorphism is a coarse equivalence.

**Definition 2.1.** Two groups \( \Gamma_1 \) and \( \Gamma_2 \) are commensurable if there exist subgroups \( H_1 \leq \Gamma_1 \) and \( H_2 \leq \Gamma_2 \), each of finite index, such that \( H_1 \) and \( H_2 \) are isomorphic.

By the comments above, \( \text{asdim} \Gamma_1 = \text{asdim} \Gamma_2 \) if \( \Gamma_1 \) and \( \Gamma_2 \) are commensurable.

**Theorem 2.2.** [25] Let \( G \) be a finitely generated, infinite group which is commensurable with its square \( G \times G \). Then \( \text{asdim} G = \infty \).

**Proof.** We first show that \( G^n \) is coarsely equivalent to \( G \) for all \( n \geq 1 \). Proceeding inductively (the \( n = 1 \) case is immediate), we assume \( G^n \) is coarsely equivalent to \( G \). But \( G^{n+1} \) is coarsely equivalent to \( G^n \times G \), which in turn is coarsely equivalent to \( G \times G \), and so by hypothesis \( G^{n+1} \) is equivalent to \( G \). This proves that \( \text{asdim} G^n = \text{asdim} G \) for all \( n \geq 1 \).
Also, by Exercise IV.A.12 of [21], there is an isometric embedding $f : \mathbb{Z} \to G$, where $G$ is taken with a word metric. Thus, for each $n \geq 1$, we have an isometric embedding $f \times f \times \cdots \times f : \mathbb{Z}^n \to G^n$, where we take the sum metrics on $\mathbb{Z}^n$ and $G^n$. Since an isometric embedding is a coarse embedding, we have that $\operatorname{asdim} G^n \geq \operatorname{asdim} \mathbb{Z}^n = n$. Thus, $\operatorname{asdim} G \geq n$ for all $n$.

**Corollary 2.3.** Let $\Gamma$ be the Grigorchuk group. Then $\operatorname{asdim} \Gamma = \infty$.

**Proof.** $\Gamma$ is finitely generated by definition. Proposition VIII.14 and Corollary VIII.15 from [21] show that $\Gamma$ satisfies the hypotheses of the theorem.

It is interesting to note that $\operatorname{asdim} \Gamma = \infty$, yet $\Gamma$ does not contain an isomorphic copy of $\mathbb{Z}^n$. However, $\mathbb{Z}^n$ does coarsely embed into $\Gamma$. Also, if one has a finitely generated group which is known to be commensurable with its square, then the asymptotic dimension is either 0 or infinity, depending on whether the group is finite or infinite.
CHAPTER 3
COUNTABLE GROUPS

Below we show that every countable group admits a left-invariant, proper metric. In view of Proposition 3.1, one can extend the invariant asdim to all countable groups (not necessarily finitely generated). More explicitly, for a countable group $G$, we define $\text{asdim} G = \text{asdim}(G, d)$, where $d$ is a left-invariant, proper metric on $G$. Propositions 3.1 and 3.3 show that asdim$G$ is well-defined. For countable groups, note that a left-invariant, proper metric induces the discrete topology (Baire Category Theorem); consequently, bounded sets are finite. The following appears in [23, Proposition 1.15], [14, Proposition 1.1], and [24, Proposition 1].

**Proposition 3.1.** For a countable group, any two left-invariant, proper metrics are coarsely equivalent.

**Definition 3.2.** Let $\Gamma$ be a countable discrete group. Let $S$ be a symmetric generating set (possibly infinite), $S = S^{-1}$, for $\Gamma$. A *weight function* $w : S \to [0, \infty)$ on $S$ is any positive, proper function such that $w(s^{-1}) = w(s)$ for all $s \in S$. The properness can essentially be viewed as the requirement that $\lim w = \infty$.

It is not hard to see that for any countable group $\Gamma$, there is a weight function. In fact, for any symmetric generating set $S$, there is a weight function with domain $S$.

**Proposition 3.3.** [14, Proposition 1.3], [24, Theorem 1] A weight function on the countable group $\Gamma$ induces a proper norm $\| \cdot \|$ given by

$$\| x \| = \inf \left\{ \sum_{i=1}^{n} w(s_i) | x = s_1 s_2 \cdots s_n, s_i \in S \right\},$$

and so a weight function induces a left-invariant, proper metric $d$.

**Proof.** Given a weight function $w : S \to [0, \infty)$, where $S$ is a generating set for the countable group $\Gamma$, define $\| x \| = \inf \left\{ \sum_{i=1}^{n} w(s_i) | x = s_1 s_2 \cdots s_n, s_i \in S \right\}$. Note that if we view $1_\Gamma$ as an empty product, $\| 1_\Gamma \| = 0$. The proof that $\| \cdot \|$ is a norm is left to the reader.
Let $R > 0$ be given. Let $r$ be a nonzero value that the weight function assumes. So 
\{s \in S | 0 < w(s) \leq r\} is nonempty and finite by definition. Thus, there is a $t \in S$ such 
that $w(t) = \min\{w(s) | s \in S, 0 < w(s) \leq r\}$. It is immediate that $0 < w(t) \leq w(s)$ for 
all $s \in S \setminus 1 \Gamma$. Now, suppose $x$ is such that $\|x\| \leq R$ and $x \neq 1 \Gamma$. Then $\|x\| < R + 1$. So 
there are $s_1, s_2, \ldots, s_n \in S$ such that $x = s_1 s_2 \cdots s_n$ and $\sum_{i=1}^n w(s_i) < R + 1$. Further, 
we may assume that $s_i \neq 1 \Gamma$ for each $i$. Thus, $s_i \in \{s \in S | w(s) \leq R + 1\}$ for all $i$.

Also, $R + 1 > \sum_{i=1}^n w(s_i) \geq nw(t)$, so that $n < (R + 1)/w(t)$. Thus, $x$ is an element of 
$\{t_1 t_2 \cdots t_m | t_i \in S \setminus 1 \Gamma, w(t_i) \leq R + 1, m < (R + 1)/w(t)\}$, a finite set. This shows that 
$\{x \mid \|x\| \leq R\}$ is finite. 

Note that the infimum in the definition of $\|x\|$ is actually a minimum. To see this, 
simply modify the argument in the last paragraph to show that the set of elements of 
$\{\sum_{i=1}^n w(s_i) | x = s_1 s_2 \cdots s_n, s_i \in S\}$ less than $\|x\| + 1$ is a finite set.

The following theorem gives a necessary and sufficient condition for a countable group 
to have asymptotic dimension zero. This condition relies only on the algebraic structure of 
the group. The following (and its corollaries) can be found in [24].

**Theorem 3.4.** Let $G$ be a countable group. Then $\operatorname{asdim} G = 0$ if and only if every finitely 
generated subgroup of $G$ is finite.

**Proof.** Let $w : G \to [0, \infty)$ be a weight function on the generating set $G$. Let $\| \cdot \|$ and $d$ be 
the induced norm and metric, respectively.

First suppose that $\operatorname{asdim} G = 0$. Let $T \subset G$ be a finite set. Take $d > \max_{g \in T} \|g\|$.

As $\operatorname{asdim} G = 0$, there is a $d$–disjoint, uniformly bounded cover $\mathcal{U}$ of $G$. Choose $U \in \mathcal{U}$ 
with $1 \in U$. We will show that $\langle T \rangle \subset U$. To do this, we will show by induction that 
every product of $k$ ($k \geq 0$) elements of $T \cup T^{-1}$ lies in $U$. This is true for $k = 0$, as 
$1 \in U$. Now suppose it is true for $k - 1$, $k \geq 1$. Consider $x = t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_k^{\epsilon_k}$, where $t_i \in T$ 
and $\epsilon_i \in \{\pm 1\}$. Set $y = t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_{k-1}^{\epsilon_{k-1}}$. By the induction assumption, $y \in U$. Since 
$d(y, x) = \|y^{-1}x\| = \|t_k^{\epsilon_k}\| = \|t_k\| < d$, and because $\mathcal{U}$ is a $d$–disjoint cover, we must have
Thus, each product of \( k \) elements of \( T \cup T^{-1} \) lies in \( U \). Therefore, \( \langle T \rangle \subset U \). As \( U \) is uniformly bounded, \( U \) is bounded, and so \( U \) and \( \langle T \rangle \) are finite.

Conversely, suppose every finitely generated subgroup of \( G \) is finite. Let \( d > 0 \) be given. Define \( T = \{ s \in G | w(s) < d \} \) and \( H = \langle T \rangle \). By definition of weight function, \( T \) is finite. By our assumption, \( H \) is finite as well. Let \( U = \{ gH | g \in G \} \) be the collection of left cosets. So \( U \) is a uniformly bounded cover, as multiplication on the left by a fixed element is an isometry of \( G \). Further, suppose \( gH \neq hH \). Let \( x \in gH \) and \( y \in hH \). It follows that \( y^{-1} x \notin H \). Hence \( y^{-1} x \) cannot be written as a product of elements of \( T \cup T^{-1} \). So if we take \( s_i \in G \) such that \( y^{-1} x = s_1 s_2 \cdots s_n \) and \( \| y^{-1} x \| = \sum w(s_i) \), then there is a \( j \) such that \( s_j \notin T \). Hence \( w(s_j) \geq d \), and so \( d(y, x) = \| y^{-1} x \| \geq d \). Therefore \( U \) is a \( d \)-disjoint, uniformly bounded cover. Since \( d > 0 \) was arbitrary, \( \text{asdim} G = 0 \). This completes the proof.

The following corollaries are immediate consequences.

**Corollary 3.5.** Let \( G \) be a finitely generated group. Then \( \text{asdim} G = 0 \) iff \( G \) is a finite group.

**Corollary 3.6.** Let \( G \) be a countable abelian group. Then \( \text{asdim} G = 0 \) if and only if \( G \) is a torsion group.

**Example 3.7.** The last corollary shows that \( \oplus_i \mathbb{Z}_{m_i} \), \( \mathbb{Q}/\mathbb{Z} \), and \( \mathbb{Z}_{p^\infty} = \lim_{\leftarrow} \mathbb{Z}_{p^k} \) all have asymptotic dimension 0.

The following theorem reduces the study of asymptotic dimension of countable groups to the asymptotic dimension of finitely generated groups, and can be found in [14].

**Theorem 3.8.** Let \( G \) be a countable group. Then

\[
\text{asdim} G = \sup \text{asdim} F,
\]

where the supremum varies over finitely generated subgroups \( F \) of \( G \).
Proof. Fix a weight function \( w : G \to [0, \infty) \); let \( \| \cdot \| \) and \( d \) denote the induced norm and metric, respectively. If \( \sup \text{asdim} F = \infty \), then \( \text{asdim} G = \infty \), and we are finished. We will now assume \( \sup \text{asdim} F < \infty \). Set \( m = \sup \text{asdim} F \).

Let \( d > 0 \) be given. Set \( T = \{ g \mid w(g) < d \} \) and \( F = \langle T \rangle \). By the definition of weight function, \( T \) is finite and so \( F \) is finitely generated. Thus, \( \text{asdim} F \leq m \). So there exist uniformly bounded, \( d \)-disjoint families \( U_0, U_1, \ldots, U_m \) of subsets of \( F \) such that \( \bigcup U_i \) is a cover of \( F \). Let \( Z \) be a system of representatives for the partition by cosets \( G/F \). For \( 0 \leq i \leq m \), define \( V_i = \{ zU \mid z \in Z, U \in U_i \} \).

It is easy to check that \( V_i \) is uniformly bounded for \( i = 0, 1, \ldots, m \), and that \( \bigcup V_i \) is a cover of \( G \). We check now that \( V_i \) is a \( d \)-disjoint family. For suppose \( zU \neq z'U' \), where \( z, z' \in Z \) and \( U, U' \in U_i \). Let \( x \in zU \) and \( y \in z'U' \). First suppose \( z \neq z' \). Note that \( zU \subset zF \) and \( z'U' \subset z'F \), so that \( x \in zF \) and \( y \in z'F \). Since \( z \neq z' \), \( zF \neq z'F \). But \( xF = zF \) and \( yF = z'F \). So \( xF \neq yF \), and hence \( x^{-1}y \notin F \). This means that \( x^{-1}y \) cannot be written as a product of the elements of \( T \), and so \( d(x, y) = \| x^{-1}y \| \geq d \) by definition of \( \| \cdot \| \). Now suppose \( z = z' \). So \( x = zu \) and \( y = zu' \) for some \( u \in U \) and \( u' \in U' \). Since we must have \( U \neq U' \), \( U \) and \( U' \) are \( d \)-disjoint; thus, \( d(x, y) = d(zu, zu') = d(u, u') \geq d \). It follows that the family \( V_i \) is \( d \)-disjoint.

Since \( d > 0 \) was arbitrary, \( \text{asdim} G \leq m \). Equality immediately follows. \( \square \)

Example 3.9. Since every finitely generated subgroup of \( \mathbb{Q} \) is cyclic, Theorem 3.8 implies in particular that \( \text{asdim} \mathbb{Q} = 1 \).

We recall the notion of the \( R \)-stabilizer from [2].

Definition 3.10. Suppose that a group \( \Gamma \) acts on a metric space \( X \) by isometries. Let \( x_0 \in X \). For every \( R > 0 \) we define the \( R \)-stabilizer of \( x_0 \) as follows:

\[
W_R(x_0) = \{ g \in \Gamma \mid d(g(x_0), x_0) \leq R \}.
\]

The following theorems are extensions of earlier results for finitely generated groups, which can be found in [4].
Theorem 3.11. Suppose that a countable group $\Gamma$ acts on a geodesic space $X$ by isometries. Let $x_0 \in X$ and suppose $\text{asdim} W_R(x_0) \leq n$ for all $R > 0$. Then $\text{asdim} \Gamma \leq n + \text{asdim} X$.

Theorem 3.12. (Hurewicz Type Formula) Let $\phi : G \to H$ be a homomorphism of countable groups with kernel $K$. Then

$$\text{asdim} G \leq \text{asdim} H + \text{asdim} K.$$ 

Both of these Theorems follow from Theorem 3.8 in combination with the corresponding Theorems for the case when the groups are finitely generated. For details see [14]. Since the notion of asymptotic dimension at the time when [4] was written was defined only for finitely generated groups, $K$ is treated in Theorem 7 of [4] as a metric space with the word metric restricted from $G$. 


CHAPTER 4
THE SUBLINEAR COARSE STRUCTURE

Throughout this section, unless stated otherwise, $X$ will be a proper metric space with metric $d$, and $x_0$ will be a basepoint of $X$. We define $\| \cdot \| : X \to [0, \infty)$, sometimes referred to as a norm, by $\| x \| = d(x, x_0)$. Also, we write $B_r = B_r(x_0)$ for the open ball of radius $r$ about $x_0$.

**Definition 4.1.** We define the sublinear coarse structure, denoted $\mathcal{E}_L$, on $X$ as follows:

$$\mathcal{E}_L = \{ E \subset X \times X : E \text{ proper}, \lim_{x \to \infty} \sup_{y \in E} \frac{d(y, x)}{\| x \|} = 0 = \lim_{x \to \infty} \sup_{y \in E} \frac{d(x, y)}{\| x \|} \}.$$ 

By the statement $\lim_{x \to \infty} \sup_{y \in E} \frac{d(y, x)}{\| x \|} = 0$, we mean that for each $\epsilon > 0$, there is a $r \geq 0$ such that

$$\sup_{y \in E_x} \frac{d(y, x)}{\| x \|} \leq \epsilon$$

for all $x$ with $\| x \| > r$. It would perhaps be better to think of this as $\lim_{\| x \| \to \infty}$. In the event that $E_x = \emptyset$, we define $\sup_{y \in E_x} d(y, x) = 0$.

We first check that $\mathcal{E}_L$ is indeed a coarse structure.

**Theorem 4.2.** $\mathcal{E}_L$ defines a proper coarse structure on a proper metric space $X$.

**Proof.** It is easy to show that $\mathcal{E}_L$ contains the diagonal and is closed under the formation of subsets, inverses, and finite unions. Also, the (coarsely) bounded sets are precisely the relatively compact sets, and as $\mathcal{E}_L$ contains the bounded coarse structure, it contains a neighborhood of the diagonal. To complete the proof, we show that it is closed under products.

Let $E, F \in \mathcal{E}_L$. Then $E \circ F$ is proper. Let $\epsilon > 0$ be given. Then there are compact sets $K, L \subset X$ (containing the basepoint) such that

$$\sup_{y \in E_x} \frac{d(y, x)}{\| x \|} \leq \min\{\epsilon/3, 1\} \quad \text{and} \quad \sup_{y \in E_z} \frac{d(y, z)}{\| z \|} \leq \epsilon/3$$

whenever $x \notin K$ and $z \notin L$. Define $K' = K \cup F^{-1}[L]$. $K'$ is relatively compact since $F$ is proper.
Now suppose \( x \notin K' \), and let \( y \in (E \circ F)_x \). So \((y, x) \in E \circ F \), and thus there is a \( z \) such that \((y, z) \in E \) and \((z, x) \in F \). We must have \( z \notin L \). For if \( z \in L \), then \((x, z) \in F^{-1} \) and so \( x \in F^{-1}[L] \subset K' \), a contradiction. Also, \( x \notin K \), \( y \in E_z \), and \( z \in F_x \). So

\[
\frac{d(y, x)}{\|x\|} \leq \frac{d(y, z)}{\|x\|} + \frac{d(z, x)}{\|x\|} = \varepsilon.
\]

Thus, \( \sup_{y \in (E \circ F)_x} d(y, x) \leq \frac{\varepsilon}{\|x\|} \), or \( \frac{\sup_{y \in (E \circ F)_x} d(y, x)}{\|x\|} \leq \varepsilon \). Since \( \varepsilon > 0 \) was arbitrary, this completes the proof.

Also, for \( x_0, x_1 \in X \), it is not difficult to show that

\[
\lim_{x \to \infty} \frac{\sup_{y \in E_x} d(y, x)}{d(x, x_0)} = 0 \quad \text{iff} \quad \lim_{x \to \infty} \frac{\sup_{y \in E_x} d(y, x)}{d(x, x_1)} = 0,
\]

where \( E \subset X \times X \). A similar statement holds when \( E_x \) is replaced by \( E^x \). This shows that the coarse structure does not depend on the choice of basepoint.

We often refer to the Higson compactification \( h_{E \circ L} X \) (corona \( \nu_{E \circ L} X \)) as the sublinear Higson compactification (sublinear Higson corona), and it will be denoted using the simpler notation \( h_{L} X \) (\( \nu_{L} X \)).

**Proposition 4.3.** [15, Proposition 2.1] Let \( X \) and \( Y \) be proper metric spaces. If \( f : X \to Y \) is a quasi-isometry, then it is a coarse equivalence with respect to the sublinear coarse structures.

**Proof.** Fix a basepoint \( x_0 \in X \); set \( y_0 = f(x_0) \in Y \). Choose \( \lambda > 0 \) and \( C \geq 0 \) such that

\[
\frac{1}{\lambda} d(x, y) - C \leq d(f(x), f(y)) \leq \lambda d(x, y) + C.
\]

Let \( g \) be a quasi-isometry such that \( f \circ g \) and \( g \circ f \) are close to the respective identity functions. It is clear that \( f \) is proper and the image under \( f \) of a bounded set is bounded. Let \( E \) be a controlled set in the sublinear coarse structure on \( X \). It is not hard to show that \( f \times f(E) \) is proper.

Let \( \varepsilon > 0 \) be given. There is a bounded \( K' \subset X \) such that \( \frac{\sup_{y \in E_x} d(y, x)}{\|x\|} \leq \frac{\varepsilon}{4\lambda^2} \) whenever \( x \notin K' \). Set \( K = B(x_0, 2\lambda C) \cup B(x_0, \frac{4C}{\varepsilon}) \cup K' \). Note that \( f(K) \) is bounded. Suppose that
$z \notin f(K)$. If $(f \times f)(E)_z = \emptyset$, then \(\sup \{d(z', z) : z' \in (f \times f)(E)_z\} / \|z\| = 0 < \epsilon\) and we are finished. Now assume that $z' \in (f \times f)(E)_z$; so $z' = f(x')$ and $z = f(x)$ for some $x, x' \in X$ with $(x', x) \in E$. We have that $x \notin K$.

Then
\[
\|f(x)\| \geq \frac{1}{\lambda} \|x\| - C = \frac{2\|x\| - 2\lambda C}{2\lambda} = \frac{\|x\|}{2\lambda} + \frac{\|x\| - 2\lambda C}{2\lambda} \geq \frac{\|x\|}{2\lambda}
\]
and so
\[
\frac{d(z', z)}{\|z\|} = \frac{d(f(x'), f(x))}{\|f(x')\|} \leq 2\lambda \frac{\lambda d(x', x) + C}{\|x\|} = 2\lambda^2 \frac{d(x', x)}{\|x\|} + \frac{2\lambda C}{\|x\|} \leq \epsilon.
\]
Thus, we have \(\sup \{d(z', z) : z' \in (f \times f)(E)_z\} / \|z\| \leq \epsilon\). Since $\epsilon > 0$ was arbitrary, $(f \times f)(E)$ is controlled and so $f$ is a coarse map.

Similarly, since $g$ is a quasi-isometry, it is a coarse map as well. Finally, it is clear that $f \circ g$ and $g \circ f$ are close to the corresponding identities when $X$ and $Y$ are equipped with the sublinear coarse structures. Thus, $f$ is a coarse equivalence. \(\square\)

**Corollary 4.4.** The sublinear coarse structure $\mathcal{E}_L$ is well-defined on finitely generated groups, i.e., the sublinear coarse structure for a given group $\Gamma$ is independent of the choices of the finite generating set and the basepoint.

**Lemma 4.5.** [15, Lemma 2.3] Let $(X, d)$ be a proper metric space with basepoint $x_0$, endowed with the sublinear coarse structure $\mathcal{E}_L$. For a finite system $E_1, \ldots, E_n$ of subsets of $X$, the following are equivalent.

1. $\nu_L X \cap [\cap_{i=1}^n \overline{E_i}] = \emptyset$;

2. there exist $c, r_0 > 0$ such that $\max_{1 \leq i \leq n} d(x, E_i) \geq c\|x\|$ whenever $\|x\| \geq r_0$.

**Proof.** We prove 1 implies 2. Assuming that 2 does not hold, then if we let $m$ be a positive integer, and if we set $c = \frac{1}{4m}$ and $r_0 = 2m$, then there is an $x_m$ such that $\|x_m\| \geq 2m$ yet
\[
\max_{1 \leq i \leq n} d(x_m, E_i) < \frac{1}{4m} \|x_m\|.
\]
Thus, for each \(i\), we have that \(d(x_m, E_i) < \frac{1}{4m} \|x_m\|\), and so there is an \(a^i_m \in E_i\) such that \(d(x_m, a^i_m) < \frac{1}{4m} \|x_m\|\). We have \(d(a^i_m, a^j_m) < \frac{1}{2m} \|x_m\|\). Also,
\[
\|x_m\| \leq \|a^i_m\| + d(a^i_m, x_m) < \|a^i_m\| + \frac{1}{4m} \|x_m\|,
\]
and hence \((1 - \frac{1}{4m}) \|x_m\| < \|a^i_m\|\) (all \(i\)). Since \(\frac{1}{4m} < 1/2\), we have \(\|a^i_m\| > \frac{1}{2} \|x_m\|\). Thus,
\[
d(a^i_m, a^j_m) < \frac{1}{2m} \|x_m\| < \frac{1}{m} \|a^i_m\| \quad \text{and} \quad \|a^i_m\| > \frac{1}{2} \|x_m\| \geq m
\]
for all \(1 \leq i, j \leq n\). Take \(F_{i,j} = \{(a^i_m, a^j_m) : m = 1, 2, \ldots\}\) for each \(1 \leq i, j \leq n\). Fixing \(i, j\), we temporarily set \(G = F_{i,j}\) for convenience, and show that \(G\) is controlled. Since \(\|a^i_m\| \to \infty\) and \(\|a^j_m\| \to \infty\) as \(m \to \infty\), it follows that \(G\) is proper. Now let \(\epsilon > 0\) be given, and take \(M\) to be a positive integer for which \(1/M < \epsilon\); set \(K = \{a^j_m : 1 \leq m < M\}\). Suppose that \(x \notin K\). If \(G_x = \emptyset\), then by our convention we have \(\sup_{y \in G_x} \frac{d(y, x)}{\|x\|} = 0\).

If \(y \in G_x\), then there is a positive integer \(m\) such that \((y, x) = (a^i_m, a^j_m)\), and since \(a^j_m = x \notin K\), we must have \(m \geq M\); it follows that \(\frac{d(y, x)}{\|x\|} = \frac{d(a^i_m, a^j_m)}{\|a^j_m\|} < \frac{1}{m} < \epsilon\) and so
\[
\sup_{y \in G_x} \frac{d(y, x)}{\|x\|} \leq \epsilon.
\]
Thus, \(\lim_{x \to \infty} \sup_{y \in G_x} \frac{d(y, x)}{\|x\|} = 0\). Similarly, \(\lim_{x \to \infty} \sup_{y \in G_x} \frac{d(x, y)}{\|x\|} = 0\), and hence \(G = F_{i,j}\) is controlled.

Consider the sequence \(\{(a^1_m, a^2_m, \ldots, a^n_m)\}_{m=1}^\infty\) consisting of points of \(X^n \subset (h_LX)^n\). Regarding this sequence as a net, there is a subnet \(\{(a^1_{m_\lambda}, a^2_{m_\lambda}, \ldots, a^n_{m_\lambda})\}_{\lambda \in \Lambda}\) which converges in \((h_LX)^n\) (\(\Lambda\) is a directed set). Set \(b = (b^1, b^2, \ldots, b^n) = \lim_{\lambda \in \Lambda}(a^1_{m_\lambda}, a^2_{m_\lambda}, \ldots, a^n_{m_\lambda})\).

So \(b^i \in h_LX\) and \(b^i = \lim_{\lambda \in \Lambda} a^i_{m_\lambda}\). By definition of the \(a^i_m\) and the definition of subnet, we have that \(\|a^i_{m_\lambda}\| \to \infty\) as \(\lambda \to \infty\). From this one concludes that \(b^i \in \nu_LX\) for each \(i\). Also, fixing \(1 \leq i, j \leq n\), we have that \((b^i, b^j) = \lim_{\lambda \in \Lambda}(a^i_{m_\lambda}, a^j_{m_\lambda}) \in \underbar{F}_{i,j}\). Since \(F_{i,j}\) is controlled, we have by Proposition 2.45 of [23] that \((b^i, b^j) \in \underbar{F}_{i,j} \setminus X \times X \subset \Delta_{\nu_LX}\), where \(\Delta_{\nu_LX}\) denotes the diagonal in \(\nu_LX \times \nu_LX\). Therefore \(b^i = b^j\). So \(b^1 = b^i = \lim_{\lambda \in \Lambda} a^i_{m_\lambda} \in \overline{E_i}\) and \(b^i \in \nu_LX\).

Thus, \(\nu_LX \cap \cap_{i=1}^n \overline{E_i} \neq \emptyset\). So 1 does not hold.

It remains to show that 2 implies 1. Define \(F_i = E_i \setminus B_{r_0+c\|x\|}\) for \(1 \leq i \leq n\). Let \(f : X \to \mathbb{R}\) be defined by \(f(x) = \sum_{i=1}^n d(x, F_i)\). Note that \(f(x) \geq c\|x\|\) when \(\|x\| \geq r_0\)
since \(d(x, F_i) \geq d(x, E_i)\). Also, \(f(x) \geq cr_0\) when \(\|x\| \leq r_0\); in particular, \(f(x) \geq c\|x\|\) for all \(x\) and \(f(x) > 0\) for all \(x\). Define \(g_i : X \to \mathbb{R}\) by \(g_i(x) = d(x, F_i)/f(x)\).

Let \(E\) be a controlled set. Since

\[
|g_i(y) - g_i(x)| \leq \frac{d(y, F_i)}{f(y)} - \frac{1}{f(x)} \leq \frac{f(x)}{f(x) f(y)} |f(x) - f(y)|
\]

we have \(\sup_{y \in E_i} |g_i(x) - g_i(y)| \to 0\) as \(x \to \infty\). Since \(E\) was an arbitrary controlled set, \(g_i\) (viewed as a map to \(\mathbb{C}\)) is a Higson function for each \(i\). Let \(G_i : \overline{X} \to \mathbb{C}\) be the extension of \(g_i\) to the Higson compactification. Since \(\sum_i g_i = 1\), it is immediate that \(\sum_i G_i = 1\) throughout \(\overline{X}\). Also, \(F_i \subset G_i^{-1}(0)\) and it is not hard to see that \(\nu X \cap F_i = \nu X \cap \overline{E_i}\). Thus,

\[
\nu X \cap (\bigcap_{i=1}^n \overline{E_i}) = \nu X \cap (\bigcap_{i=1}^n \overline{F_i}) \subset \nu X \cap (\bigcap_{i=1}^n G_i^{-1}(0)) = \emptyset
\]

since \(\sum_i G_i = 1\) on \(\nu X\). \(\square\)

A system satisfying one of these two properties is said to diverge. In the case that \(n = 2\), we can add another condition.

**Lemma 4.6.** Let \(A\) and \(B\) be subsets of a metric space \(X\) with basepoint \(x_0 \in X\). The following are equivalent.

1. There exist \(C, r_0 > 0\) such that \(\max \{d(x, A), d(x, B)\} \geq C\|x\|\) whenever \(\|x\| \geq r_0\);
2. there exist \(D, r_1 > 0\) such that \(d(A \setminus B_r, B \setminus B_r) \geq Dr\) whenever \(r \geq r_1\).

**Proof.** We show 1 implies 2. Given \(C\) and \(r_0\), take \(D = C\) and \(r_1 = r_0\). Let \(r \geq r_1\), \(a \in A \setminus B_r\), and \(b \in B \setminus B_r\). So \(\|a\| \geq r \geq r_0\). Thus,

\[
Cr \leq C\|a\| \leq \max \{d(a, A), d(a, B)\} = \max \{0, d(a, B)\} = d(a, B) \leq d(a, b)
\]

by 1. So \(d(A \setminus B_r, B \setminus B_r) \geq Dr\).

We show 2 implies 1. Let \(D, r_1\) be positive numbers satisfying 2. Set \(r_0 = 2r_1\) and take \(C\) to be a positive number satisfying \(C < \min \{1/2, D/4\}\). Now let \(x \in X\) be such
that \( \|x\| \geq r_0 = 2r_1 \). To get a contradiction, suppose that \( \max \{d(x, A), d(x, B)\} < C\|x\| \). Then \( d(x, A) < C\|x\| \) and \( d(x, B) < C\|x\| \). Thus, there exist \( a \in A \) and \( b \in B \) such that \( d(x, a) < C\|x\| \) and \( d(x, b) < C\|x\| \). So \( d(a, b) < 2C\|x\| \). We then have

\[
\|a\| \geq \|x\| - d(x, a) > \|x\| - C\|x\| = (1 - C)\|x\| \geq \|x\| / 2.
\]

Similarly, \( \|b\| \geq \|x\| / 2 \). Since \( \|x\| / 2 \geq r_1 \), we have by 2 that

\[
\frac{D\|x\|}{2} \leq d(A \setminus B_{\|x\| / 2}, B \setminus B_{\|x\| / 2}) \leq d(a, b) < 2C\|x\| \leq \frac{D\|x\|}{2},
\]

a contradiction. Therefore, \( \max \{d(x, A), d(x, B)\} \geq C\|x\| \) when \( \|x\| \geq r_0 \).

**Definition 4.7.** Let \( (X, d) \) be a metric space, and let \( V \) be a family of open subsets of \( X \). We define the Lebesgue function associated with the cover \( V \), denoted \( L^V \), by

\[
L^V(x) = \sup_{V \in V} d(x, X \setminus V).
\]

**Definition 4.8.** For a proper metric space \( (X, d) \) with basepoint \( x_0 \), we say a function \( f : X \to [0, \infty) \) is (eventually) at least linear if there exist \( c, r_0 > 0 \) such that \( f(x) \geq c\|x\| \) whenever \( \|x\| \geq r_0 \).

**Corollary 4.9.** Let \( (X, d) \) be a proper metric space endowed with the coarse structure \( \mathcal{E}_L \). Let \( \alpha = \{O_1, \ldots, O_n\} \) be a finite family of open subsets of \( X \). Then \( \tilde{\alpha} = \{\tilde{O}_1, \ldots, \tilde{O}_n\} \) covers the corona \( \nu_L X \) if and only if the Lebesgue function \( L^\alpha \) is at least linear.

**Proof.** \( \tilde{\alpha} = \{\tilde{O}_1, \ldots, \tilde{O}_n\} \) covers the corona \( \nu_L X \) iff \( \nu_L X \setminus (\cup_i \tilde{O}_i) = \emptyset \), iff \( \nu_L X \cap (X \setminus \cup_i \tilde{O}_i) = \emptyset \), iff \( \nu_L X \cap (\cap_i (X \setminus \tilde{O}_i)) = \emptyset \), iff \( \nu_L X \cap (\cap_i X \setminus O_i) = \emptyset \) by the discussion of compactifications in §1, iff the system \( X \setminus O_1, \ldots, X \setminus O_n \) diverges, which, by lemma 4.5 above, is true if and only if \( L^\alpha \) is at least linear.

**Corollary 4.10.** Let \( (X, d) \) be a proper metric space, and let \( A \) be a closed subspace of \( X \) equipped with the restricted metric. Then the embedding \( A \hookrightarrow X \) extends to an embedding
$h_L A \hookrightarrow h_L X$ on the compactifications and induces an embedding $\nu_L A \hookrightarrow \nu_L X$ on the coronas.

It follows from this corollary that if $A$ is a closed subset of a proper metric space $X$, then $h_L A$ is homeomorphic to $\text{cl}_{h_L} X A$, the closure of $A$ in $h_L X$. Further, $\nu_L A$ is homeomorphic to $\text{cl}_{h_L} X A \setminus A$.

**Algebra of functions.** We define a subalgebra $U(X) = U(X, x_0)$ of $C(X)$ as follows: $f : X \to \mathbb{C}$ is in $U(X)$ if and only if $f$ is bounded, continuous, and there exists a $c = c_f$ such that

$$|f(x) - f(y)||x|| \leq cd(x, y).$$

It is not difficult to check that $U(X)$ is closed under addition, multiplication, and complex conjugation.

**Remark 4.11.** In the definition, the continuity condition is almost unnecessary. It is not hard to see that the property $|f(x) - f(y)||x|| \leq cd(x, y)$ implies that $f$ is continuous for $x \neq x_0$.

It is easy to show that $U(X)$ separates points and closed sets. It is also clear that $U(X)$ is, in general, not complete. We set $C'(X) = \overline{U(X)}$, where the bar represents closure in $C(X)$ with the uniform metric. So $C'(X)$ is a $C^*$-algebra which separates points and closed sets. Thus, by the GNS Theorem we can extract a compactification of $X$, denoted $\overline{X}$, which satisfies $C'(X) = C(\overline{X})$.

It is not hard to prove that $C'(X) \subset C_h(X, \mathcal{E}_L)$; hence there is a surjective, continuous map $h_L X \to \overline{X}$ which extends the identity. This map is actually a homeomorphism.

**Theorem 4.12.** [15, Theorem 2.11] Let $(X, d)$ be a proper metric space. Then the compactification $\overline{X}$ is homeomorphic to the Higson compactification $h_L X$ for the sublinear coarse structure $\mathcal{E}_L$ via a homeomorphism extending the identity on $X$. 

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CHAPTER 5
THE DIMENSION OF THE SUBLINEAR HIGSON CORONA

We recall that a metric space \((X, d)\) is called \textit{cocompact} if there is a compact subset \(K\) of \(X\) such that \(X = \bigcup_{\gamma \in \text{Isom}(X)} \gamma(K)\), where \(\text{Isom}(X)\) is the set of all isometries of \(X\). The proof of the following is long and requires numerous results concerning extensions, and will be omitted here.

**Theorem 5.1.** Let \(X\) be a cocompact, connected, proper metric space which has finite asymptotic Assouad-Nagata dimension. Then \(\dim \nu_\ell X \geq \text{AN-asdim}X\).

**Proof.** See Theorem 3.7 of [15].

If \(\mathcal{U}\) is a cover of \(X\) and \(A \subset X\), then we write \(\mathcal{U}_A = \{U \in \mathcal{U} : U \cap A \neq \emptyset\}\).

**Lemma 5.2.** Let \(\mathcal{U}\) and \(\mathcal{V}\) be covers of \(X\), and suppose that \(\mathcal{U}\) refines \(\mathcal{V}\). Let \(K\) be a subset of \(X\). So for each \(U \in \mathcal{U}\) with \(U \cap (X \setminus K) \neq \emptyset\), there is a \(V_U \in \mathcal{V}\) with \(U \subset V_U\).

For \(V \in \mathcal{V}\), set

\[
V' = [V \cap (X \setminus K)] \bigcup_{U \in \mathcal{U}} \bigcup_{U \setminus K, V_U = V} U
\]

and define

\[
\mathcal{W} = \{U \in \mathcal{U} | U \subset K\} \cup \{V'|V \in \mathcal{V}_{X \setminus K}\}.
\]

Then

1. \(\mathcal{W}\) is a cover of \(X\);
2. \(\text{mult}\mathcal{W} \leq n + 1\) if \(\text{mult}\mathcal{U} \leq n + 1\) and \(\text{mult}\mathcal{V} \leq n + 1\);
3. \(\mathcal{W}\) refines \(\mathcal{V}\);
4. \(\mathcal{U}\) refines \(\mathcal{W}\);
5. if \(W \in \mathcal{W}\) and \(W \subset K\), then \(W \in \mathcal{U}\);
6. if \(\mathcal{U}\) and \(\mathcal{V}\) are open covers and \(K\) is closed in \(X\), then \(\mathcal{W}\) is also an open cover;
7. if \(V \in \mathcal{V}\) and \(V \cap K = \emptyset\), then \(V = V' \in \mathcal{W}\).
The proof is straightforward. For the above $\mathcal{W}$ we will write
\[ \mathcal{W} = U \ast_K \mathcal{V}. \]

The following Theorem is a modification of Lemma 2.9 of [13].

**Theorem 5.3.** [15, Theorem 3.10] Let $(X, d)$ be a proper metric space. Then $\dim \nu_X \leq \text{AN-asdim} X$.

**Proof.** We write $\nu_X = \nu_{\mathcal{L} X}$ and use $\hat{B}(x, r)$ to denote a closed ball of radius $r$. Set $n = \text{AN-asdim} X$ (if $\text{AN-asdim} X = \infty$, the inequality is immediate). As $\{ \tilde{U} \cap \nu X : U \subset X \text{ open} \}$ is a basis for $\nu X$, it suffices to prove that each cover of the form $\{ \tilde{U}_i \cap \nu X : 1 \leq i \leq m \}$, where each $U_i \subset X$ is open, admits a finite refinement of multiplicity $\leq n + 1$.

So let $\{ \tilde{U}_i \cap \nu X : 1 \leq i \leq m \}$ be a cover of $X$, and set $\mathcal{U} = \{ U_i : 1 \leq i \leq m \}$. Since $\text{AN-asdim} X = n$, there exist $C > 0$ and $r_{-1} > 0$ such that whenever $r \geq r_{-1}$, there is an open cover $\mathcal{U}(r)$ of $X$ satisfying $\text{mult} \mathcal{U}(r) \leq n + 1$, $\text{mesh} \mathcal{U}(r) < Cr$, and $L^{\mathcal{U}(r)} > r$.

Without loss of generality, we take $C > 1$. Also, there is a $D > 0$ and an $r_{-2} > 0$ such that $L^{\mathcal{U}(x)} \geq D \|x\|$ whenever $x$ is such that $\|x\| \geq r_{-2}$. We may take $D < 1$.

Now, choose $r_0 > \max \{ r_{-2}, r_{-1}, 1 \}$. Define $r_i = (\frac{C}{D})^i r_0$ for $i \geq 1$. Observe that $r_{i+1} = \frac{C}{D} r_i > r_i > r_0 > 1$. Since $r_i > r_0 > r_{-1}$ for $i \geq 1$, there is a cover $\mathcal{U}_i$ of $X$ such that $\text{mult} \mathcal{U}_i \leq n + 1$, $\text{mesh} \mathcal{U}_i < Cr_i$, and $L^{\mathcal{U}_i} > r_i$.

Define $\mathcal{V}_1 = \mathcal{U}_1$, and note that $\text{mesh} \mathcal{V}_1 < Cr_1$. Now, supposing we have defined $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_i$ satisfying $\text{mesh} \mathcal{V}_j < Cr_j < r_{j+1}$ for all $1 \leq j \leq i$, then $\mathcal{V}_i$ refines $\mathcal{U}_{i+1}$, and so we can define
\[ \mathcal{V}_{i+1} = \mathcal{V}_i \ast_{\mathcal{B}(x_0, 2r_{i+2})} \mathcal{U}_{i+1}. \]

By Lemma 5.2, $\mathcal{V}_{i+1}$ refines $\mathcal{U}_{i+1}$, and so $\text{mesh} \mathcal{V}_{i+1} < Cr_{i+1}$. Thus, we have constructed $\mathcal{V}_i$ for all positive integers $i$.

Set $\mathcal{V} = \lim \inf_i \mathcal{V}_i = \bigcup_{s} \cap_{t \geq s} \mathcal{V}_t$. We now investigate some properties of $\mathcal{V}$.
Using the definition of $V_{i+1}$, it is easy to show that if $U \in V_i$ and $U \cap \hat{B}(x_0, r_{i+2}) \neq \emptyset$, then $U \in V_{i+1}$. We conclude that \{ $U \in V_i : U \cap \hat{B}(x_0, r_{i+2}) \neq \emptyset$ \} $\subset V_i$. As $V_i$ is a cover of $X$ and hence of $\hat{B}_{r_{i+2}}$, we have that $V$ covers $\hat{B}_{r_{i+2}}$; as $i$ here is arbitrary, $V$ covers $X$.

We now show that if $V \in V$ and $V \cap \hat{B}_{r_{i+1}} \neq \emptyset$, then $V \in V_{i-1}$. First suppose that $V \in V_i$ and $V \cap \hat{B}_{r_{i+1}} \neq \emptyset$; then mesh$V_i < C r_i < r_{i+1}$ implies that $V \subset \hat{B}_{2r_{i+1}}$ and so $V \in V_{i-1}$ by 5 of the lemma. Now suppose $V \in V_i$. This means there is an $s \geq i - 1$ such that $V \in V_s$. Applying the result we just found and proceeding inductively, one can show that $V \in V_j$ for all $j$ such that $i - 1 \leq j \leq s$.

We show that $V_i$ refines $V$ for all $i \geq 1$. We know by 4 of the lemma that $V_i$ refines $V_{i+1}$ for all $i \geq 1$. Fixing $i$, let $V \in V_i$. Choose $j \geq i$ such that $V \cap \hat{B}_{r_{j+2}} \neq \emptyset$. As $V_i$ refines $V_j$, there is a $U \in V_j$ such that $V \subset U$. Also, $U \cap \hat{B}_{r_{j+2}} \subset V \cap \hat{B}_{r_{j+2}} \neq \emptyset$. Thus, $V \subset U \in V$. So $V_i$ refines $V$.

Since each $V_i$ has multiplicity $n + 1$ for each $i$, it is clear from the definition that $V$ has multiplicity $\leq n + 1$.

Set $W = V_{X \setminus \hat{B}_{r_2}}$. We have that ($V_i)_{X \setminus \hat{B}_{r_2}}$ refines $W$.

We show that $W$ refines $U$. Let $W \in W$. So there is an $x \in W$ such that $\|x\| > r_2$. Take $i = \max \{ j : \|x\| > r_j \}$, and note that $i \geq 2$. Thus, $\|x\| \leq r_{i+1}$, or $x \in \hat{B}_{r_{i+1}}$. Hence $W \cap \hat{B}_{r_{i+1}} \neq \emptyset$. Since $W \in V$, we have $W \in V_{i-1}$. So

$$\text{diam } W < C r_{i-1} = D \left( \frac{C}{D} \right) r_{i-1} = D r_i \leq D \|x\| \leq L^U (x),$$

and so there is a $U \in U$ with $W \subset U$.

We show that $L^W : X \rightarrow [0, \infty)$ is at least linear. Set $a = 3r_2$, and let $x$ be an element of $X$ with $\|x\| \geq a = 3r_2$. Set $i = \max \{ j : 3r_{j+1} \leq \|x\| \}$, and note that $i \geq 1$ and $3r_{i+1} \leq \|x\| < 3r_{i+2}$. As $L^U > r_i$, there is a $U \in U_i$ such that $B(x, r_i) \subset U$. Since $\|x\| \geq 3r_{i+1}$ and $\text{diam } U \leq \text{mesh } U_i < C r_i < r_{i+1}$, we have that $U \subset X \setminus \hat{B}(x_0, 2r_{i+1})$. By definition of $V_i$, and by 7 of the lemma, we have that $U \in V_i$. In fact, as $\|x\| > r_2$, we have
Since \((\mathcal{V}_i)_{X\setminus \hat{B}_{r_2}}\) refines \(\mathcal{W}\), we have that
\[
L^W(x) \geq d(x, X \setminus U) \geq r_i.
\]

But \(\|x\| < 3r_{i+2} = 3\frac{C^2}{D}r_i\), so \(L^W(x) > \frac{D^2}{3C^2}\|x\|\). Therefore, \(L^W\) is at least linear.

To summarize, \(\mathcal{W}\) covers \(X \setminus \hat{B}_{r_2}\), \(\text{mult} \mathcal{W} \leq n + 1\), \(\mathcal{W}\) refines \(\mathcal{U}\), and \(L^W\) is at least linear. Thus, for \(W \in \mathcal{W}\), there is a \(U_W \in \mathcal{U}\) for which \(W \subset U_W\). So for each \(1 \leq i \leq m\), we define \(W_i = \cup_{U_W = U_i} W\).

Now set \(W' = \{W_i\}\). It follows that \(\mathcal{W}\) refines \(W'\) and \(W'\) has multiplicity \(\leq n + 1\). Thus, \(L^{W'} \geq L^W\) and hence \(L^{W'}\) is at least linear. As a consequence, if we define \(\tilde{W}' = \{\tilde{W}_i \cap \nu X\}\), we have that \(\tilde{W}'\) is a cover of \(\nu X\). Since \(\mathcal{W}'\) refines \(\mathcal{U}\), we have that \(\tilde{W}'\) refines \(\tilde{U}\). Finally, as \(W'\) has multiplicity \(\leq n + 1\), so does \(\tilde{W}'\).

Since a finitely generated group is quasi-isometric to its Cayley graph, a cocompact geodesic metric space, one has the following corollary.

**Corollary 5.4.** [15, 3.8] For a finitely generated group \(\Gamma\) with word metric, \(\dim \nu_L \Gamma = \text{AN-asdim} \Gamma\) provided \(\text{AN-asdim} \Gamma < \infty\).

**Example 5.5.** Consider the parabolic region \(X = \{(x, y) \in \mathbb{R}^2 : x \geq 0, |y| \leq \sqrt{x}\}\), which we will equip with the (restricted) Euclidean metric. Let \(i : [0, \infty) \to X\) be the map \(i(x) = (x, 0)\). Taking the usual metric on \([0, \infty)\), it is not hard to show that \(i\) is a coarse equivalence for the sublinear coarse structures, and hence there is a homeomorphism \(\nu_L[0, \infty) \to \nu_L X\). In particular, \(\dim \nu_L X = 1\). But \(\text{AN-asdim} X = 2\). Since \(X\) is a connected proper metric space, this shows that we can not drop the requirement in Theorem 5.1 that the space be cocompact.

**Lemma 5.6.** [15, Lemma 4.2] Let \(X\) be a proper metric space. Then there is an embedding
\[
\nu_L X \times [0, 1] \hookrightarrow \nu_L (X \times \mathbb{R}_+).
\]

**Proof.** We use \(\| \cdot \|\) to denote the norm on both \(X \times \mathbb{R}_+\) and \(X\). Let \(Y = \{(x, y) \in X \times \mathbb{R}_+ : 0 \leq y \leq \|x\|, \|x\| \geq 1\}\). Consider the maps \(\pi : Y \to X\) and \(\phi : Y \to [0, 1]\) given
by \( \pi(x, y) = x \) and \( \phi(x, y) = \frac{y}{\|x\|} \), respectively. The reason for \( \|x\| \geq 1 \) in the definition of \( Y \) is the desire to make \( \phi \) well-defined in the domain of definition; note that requiring \( \|x\| \geq 1 \) only cuts off a compact set in our cone, and hence leaves the corona unchanged.

We first show that \( \pi \) is a coarse map for the sublinear coarse structures. It is easy to check that \( \pi \) is proper. Let \( E \subset Y \times Y \) be controlled; we show that \( \pi \times \pi(E) \) is controlled. First notice that \( \pi \times \pi(E) \) is proper since \( E \) is proper and since \( (\pi \times \pi(E))[K] \subset \pi(E[\pi^{-1}K]) \). Let \( \epsilon > 0 \). Then there is an \( R > 0 \) such that

\[
\left\| (x, y) - (w, z) \right\| < \epsilon / 2
\]

whenever \((x, y), (w, z) \in E \) and \( \left\| (w, z) \right\| \geq R \). Now suppose \((x, w) \in \pi \times \pi(E) \) and \( \|w\| \geq R \). Then for some \( y, z \in \mathbb{R}_+ \) (indeed for any appropriate pair of elements of \( \mathbb{R}_+ \)) one has \( \pi(x, y) = x \) and \( \pi(w, z) = w \). Since \( \left\| (w, z) \right\| \geq \|w\| \geq R \) and \( |z| \leq \|w\| \), we have

\[
\frac{d(x, w)}{\|w\|} \leq \frac{d(x, w) + |y - z|}{\left(\frac{\|w\| + |z|}{2}\right)} = 2 \frac{d((x, y), (w, z))}{\left\| (w, z) \right\|} < \epsilon.
\]

Since \( \epsilon \) was arbitrary, \( \pi \times \pi(E) \) is controlled. Since \( E \) was an arbitrary controlled set on \( Y \), we have that \( \pi \) is a coarse map.

We now show that \( \phi \) is a sublinear Higson function. For \((x, y), (w, z) \in Y \), one has

\[
|\phi(x, y) - \phi(w, z)| = \left| \frac{y}{\|x\|} - \frac{z}{\|w\|} \right| = \left| y - z \right| \left( \frac{1}{\|x\|} - \frac{1}{\|w\|} \right) \\
\leq \left| y - z \right| \left| \frac{\|w\| - \|x\|}{\|x\||\|w\|} \right| \leq \left| y - z \right| \left( \frac{\|w\| - \|x\|}{\|x\|} \right) \\
\leq \frac{d((x, y), (w, z))}{\|x\|} \leq 2 \frac{d((x, y), (w, z))}{\| (x, y) \|}.
\]

By the alternative characterization of the sublinear Higson compactification given at the end of \$4\$, we have that \( \phi \) is a sublinear Higson function defined on \( Y \).

Define \( P : \nu_L Y \rightarrow \nu_L X \) to be the map induced by the coarse map \( \pi \), and define \( \Phi : \nu_L Y \rightarrow [0, 1] \) to be map induced by the sublinear Higson function \( \phi \). We will also have occasion to consider the extension of \( \phi \) to \( h_L Y \), which we will also denote by \( \phi \). That is,
we write $\phi : h_L Y \to [0, 1]$ for the extension, and note that $\phi|_{\nu L X} = \Phi$. In particular note that $\phi(A) = \overline{\phi(A)}$ for $A \subset h_L X$, where closure takes place in $h_L X$.

We define $f = P \times \Phi : \nu_L Y \to \nu_L X \times [0, 1]$. We show this map is a homeomorphism. The inverse will be the desired embedding.

Define $O_t = \{(x, t\|x\|) : x \in X, \|x\| \geq 1\}$ for $t \in [0, 1]$. Note that $O_t \subset Y$. The map $\pi|_{O_t} : O_t \to X \setminus B_1$ is bi-Lipschitz since

\[
d(x, y) \leq d((x, t\|x\|), (y, t\|y\|)) = d(x, y) + t\|x\| - \|y\| \leq (1 + t)d(x, y);
\]

this means that $\pi|_{O_t}$ extends to a homeomorphism $\nu_L(\pi|_{O_t}) : \nu_L O_t \to \nu_L X$. Since $\phi|_{O_t} : O_t \to [0, 1]$ takes every point of $O_t$ to $t$, the image of $\Phi|_{\nu_L O_t} : \nu_L O_t \to [0, 1]$ is also $\{t\}$. This means that the map $(\pi \times \phi)|_{O_t} : O_t \to (X \setminus B_1) \times t$ induces a homeomorphism $f|_{\nu_L O_t} : \nu_L O_t \to \nu_L X \times t$.

It is clear from the discussion above that $f$ is surjective. It remains only to show that it is injective. Before doing this, we show that $f^{-1}(\nu_L X \times t) = \nu_L O_t$. It might be helpful to note that $f^{-1}(\nu_L X \times t) = \Phi^{-1}(t)$. That $f^{-1}(\nu_L X \times t) \supset \nu_L O_t$ follows from the definitions above. To get a contradiction, suppose that there is a $z \in f^{-1}(\nu_L X \times t) \setminus \nu_L O_t$. It follows from Proposition 1.12 that there is a closed set $A \subset X$ such that $z \in \overline{A} \cap \nu_L X(= \nu_L A)$ and $\nu_L A \cap \nu_L O_t = \emptyset$. Thus, by Lemma 4.5, there are $C, r_0 > 0$ such that $L(u) := \max\{d(u, A), d(u, O_t)\} \geq C\|u\|$ whenever $u \in Y$ and $\|u\| \geq r_0$. In particular, for $(x, y) \in A$, we have

\[
|y - t\|x\|| = d((x, y), (x, t\|x\|)) \geq d((x, y), O_t) \geq \max\{d((x, y), O_t), d((x, y), A)\}
\]

\[
\geq C\|x, y\| = C(\|x\| + |y|),
\]

and so

\[
|\phi(x, y) - t| = \frac{|y|}{\|x\|} - t \geq C.
\]
Since this is true for all \((x, y) \in A\), we have that \((t - C, t + C) \cap \phi(A) = \emptyset\), and so 
\[ t \notin \overline{\phi(A)} = \phi(A) \supset \Phi(\nu_LA) \] by the comments following Corollary 4.10 and the comments above concerning the extension of \(\phi\). Thus, \(\Phi(z) \neq t\), contradicting \(z \in f^{-1}(\nu_LX \times t)\).

To prove injectivity, assume \(f(z) = f(z')\) for some \(z, z' \in Y\). Let \(t = \Phi(z) = \Phi(z')\). Then by the preceding fact, we have \(z, z' \in f^{-1}(\nu_LX \times t) = \nu_L(O_t)\). But \(f|_{\nu_LO_t}\) is a homeomorphism; consequently, \(z = z'\). \(\square\)

**Theorem 5.7.** [15, Theorem 4.3] Let \(X\) be cocompact connected proper metric space. Then

\[ \text{AN-asdim}(X \times \mathbb{R}) = \text{AN-asdim}X + 1. \]

**Proof.** By Theorem 5.3, Lemma 5.6, the classical Morita theorem, and by Theorem 5.1 we obtain

\[ \text{AN-asdim}(X \times \mathbb{R}) \geq \dim \nu_L(X \times \mathbb{R}) \geq \dim(\nu_LX \times [0, 1]) = \dim \nu_LX + 1 \geq \text{AN-asdim}X + 1. \]

The opposite inequality is well-established. \(\square\)

Again noting that a finitely generated group is quasi-isometric to its Cayley graph, we have the following.

**Corollary 5.8.** For a finitely generated group \(G\), \(\text{AN-asdim}(G \times \mathbb{Z}) = \text{AN-asdim}G + 1\).

It was observed in [5] that \(\text{AN-asdim}G = \dim_{\text{AN}}G\) for finitely generated groups. Consequently, for these spaces, the formula given above holds for \(\dim_{\text{AN}}\): \(\dim_{\text{AN}}(G \times \mathbb{Z}) = \dim_{\text{AN}}G + 1\).

As mentioned in the introduction, along with results of Dydak et al. [5], this result inclines one to believe that \(\text{asdim}G \times \mathbb{Z} = \text{asdim}G + 1\) for finitely generated groups \(G\).
REFERENCES


BIOGRAPHICAL SKETCH

Justin Ian Smith was born on January 2, 1980 in Avon Park, Florida. He grew up in Lakeland, Florida, and graduated from George Jenkins High School in 1998. He earned his B.S. in physics with a minor in mathematics from Florida Atlantic University in 2002. Justin then entered graduate school at the University of Florida in order to continue his studies in mathematics. At the University of Florida, he earned an M.S. in 2004 and completed his Ph.D. in 2007.