BORSUK-ULAM PROPERTY OF FINITE GROUP ACTIONS ON MANIFOLDS AND APPLICATIONS

By

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To my parents
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I would like to thanks my advisor Alexander Dranishnikov for many encouraging conversations over the years on various topics in topology which were of a great influence on my mathematical education. I also would like to thank Yuli Rudyak for being always able to find time to discuss topology with me. His influence on my mathematical education has also been substantial.
This dissertation is devoted to several topics in geometric topology and dimension theory. In the first chapter we discuss Borsuk-Ulam theorems. We viewed the history of the subject, stated a few classical results in this area and described a general approach to proving Borsuk-Ulam type theorems. The results of the author in this area are also stated and proved in this chapter.

In the second chapter we discuss two closely related questions in dimension theory. Namely, a fiberwise version of the classical theorem by Hurewicz about 0-dimensional maps of $k$-dimensional compacta into $k$-dimensional cube and a conjecture by V.V. Uspenskij about approximation of $k$-dimensional maps between compacta by $k$-dimensional simplicial maps of polyhedra.

In the third chapter we outline a general geometric construction which shows how it might be possible to use Borsuk-Ulam type theorems for constructing an example of a 1-dimensional map between compacta which cannot be approximated by 1-dimensional simplicial maps of polyhedra.
CHAPTER 1
BORSUK-ULAM THEOREMS

1.1 Introduction

The famous Borsuk-Ulam theorem is well known. It states that every continuous map of a sphere \( S^n \) into Euclidean space \( \mathbb{R}^n \) will necessarily collapse at least one pair of antipodal points. It has been generalized by many authors. Among the first and most memorable generalizations were by C.T. Yang [29] and D.G. Bourgin [1]:

**Theorem 1.1.1.** Let \( T \) be a fixed point free involution on a sphere \( S^n \) and let \( f : S^n \to \mathbb{R}^m \) be a continuous map into Euclidean space. Then the dimension of the coincidence set \( A(f) = \{ x \in S^n | f(x) = f(Tx) \} \) is at least \( n - m \).

The later theorem stimulated a lot of interest in generalizations of the Borsuk-Ulam theorem. It became a starting point in the research of many authors on this subject. The next important generalization belongs to P.E. Conner and E.E. Floyd [2] and it has first appeared in their famous book.

**Theorem 1.1.2.** Let \( T \) be a differentiable involution on a sphere \( S^n \) and let \( f : S^n \to M^m \) be a continuous map into a differentiable manifold \( M^m \) of dimension \( m \). Assume that \( f_* : H_n(S^n; \mathbb{Z}_2) \to H_n(M; \mathbb{Z}_2) \) is trivial. Then the dimension of the coincidence set \( A(f) = \{ x \in S^n | f(x) = f(Tx) \} \) is at least \( n - m \).

The Theorem 1.1.2 became a cornerstone in the development of the Borsuk-Ulam type theorems. Its proof helped to shape up the general approach to proving generalizations of the Borsuk-Ulam theorem. Its importance can hardly be overestimated also due to the fact that it has the famous theorem by J. Milnor [11] as one of its corollaries. The theorem of J. Milnor asserts that every element of order two in a group which acts freely on a sphere must be central (see [15] for details). The later theorem played an important role in the solution of the so called ”spherical space form problem” which aim was to give a classification of all finite groups which admit a free action on a sphere.

In their consequent works H. Munkholm [12] and M. Nakaoka [15] showed that the differentiability condition on the involution \( T \) in the formulation of Theorem 1.1.2 can
be dropped provided the target topological manifold $M^m$ is assumed to be compact.

Moreover, they generalized the previous theorem to the case of free actions of a cyclic group $\mathbb{Z}_p$ on (mod p) homology spheres. Their result reads as follows:

**Theorem 1.1.3.** Let a cyclic group $\mathbb{Z}_p$ of a prime order act freely on a (mod p) homology $n$-sphere $N^n$, and let $f: N^n \to M^m$ be a continuous map into a compact topological manifold $M^m$ of dimension $m$. If $p$ is odd also assume that $M$ is orientable. Suppose that $f_*: H_n(N; \mathbb{Z}_p) \to H_n(M; \mathbb{Z}_p)$ is trivial. Then the dimension of the coincidence set $A(f) = \{ x \in N | f(x) = f(gx) \forall g \in \mathbb{Z}_p \}$ is at least $n - m(p - 1)$.

### 1.2 Borsuk-Ulam Theorem for $(\mathbb{Z}_p)^k$-actions

The purpose of this section is to suggest another generalization of the Borsuk-Ulam theorem which initially appeared in [24]. Further and until the rest of the dissertation $p$ is always assumed to be a prime number.

**Theorem 1.2.1.** Let $M := N^{n_1} \times \ldots \times N^{n_k}$ be a product of (mod p) homology $n_i$-spheres and let $\mu: (\mathbb{Z}_p)^k \circ M$ be the product of free actions $\mu_i: \mathbb{Z}_p \circ N^{n_i}$ ($1 \leq i \leq k$). If $p$ is odd also assume that all $n_i$’s are odd. For a map $f: M \to \mathbb{R}^m$ define a coincidence set $A(f) := \{ x \in M | f(x) = f(gx) \forall g \in (\mathbb{Z}_p)^k \}$. Then

$$\dim A(f) \geq \dim M - m(p^k - 1)$$

provided $n_i \geq mp^{i-1}(p - 1)$ for all $i(1 \leq i \leq k)$.

**Remark.** For $p = 2$ and $m = 1$ the theorem above was implicitly proved by A.N. Dranishnikov in [3]. In the case $n_i \geq mp^i - 1$ for all $i(1 \leq i \leq k)$ the theorem above was proved by V.V. Volovikov in [27]. Moreover, in the Volovikov’s theorem the action $\mu$ can be assumed an arbitrary free action.

Let $G$ be a group and let $R$ be a commutative ring with a unit. Then by $I_R(G)$ we denote the augmentation ideal of the group ring $R[G]$, i.e. the kernel of the augmentation homomorphism $R[G] \to R$. In this paper we assume $\mathbb{R}^m$ to be a ring where multiplication structure is given by multiplication of the coordinates.
The key ingredient in the proofs of the most Borsuk-Ulam type theorems for maps into Euclidean spaces is the following basic observation:

**Lemma 1.2.1.** Let $G \acts M$ be a free action of a finite group $G$ on a topological manifold $M$. For a continuous map $f : M \to \mathbb{R}^m$ define a coincidence set $A(f) := \{ x \in M | f(x) = f(gx) \forall g \in G \}$. Then $A(f) \neq \emptyset$ if and only if the vector bundle $\xi : M \times_G I_{\mathbb{R}^m}(G) \to M/G$ does not have a non-vanishing section.

**Proof.** First, note that every continuous map $f : M \to \mathbb{R}^m$ gives rise to a continuous section $\hat{s}(f) : M/G \to M \times G \mathbb{R}^m[G]$ of the vector bundle $\hat{\xi} : M \times G \mathbb{R}^m[G] \to M/G$ defined by a formula:

$$\hat{s}(f)(xG) = (x, \sum_{g \in G} f(xg^{-1})g)G.$$ 

Observe that $\hat{\xi} = \xi \oplus \varepsilon^m_{\mathbb{R}}$ where $\varepsilon^m_{\mathbb{R}}$ is a trivial $m$-dimensional real vector bundle. Therefore a projection $\pi : M \times_G \mathbb{R}^m[G] \to M \times_G I_{\mathbb{R}^m}(G)$ is well defined. Now define a continuous section $s(f) : M/G \to M \times_G I_{\mathbb{R}^m}(G)$ of $\xi$ by a formula $s(f) := \pi \circ \hat{s}(f)$. It is easy to see that $s(f)(xG) = 0$ if and only if the orbit of $x \in M$ is mapped by $f$ to a point.

Conversely, given a continuous section $s$ of $\xi$, it defines a $G$-equivariant map $\bar{s} : M \to M \times \mathbb{R}^m[G]$ which is due to its equivariance must be of the form $\bar{s}(x) = (x, \sum_{g \in G} f(xg^{-1})g)$ for some $f : M \to \mathbb{R}^m$, and the lemma follows. \qed

Usually, to prove a Borsuk-Ulam type theorem for maps into Euclidean spaces one shows that the Euler class of the vector bundle $\xi : M \times_G I_{\mathbb{R}^m}(G) \to M/G$ in a suitable cohomology theory is non-trivial. Then the dimension restrictions on the coincidence set $A(f)$ follow (see the proof of Theorem 1.2.1). The theorems from [12, 13] were proved in this way. Unfortunately, when one uses ordinary cohomology theory, Euler class of $\xi$ very often turns out to be trivial (see [12]). This, in fact, is the reason why all available results in the area are restricted to the actions of so few groups. In this setting the results of H. Munkholm from [13] (also see [14]) are especially interesting. In that paper he proves a Borsuk-Ulam type theorem for $\mathbb{Z}_p\alpha$-actions, $p$ is odd, on odd dimensional spheres using a
\(KU\)-theory Euler class. The remaining case of \(\mathbb{Z}_{2k}\)-actions on spheres, \(k > 1\), is considered in [25] and in this dissertation (see Theorem 1.6.2).

The proof the Theorem 1.2.1 is based on the non-triviality of the \((\text{mod } p)\) Euler class of a corresponding vector bundle. The next two sections will be devoted to the calculation of Euler classes of relevant vector bundles.

1.3 Calculation of \(w_{2k-1}(\eta)\)

In this section assume that \(G = (\mathbb{Z}_2)^k\). As usual \(BG\) stands for the classifying space of \(G\) and \(EG\) stands for the total space of the universal \(G\)-bundle. This section is devoted to the calculation of the \((\text{mod } 2)\) Euler class of a vector bundle \(\eta: EG \times_G I_\mathbb{R}(G) \rightarrow BG\), i.e. its Stiefel-Whitney class \(w_{2k-1}(\eta)\). These calculations are then needed in the proof of Theorem 1.2.1 in case \(p = 2\). Recall that \(H^*(BG; \mathbb{Z}_2)\) is a polynomial algebra \(\mathbb{Z}_2[x_1, \ldots, x_k]\) on 1-dimensional generators.

**Lemma 1.3.1.** \(w_{2k-1}(\eta) = \prod_{q=1}^{k} \prod_{1 \leq i_1 < \ldots < i_q \leq k} (x_{i_1} + \ldots + x_{i_q})\)

**Proof.** Let \(\mathbb{Z}_2\) act on \(\mathbb{R}\) by an obvious involution. This involution induces on \(\mathbb{R}\) a structure of an \(\mathbb{R}[\mathbb{Z}_2]\)-module which we will denote by \(V\). Denote by \(pr_i: BG \rightarrow \mathbb{R}P^\infty\) a projection on the \(i^{th}\) coordinate. Then by \(\lambda_i\) we denote a 1-dimensional real vector bundle obtained from the following diagram:

\[
\begin{array}{ccc}
E(\lambda_i) & \rightarrow & S^\infty \times_{\mathbb{Z}_2} V \\
\lambda_i \downarrow & & \downarrow \\
BG & \xrightarrow{pr_i} & \mathbb{R}P^\infty
\end{array}
\]

Here \(S^\infty\) stands for the infinite dimensional sphere. From the construction of \(\lambda_i\) it follows that \(w_1(\lambda_i) = x_i\).

Let \(\eta_i\) be a vector bundle obtained from the following diagram:

\[
\begin{array}{ccc}
E(\eta_i) & \rightarrow & S^\infty \times_{\mathbb{Z}_2} \mathbb{R}[\mathbb{Z}_2] \\
\eta_i \downarrow & & \downarrow \\
BG & \xrightarrow{pr_i} & \mathbb{R}P^\infty
\end{array}
\]
From the isomorphism $\mathbb{R}[\mathbb{Z}_2] \cong V \oplus (V \otimes_{\mathbb{R}[\mathbb{Z}_2]} V) \cong V \oplus V^2$ it follows that $\eta_i \cong \lambda_i^2 \oplus \lambda_i$ where $\lambda_i^2 = \lambda_i \otimes \lambda_i$ is a trivial 1-dimensional bundle. Recall the isomorphism of $\mathbb{R}$-modules: $\mathbb{R}[G] \cong \mathbb{R}[\mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2] \cong \mathbb{R}[\mathbb{Z}_2] \otimes_{\mathbb{R}} \mathbb{R}[\mathbb{Z}_2]$. From this isomorphism it follows that $\eta \oplus \varepsilon^1_\mathbb{R} \cong \eta_1 \otimes \ldots \otimes \eta_k$. Therefore, there exists the following chain of isomorphisms of vector bundles:

$$\eta \oplus \varepsilon^1_\mathbb{R} \cong \bigotimes_{i=1}^k (\lambda_i \oplus \lambda_i^2) \cong \bigoplus_{(\alpha_1,\ldots,\alpha_k) \in G} (\lambda_1^{\alpha_1} \otimes \ldots \otimes \lambda_k^{\alpha_k}).$$

It is a well known that the first Stiefel-Whitney class of a tensor product of 1-dimensional real vector bundles equals to the sum of the first Stiefel-Whitney classes of the multiplies. Then by this fact and a formula of Whitney we get the following chain of equalities:

$$w_{2k-1}(\eta) = w_{2k-1}(\eta \oplus \varepsilon^1_\mathbb{R}) = \prod_{(\alpha_1,\ldots,\alpha_k) \neq 0} (\alpha_1 x_1 + \ldots + \alpha_k x_k) = \prod_{q=1}^k \prod_{1 \leq i_1 < \ldots < i_q \leq k} (x_{i_1} + \ldots + x_{i_q}).$$

1.4 Euler Class of $\eta_C$: $EG \times_G I_C(G) \to BG$

Throughout this section assume that $p$ is a fixed odd prime and that $G = (\mathbb{Z}_p)^k$. In this section we will calculate the $(\text{mod } p)$ Euler class of a complex vector bundle $\eta_C: EG \times_G I_C(G) \to BG$ which equals to its Chern class $c_{p,k-1}(\eta_C)$. These calculations are then needed in the proof of Theorem 1.2.1 in case of odd primes. Recall that:

$$H^*(BG; \mathbb{Z}_p) = \Lambda_{\mathbb{Z}_p}(y_1,\ldots,y_k) \otimes \mathbb{Z}_p[x_1,\ldots,x_k],$$

where $\Lambda_{\mathbb{Z}_p}(y_1,\ldots,y_k)$ is an exterior algebra on 1-dimensional generators and $\mathbb{Z}_p[x_1,\ldots,x_k]$ is a polynomial algebra on 2-dimensional generators.

Chern classes of a regular representation of $G$, i.e. Chern classes of the vector bundle $\eta_C \oplus \varepsilon^C_C: EG \times_G \mathbb{C}[G] \to BG$, were first computed by B.M. Mann and R.J. Milgram in
The lemma which is stated after the next definition is essentially borrowed from their paper.

**Definition 1.4.1.** $L_k = \prod_{i=1}^{k} \prod_{\alpha_j \in \mathbb{Z}/p} (\alpha_1 x_1 + ... + \alpha_{i-1} x_{i-1} + x_i)$

The polynomial defined above is called the $k^{th}$ Dickson’s polynomial (see [10] for more details).

**Lemma 1.4.1.** $e(\eta_C) = (-1)^k L_p^{k-1}$

**Proof.** The action of $\mathbb{Z}_p$ on $\mathbb{C}$ by rotations by $\frac{2\pi}{p}$ induces on $\mathbb{C}$ a structure of a $\mathbb{C}[\mathbb{Z}_p]$-module which we will denote by $L$. Let $pr_i: BG \to B\mathbb{Z}_p$ be a projection on the $i^{th}$ coordinate.

Then let $\lambda_i$ be a 1-dimensional complex vector bundle obtained from the following diagram:

$$
\begin{align*}
E(\lambda_i) &\longrightarrow S^\infty \times_{\mathbb{Z}_p} L \\
\lambda_i &\downarrow \\
BG &\xrightarrow{pr_i} B\mathbb{Z}_p
\end{align*}
$$

It is not very difficult to show that $c_1(\lambda_i) = x_i$.

Let $\eta_i$ be a vector bundle obtained from the following diagram:

$$
\begin{align*}
E(\eta_i) &\longrightarrow S^\infty \times_{\mathbb{Z}_p} \mathbb{C}[\mathbb{Z}_p] \\
\eta_i &\downarrow \\
BG &\xrightarrow{pr_i} B\mathbb{Z}_p
\end{align*}
$$

It follows from the isomorphism $\mathbb{C}[\mathbb{Z}_p] \cong L \oplus ... \oplus L^p$, where $L^j = L \otimes_{\mathbb{C}[\mathbb{Z}_p]} ... \otimes_{\mathbb{C}[\mathbb{Z}_p]} L$, that $\eta_i \cong \lambda_i \oplus ... \oplus \lambda_i^p$. Here $\lambda_i^j = \lambda_i \otimes_{\mathbb{C}} ... \otimes_{\mathbb{C}} \lambda_i$. Also note that $\lambda_i^p$ is a trivial 1-dimensional complex bundle. Recall the isomorphism of $\mathbb{C}$-modules: $\mathbb{C}[G] \cong \mathbb{C}[\mathbb{Z}_p \oplus ... \oplus \mathbb{Z}_p] \cong \mathbb{C}[\mathbb{Z}_p] \otimes_{\mathbb{C}} ... \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{Z}_p]$. From this isomorphism it follows that $\eta_C \oplus \varepsilon_C \cong \eta_1 \otimes ... \otimes \eta_k$.

Therefore there exists the following chain of isomorphisms of vector bundles:

$$
\eta_C \oplus \varepsilon_C \cong \bigotimes_{i=1}^{k} (\lambda_i \oplus ... \oplus \lambda_i^p) \cong \bigoplus_{(\alpha_1,...,\alpha_k) \in G} \lambda_i^{a_1} \otimes ... \otimes \lambda_i^{a_k}.
$$
From a formula by Whitney and the fact that the first Chern class of a tensor product of 1-dimensional complex bundles equals to the sum of the first Chern classes of the multiples, it follows that

\[ c_{p^k-1}(\eta_C) = c_{p^k-1}(\eta_C \oplus \varepsilon^1_C) = \prod_{(\alpha_1, \ldots, \alpha_k) \neq 0} (\alpha_1 x_1 + \ldots + \alpha_k x_k) = \]

\[ = \prod_{i=1}^{k} [(p-1)!]^k \prod_{(\alpha_1, \ldots, \alpha_k, 1, 0, \ldots, 0, 0, \ldots, 0)} (\alpha_1 x_1 + \ldots + \alpha_k x_k)^{p-1} = \]

\[ = [(p-1)!]^k L_k^{p-1} = (-1)^k L_k^{p-1}. \]

The last equality follows from a theorem of Wilson which states that \((p-1)! \equiv (-1)(\text{mod } p)\). Thus \(e(\eta_C) = c_{p^k-1}(\eta_C) = (-1)^k L_k^{p-1}. \)

**1.5 Proof of Theorem 1.2.1**

In this section we use the results of the previous sections to finish the proof of Theorem 1.2.1. Here assume that \(p\) is any prime number and that \(G = (\mathbb{Z}_p)^k\).

**Proof of Theorem 1.2.1.** Recall that \(M = N_1 \times \ldots \times N^n\) is a product of (mod \(p\)) homology \(n_i\)-spheres. We will begin the proof by showing that under assumptions of the theorem the (mod \(p\)) Euler class of \(\xi_M: M \times \mathbb{I}_\mathbb{R}^m(G) \to M/G\) is non-trivial.

By universality property there exists the following commutative diagram:

\[
\begin{array}{ccc}
M \times \mathbb{I}_\mathbb{R}^m(G) & \longrightarrow & EG \times_G \mathbb{I}_\mathbb{R}^m(G) \\
\xi_M \downarrow & & \downarrow \xi \\
M/G & \xrightarrow{\phi} & BG.
\end{array}
\]

**Case \(p=2\).** Let \(\eta\) be the vector bundle from section 1.3. Then from the isomorphism \(I_\mathbb{R}^m \cong I_\mathbb{R} \oplus \ldots \oplus I_\mathbb{R} \cong mI_\mathbb{R}\) it follows that \(\xi \cong \eta \oplus \ldots \oplus \eta \cong m\eta\). Thus \(e_2(\xi) = w_{2^{k-1}}(\xi) = w_{2^{k-1}}(\eta)^m\).

By Lemma 1.3.1 we have

\[ w_{2^{k-1}}(\eta) = \prod_{q=1}^{k} \prod_{i_1 \leq i_2 < \ldots < i_q \leq k} (x_{i_1} + \ldots + x_{i_q}) = \]
\[ x_k^{2^k-1} \prod_{q=1}^{k-1} \prod_{1 \leq i_1 < \ldots < i_q \leq k-1} (x_{i_1} + \ldots + x_{i_q}) + R_k, \]

where \( R_k \) contains monomials in powers less than \( 2^{k-1} \). Therefore

\[ e_2(\xi) = x_1^{m_1} x_2^{2m_1} \cdot \ldots \cdot x_k^{2^{k-1}m} + Q_k, \quad (1) \]

where \( Q_k \) does not contain monomials of the form \( x_1^{m_1} x_2^{2m_1} \cdot \ldots \cdot x_k^{2^{k-1}m} \). It is easy to verify that

\[ H^*(M/G; \mathbb{Z}_2) = \mathbb{Z}_2[x_1, \ldots, x_k]/(x_1^{n_1+1}, \ldots, x_k^{n_k+1}), \]

and \( \varphi^*: H^*(BG; \mathbb{Z}_2) \to H^*(M/G; \mathbb{Z}_2) \) is an epimorphism with

\[ \text{Ker } \varphi^* = (x_1^{n_1+1}, \ldots, x_k^{n_k+1}). \]

Thus from (1) and the assumption \( n_i \geq m2^{i-1} \) for all \( i(1 \leq i \leq k) \) it follows that \( e_2(\xi_M) = \varphi^*(e_2(\xi)) \neq 0 \).

Case \( p > 2 \). Let \( \eta_C \) be a vector bundle from section 1.4. Then from the isomorphism \( I_{C^m} \cong I_C \oplus \ldots \oplus I_C \cong mI_C \) it follows that \( C\xi \cong \eta \oplus \ldots \oplus \eta \cong m\eta \), where \( C\xi \) is a complexification of the vector bundle \( \xi \). We have the following chain of equalities:

\[ e_p(\xi)^2 = e_p(C\xi) = c_{2^{k-1}}(C\xi) = c_{2^{k-1}}(\eta_C)^m. \quad (2) \]

By Lemma 1.4.1 we have

\[ e_p(\eta_C) = (-1)^k L_k^{p-1} = (-1)^k L_k^{p-1} \left[ \prod_{\alpha_j \in \mathbb{Z}_p} (\alpha_1 x_1 + \ldots + \alpha_{k-1} x_{k-1} + x_k) \right]^{p-1} = \]

\[ = (-1)^k x_k^{p^{k-1}(p-1)} L_k^{p-1} + R_k, \]

where \( R_k \) contains \( x_k \) in powers less than \( p^{k-1}(p-1) \). Thus

\[ e_p(C\xi) = (-1)^{km} x_k^{mp^{k-1}(p-1)} L_k^{m(p-1)} + \hat{R}_k, \]

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where $\hat{R}_k$ contains $x_k$ in powers less than $mp^{k-1}(p-1)$. Then by induction it follows that

$$e_p(\mathbb{C}\xi) = (-1)^k m x_1^{m(p-1)} x_2^{mp(p-1)} \cdots x_k^{mp^{k-1}(p-1)} + Q_k,$$

where $Q_k$ contains no monomials of the form

$$bx_1^{m(p-1)} x_2^{mp(p-1)} \cdots x_k^{mp^{k-1}(p-1)}, \quad b \neq 0, \quad b \in \mathbb{Z}_p.$$

Therefore from the previous and (2) it follows that

$$e_p(\xi_M) = ax_1^{m(p-1)} x_2^{mp(p-1)} \cdots x_k^{mp^{k-1}(p-1)} + \hat{Q}_k,$$

where $a^2 \equiv (-1)^q \pmod{p}$ for some $q \geq 0$ and $\hat{Q}_k$ contains no monomials of the form

$$bx_1^{m(p-1)} x_2^{mp(p-1)} \cdots x_k^{mp^{k-1}(p-1)}, \quad b \neq 0, \quad b \in \mathbb{Z}_p.$$

It is not very difficult to see that

$$H^*(M/G; \mathbb{Z}_p) = \Lambda_{\mathbb{Z}_p}(y_1, \ldots, y_k) \otimes_{\mathbb{Z}_p} \mathbb{Z}_2[x_1, \ldots, x_k]/(x_1^{n_1+1}, \ldots, x_k^{n_k+1}),$$

where $\dim x_i = 2$, and $\varphi^*: H^*(BG; \mathbb{Z}_p) \to H^*(M/G; \mathbb{Z}_p)$ is an epimorphism with

$$\text{Ker } \varphi^* = (x_1^{n_1+1}, \ldots, x_k^{n_k+1}).$$

Then from (3) and the assumption $n_i \geq mp^{i-1}(p-1)$ for all $i(1 \leq i \leq k)$ it follows that $e_p(\xi_M) = \varphi^*(e_p(\xi)) \neq 0$.

Since $A(f)$ is closed and $G$-invariant, the set $M \setminus A(f)$ is also $G$-invariant, and therefore we can consider the following exact sequence of a pair:

$$\cdots \to H^i(M/G, (M \setminus A(f))/G) \xrightarrow{\alpha} H^i(M/G) \xrightarrow{\beta} H^i((M \setminus A(f))/G) \to \cdots$$

By Lemma 1.2.1 the vector bundle $\xi_M$ has a non-vanishing section over $M \setminus A(f)$. Thus $\beta(e_p(\xi_M)) = 0$. Therefore there exists a non-trivial element

$$\mu \in H^{mp^{k-1}}(M/G, (M \setminus A(f))/G)$$
such that \( \alpha(\mu) = e_p(\xi_M) \). Since we are working over coefficients in a field \( \mathbb{Z}_p \) there exists a corresponding non-trivial element \( \tilde{\mu} \in H_{m(p^k-1)}(M/G, (M \setminus A(f))/G) \). Then by Alexander duality we have

\[
H^{\dim M - m(p^k-1)}(A(f)/G; \mathbb{Z}_p) \neq 0,
\]

and thus \( \dim_{\mathbb{Z}_p} A(f)/G \geq \dim M - m(p^k - 1) \) (see [7]). Since the group \( G \) is finite it easily follows that

\[
\dim A(f) \geq \dim_{\mathbb{Z}_p} A(f) \geq \dim M - m(p^k - 1),
\]

and we are done.

1.6 Borsuk-Ulam Theorem for \( \mathbb{Z}_{2^k} \)-actions

In [14] H. Munkholm and M. Nakaoka proved the following generalization of the Borsuk-Ulam theorem:

**Theorem 1.6.1.** Let \( G \) be a cyclic group of odd order \( p^k \), where \( p \) is a prime, and \( \Sigma \) be a homotopy \( 2n + 1 \) sphere on which a free differentiable \( G \)-action is given. Let \( M \) be a differentiable \( m \)-manifold and let \( f: \Sigma \to M \) be a continuous map. Then the set \( A(f) = \{ x \in \Sigma | f(x) = f(gx) \ \forall g \in G \} \) has dimension at least \( 2n + 1 - (p^k - 1)m - [m(k - 1)p^k - (mk + 2)p^{k-1} + m + 3] \).

The proof of the theorem above is essentially based on non-triviality of a \( \widetilde{KU} \)-theoretic Euler class of a certain complex vector bundle. Let us sketch here the main ideas of the proof needed to show that \( A(f) \neq \emptyset \) provided that dimension of \( \Sigma \) is sufficiently large. For simplicity, we will omit the tricks used to estimate the dimension of \( A(f) \).

First, consider a bundle \( \hat{\xi}: \Sigma \times_G MG \to \Sigma/G \), where \( MG = \prod_{i=1}^{[G]} M \) and \( G \) acts on \( MG \) by permuting the coordinates. Every continuous map \( f: \Sigma \to M \) induces a section \( s(f) \) of the bundle \( \hat{\xi} \) given by the formula:

\[
s(f)(xG) = (x, \sum_{g \in G} f(xg^{-1})g)G.
\]
One can easily show that functions \( \{ f: \Sigma \to M \} \) which do not collapse any orbit of \( G \) to a point are in one-to-one correspondence with sections \( \{ s(f) \} \) such that \( s(f)(\Sigma/G) \cap (\Sigma/G \times \Delta M) = \emptyset \), where \( \Delta M \) is the diagonal in the product \( MG \). See Lemma 1.2.1 for details.

Now consider the normal bundle \( \nu \) of \( \Sigma/G \times \Delta M \) in \( \Sigma \times_G MG \). It was shown in [14] that \( \nu \) has a structure of a complex vector bundle (Proposition 2 [14]). Mainly it follows from the fact that all irreducible representations of \( G \) are complex. Let \( \theta \) be the \( \widetilde{KU} \)-theoretic Thom class of \( \nu \). It turns out that for any continuous map \( f: \Sigma \to M \) the induced homomorphism

\[
s(f)^*: \widetilde{KU}^*(\Sigma \times_G MG, \Sigma/G \times \Delta M) \to \widetilde{KU}^*(\Sigma/G)
\]

in K-theory maps \( \theta \) to the Euler class of the vector bundle \( \xi: \Sigma_G \times I_{\mathbb{R}^m}(G) \to \Sigma/G \), where \( I_{\mathbb{R}^m} \) is the kernel of the augmentation homomorphism \( \mathbb{R}^m[G] \to \mathbb{R}^m \) (see Proposition 3 of [14]). Here and throughout the paper \( \mathbb{R}^m \) is assumed to be a ring with multiplication given by the multiplication of the coordinates.

It follows from our constructions that if \( f: \Sigma \to M \) is a map which does not collapse any orbit of \( G \) to a point, then \( s(f)^*(\theta) = e(\xi) = 0 \).

The complex \( G \)-module \( I_{\mathbb{R}^m}(G) \) is a sum of all non-trivial irreducible complex \( G \)-modules, which makes \( \xi \) a sum of one-dimensional complex vector bundles. This decomposition allows to compute the \( \widetilde{KU} \)-theoretic Euler class of \( \xi \). From certain considerations in elementary algebraic number theory it follows that \( e(\xi) \neq 0 \) provided \( \text{dim} \Sigma \) is sufficiently large, which completes the proof of the theorem.

In case \( G = \mathbb{Z}_{2^k}, k > 1 \), neither \( \nu \) nor \( \xi \) have a complex structure, simply, because the dimension of \( \xi \) and \( \nu \) is odd. This fact does not allow to use complex K-theory in order to prove that \( \xi \) does not have a non-vanishing section provided dimension of \( \Sigma \) is sufficiently large. Presumably, this is the reason the above result of H. Munkholm and M. Nakaoka is restricted to the case of odd order groups \( G = \mathbb{Z}_{p^k} \). Note that the Euler class of \( \xi \) in ordinary cohomology theory is trivial if \( k > 2 \) (see [12],[14] for details).
**Remark.** The manifolds $\Sigma$ and $M$ in Theorem 1.6.1 do not need to be assumed differentiable. The proof works without any changes if one assumes $\Sigma = S^{2n+1}$, the action $G \acts \Sigma$ to be free and the manifold $M$ to be an $m$-dimensional topological manifold.

To the best knowledge of the author a Borsuk-Ulam type theorem for free $\mathbb{Z}_{2k}$-actions on spheres has not been published yet. Theorem 1.6.2 which first appeared in [25] covers this gap.

**Theorem 1.6.2.** Let $S^{2n+1}$ be a $(2n + 1)$-dimensional sphere endowed with a free action of a cyclic group $\mathbb{Z}_{2k}$, where $k > 1$. Let $M$ be an $m$-dimensional topological manifold and let $f : S^{2n+1} \to M$ be a continuous map. Then the coincidence set $A(f) = \{x \in \Sigma | f(x) = f(gx) \ \forall g \in \mathbb{Z}_{2k}\}$ has dimension at least $(2n + 1) - \lfloor 2^k(m - 1) + m2^{k-1}(k - 1) + 1 \rfloor$.

We will postpone the proof of Theorem 1.6.2 until section 1.10. In the next sections we will state and prove all the necessary results which are needed for the proof of Theorem 1.6.2.

### 1.7 Necessary Lemmas

Let $G = \mathbb{Z}_{2k}$ and suppose that a free action of $G$ on $S^{2n+1}$ is given. Then consider a bundle

$$\hat{\xi}: S^{2n+1} \times_G MG \to S^{2n+1}/G,$$

where $M$ is a topological manifold of dimension $m$ and $MG = \prod_{i=1}^{[G]} M$. It is assumed here that $G$ acts on $MG$ by permutation of coordinates. Let

$$\nu: E(\nu) \to S^{2n+1}/G \times \Delta M$$

be a normal bundle of $S^{2n+1}/G \times \Delta M$ in $S^{2n+1} \times_G MG$. Here $\Delta M$ is the diagonal of $MG$ invariant under the action of $G$. Let

$$\xi: S^{2n+1} \times_G I_{\mathbb{R}^m}(G) \to S^{2n+1}/G$$

be a vector bundle with $I_{\mathbb{R}^m}(G) = Ker(\mathbb{R}^m[G] \to \mathbb{R}^m)$ as a fiber. Then the following lemma holds:
Lemma 1.7.1. Let $i: S^{2n+1}/G \hookrightarrow S^{2n+1}/G \times \Delta M$ be an obvious inclusion. Then $i^*(\nu) = \xi$.

As it was mentioned in section 1.6, the proof of Theorem 1.6.2 heavily relies on the geometry of the vector bundle $\xi$. In the next lemma we will give a full description of $\xi$ in terms of ”smaller” vector bundles whose geometry is fairly simple.

Let $G$ act on $\mathbb{C}$ by rotations by $\frac{2\pi}{2k}$. This action gives $\mathbb{C}$ a structure of a $\mathbb{C}[G]$-module which in its turn gives rise to a vector bundle $\lambda: S^{2n+1} \times_G \mathbb{C} \to S^{2n+1}/G$. Similarly, the group $G$ acts on $\mathbb{R}$ by involutions which gives rise to a vector bundle $\mu: S^{2n+1} \times_G \mathbb{R} \to S^{2n+1}/G$.

Lemma 1.7.2. $\xi = m(\mu \oplus \lambda \oplus \lambda^2 \oplus ... \oplus \lambda^{2k-1})$

Proof. The proof of this lemma follows immediately from elementary representation theory.

Lemma 1.7.3. $\xi \oplus m\mu = m(\lambda \oplus \lambda^2 \oplus ... \oplus \lambda^{2k-1})$

Proof. This fact is an immediate consequence of Lemma 1.7.2 and the following obvious equality:

$$\mu \oplus \mu = \lambda^{2k-1}.$$
to see that there exists the following commutative diagram:

\[
\begin{align*}
E(\xi \oplus m\mu) & \longrightarrow E(\nu \oplus m\mu^*) \\
\xi \oplus m\mu \downarrow & \quad \downarrow \nu \oplus m\mu^* \\
S^{2n+1}/G & \hookrightarrow S^{2n+1}/G \times \Delta M.
\end{align*}
\]

The existence of this diagram is equivalent to the statement of the following lemma:

**Lemma 1.7.4.** \(\xi \oplus m\mu = i^*(\nu \oplus m\mu^*)\).

It follows now that a vector bundle \(\nu \oplus m\mu^*\) admits a structure of a complex vector bundle.

It is easy to see that there exists the following commutative diagram:

\[
\begin{align*}
S^{2n+1} \times_G (MG \times \mathbb{R}^m) & \longrightarrow S^{2n+1} \times_G \mathbb{R}^m \\
\pi \downarrow \uparrow j & \quad \downarrow m\mu \\
S^{2n+1} \times_G MG & \longrightarrow S^{2n+1}/G,
\end{align*}
\]

where \(j: S^{2n+1} \times_G MG \to S^{2n+1} \times_G (MG \times \mathbb{R}^m)\) is the obvious inclusion and \(\pi \circ j = id\). It is not very difficult to show that the normal bundle of \(S^{2n+1}/G \times \Delta M\) in \(S^{2n+1} \times_G (MG \times \mathbb{R}^m)\) is isomorphic to

\[
\nu \oplus m\mu^*: E(\nu \oplus m\mu^*) \to S^{2n+1}/G \times \Delta M.
\]

Now let \(f: S^{2n+1} \to M\) be a continuous map and let

\[
\hat{s}(f): S^{2n+1}/G \to S^{2n+1} \times_G MG
\]

be a section of the bundle \(\hat{\xi}: S^{2n+1} \times MG \to S^{2n+1}/G\) associated to it. Define a map

\[
s(f): S^{2n+1}/G \to S^{2n+1} \times_G (MG \times \mathbb{R}^m)
\]

as a composition \(s(f) = j \circ \hat{s}(f)\). Consider the Thom class of the complex vector bundle \(\nu \oplus m\mu^*\):

\[
T(\nu \oplus m\mu^*) \in \widetilde{KU}^0(S^{2n+1} \times_G (MG \times \mathbb{R}^m), S^{2n+1}/G \times \Delta M).
\]
Note that by excision there exists the following isomorphism:

\[ \tilde{KU}^0(E(\nu \oplus \mu^*), S^{2n+1}/G \times \Delta M) \simeq \tilde{KU}^0(S^{2n+1} \times_G (MG \times \mathbb{R}^m), S^{2n+1}/G \times \Delta M). \]

Define \( \theta \in \tilde{KU}^0(S^{2n+1} \times (MG \times \mathbb{R}^m)) \) by a formula

\[ \theta = i'^*T(\nu \oplus m\mu^*), \]

where

\[ i'^*: (S^{2n+1} \times (MG \times \mathbb{R}^m), \emptyset) \hookrightarrow (S^{2n+1} \times (MG \times \mathbb{R}^m), S^{2n+1}/G \times \Delta M) \]

is an obvious inclusion. The following proposition plays an important role in the proof of Theorem 1.6.2:

**Proposition 1.7.1.** In the notations above, one has the following equality:

\[ s(f)^*\theta = e(\xi \oplus m\mu). \]

**Proof.** The idea of the proof is to compare the map \( s(f) \) with the ”zero” section of \( \tilde{\xi} \circ \pi \) on the cohomological level.

1.8 Computation of Norms

Let \( F/K \) be a field extension. Recall that a norm map \( N: F \to K \) is a map defined as \( N(u) = ((-1)^na_0)^{[F:K(u)]} \), where \( f(x) = x^n + \ldots + a_0 \in K[x] \) is the minimal polynomial of \( u \in F \). Also recall that \( N: F \to K \) is a multiplicative map, i.e. for any \( u, v \in F \) we have \( N(uv) = N(u)N(v) \). Throughout this section and further in the paper \( \Phi_n(x) \) stands for the \( n^{th} \) cyclotomic polynomial.

**Lemma 1.8.1.** For a prime \( p \), let \( \gamma \) be a primitive root of unity of order \( p^k \) and let \( \mathbb{Q}(\gamma)/\mathbb{Q} \) be the corresponding cyclotomic extension. Then

(a) \( N(\gamma^{p^l} - 1) = p^{l^2}, 0 \leq l < k, \)

(b) \( N(\Phi_{p^l}(\gamma)) = p^{l^2 - p^{l-1}}, 0 \leq l < k, \)

(c) \( N(\Phi_{mp^l}(\gamma)) = 1, \) if \( p \nmid m, m > 1. \)

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Proof. (a) First, note that $\gamma^{p^l}$ is a primitive root of unity of order $p^{k-l}$. Therefore, the minimal polynomial for $(\gamma^{p^l} - 1)$ is $\Phi_{p^{k-l}}(x + 1)$. Thus $a_0 = \Phi_{p^{k-l}}(1) = p$. Also, note that

$$[\mathbb{Q}(\gamma) : \mathbb{Q}(\gamma^{p^l} - 1)] = \frac{[\mathbb{Q}(\gamma) : \mathbb{Q}]}{[\mathbb{Q}(\gamma^{p^l} - 1) : \mathbb{Q}]} = \frac{p^k - p^{k-l}}{p^{k-l} - p^{k-l-1}} = p^l.$$

Therefore,

$$N(\gamma^{p^l} - 1) = ((-1)^{p^{k-l-1}}p)^{p^l} = p^{p^l}.$$

(b) From (a) it follows that

$$N(\Phi_{p^l}(\gamma)) = \frac{N(\gamma^{p^l} - 1)}{N(\gamma^{p^l-1} - 1)} = p^{p^l-1}.$$

(c) Note that, if $p \nmid m$, then $\gamma^{mp^l}$ is a primitive root of unity of order $p^{k-l}$. Following the same reasoning as in (a), one can show that $N(\gamma^{mp^l} - 1) = N(\gamma^{p^l} - 1) = p^l$. Thus, we have

$$N(\Phi_{mp^l}(\gamma)) = \frac{N(\gamma^{mp^l} - 1)}{\prod_{d\mid mp^l, d < mp^l} \Phi_d(\gamma)} = \frac{N(\gamma^{mp^l} - 1)}{N(\gamma^{p^l} - 1)} = 1.$$

1.9 Computation of Euler Class $e(\xi \oplus m\mu)$

In this section we will compute the Euler class $e(\xi \oplus m\mu) \in \widetilde{KU}^0(S^{2n+1}/G)$ and will prove that it is non-trivial provided the dimension of the sphere $S^{2n+1}$ is sufficiently large.

**Proposition 1.9.1.** Let $\mathbb{Z}_m$ act on $\mathbb{C}$ by rotations by $\frac{2\pi}{m}$ and let

$$\lambda: S^{2n+1} \times_{\mathbb{Z}_m} \mathbb{C} \to S^{2n+1}/\mathbb{Z}_m$$

be a vector bundle associated to this action. Then

$$\widetilde{KU}^0(S^{2n+1}/\mathbb{Z}_m) \cong \mathbb{Z}[x]/((x + 1)^{2^k} - 1, x^{n+1}),$$

where $x = e(\lambda)$.

**Proof.** Let $\eta$ be the universal one-dimensional complex vector bundle over $\mathbb{C}P^n$. It is a well-known fact that the total space of the spherization of the $m$-fold tensor product of $\eta$
is homeomorphic to $S^{2n+1}/\mathbb{Z}_m$. The later fact allows to write down the following Gysin long exact sequence:

$$
\cdots \to \widetilde{K}^0(U(CP^n)) \xrightarrow{\cup c_1} \widetilde{K}^0(CP^n) \to \widetilde{K}^0(S^{2n+1}/\mathbb{Z}_m) \to \cdots
$$

The ring $\widetilde{K}^0(U(CP^n))$ is a truncated polynomial algebra $\mathbb{Z}[[x]]/(x^{n+1})$ where $x = e(\eta)$. From a formula $e(\eta \otimes \eta) = x^2 + 2x$ it follows that

$$
e(\eta^m) = (x + 1)^m - 1.
$$

Therefore, $\widetilde{K}^0(U(S^{2n+1}/\mathbb{Z}_m)) \cong \mathbb{Z}[[x]]/((x + 1)^m - 1, x^{n+1})$. The equality $x = e(\lambda)$ in $\widetilde{K}^0(U(S^{2n+1}/\mathbb{Z}_m))$ follows from the fact that the homomorphism

$$
(\eta^m)^* : H^2(CP^n) \to H^2(S^{2n+1}/\mathbb{Z}_m)
$$

map the first Chern class of $\eta$ to the first Chern class of $\lambda$.

**Proposition 1.9.2.** $e(\xi \oplus m\mu) = [(x - 1)(x^2 - 1) \cdot \ldots \cdot (x^{2^{k-1}} - 1)]^m$.

**Proof.** The proposition follows from Lemma 1.7.2 and Proposition 1.9.1.

**Proposition 1.9.3.** Let $d \geq 0$ and $I = ((x - 1)^{n+1}, x^{2^k} - 1)$ be an ideal in $\mathbb{Z}[x]$ generated by polynomials $(x - 1)^{n+1}$ and $x^{2^k} - 1$. Suppose that

$$
P(x) = (x - 1)^d[(x - 1)(x^2 - 1) \cdot \ldots \cdot (x^{2^{k-1}} - 1)]^m \in I,
$$

then $d \geq n - 2^{k-1}(m - 1) - m2^{k-2}(k - 1)$.

**Proof.** The polynomial $P(x)$ lies in the ideal $I$ if and only if there exist polynomials $h(x)$ and $g(x)$ such that

$$
(x - 1)^d[(x - 1)(x^2 - 1) \cdot \ldots \cdot (x^{2^{k-1}} - 1)]^m = h(x)(x - 1)^{n+1} + g(x)(x^{2^k} - 1)
$$
Let $\Phi_j(x)$ be the $j^{th}$ cyclotomic polynomial. Then the equality above can be rewritten in the following form:

$$(x - 1)^{d + m^{2k-1}} \prod_{j=2}^{2k-1} \Phi_j(x)^{\frac{x^{2k-1} - 1}{j}} = h(x)(x - 1)^{n+1} + g(x)(x - 1) \prod_{j=2}^{k} \Phi_{2j}(x).$$

Let $\epsilon_j$ be defined as follows:

$$\epsilon_j = \begin{cases} 
1, & \text{if } j|2^k \\
0, & \text{if } j \nmid 2^k.
\end{cases}$$

There exist polynomials $\bar{g}(x)$ and $\bar{h}(x)$ such that:

$$\prod_{j=2}^{2k-1} \Phi_j(x)^{\frac{x^{2k-1} - 1}{j} - \epsilon_j} = \bar{h}(x)(x - 1)^{n+1-d-m^{2k-1}} + \bar{g}(x)\Phi_{2k}(x).$$

Let $\gamma$ be a primitive root of unity of order $2^k$ and let $Q(\gamma)/Q$ be the corresponding cyclotomic extension. Then in $Q(\gamma)$ we have:

$$\prod_{j=2}^{2k-1} \Phi_j(\gamma)^{\frac{x^{2k-1} - 1}{j} - \epsilon_j} = \bar{h}(\gamma)(\gamma - 1)^{n+1-d-m^{2k-1}}.$$ 

$$\prod_{j=2}^{2k-1} N(\Phi_j(\gamma))^{\frac{x^{2k-1} - 1}{j} - \epsilon_j} = N(\bar{h}(\gamma))N((\gamma - 1))^{n+1-d-m^{2k-1}}.$$ 

So, it follows that

$$N((\gamma - 1))^{n+1-d-m^{2k-1}} \mid \prod_{j=2}^{2k-1} N(\Phi_j(\gamma))^{\frac{x^{2k-1} - 1}{j} - \epsilon_j}.$$ 

By Lemma 1.8.1 we have:

$$\prod_{j=2}^{2k-1} N(\Phi_j(\gamma))^{\frac{x^{2k-1} - 1}{j} - \epsilon_j} = \prod_{j=1}^{k-1} N(\Phi_{2j}(\gamma))^{m^{2k-j-1} - 1} = \prod_{j=1}^{k-1} 2^{(2j-2j-1)(m^{2k-j-1} - 1)} = 2^{\sum_{j=1}^{k-1} 2j-1(m^{2k-j-1} - 1)} = 2^{m^{2k-2}(k-1)-2k-1+1}.$$ 

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Note that according to Lemma 1.8.1, \( N(\gamma - 1) = p \). Thus, we have
\[
n + 1 - d - m2^{k-1} \leq m2^{k-2}(k - 1) - 2^{k-1} + 1
\]
and
\[
d \geq n - 2^{k-1}(m - 1) - m2^{k-2}(k - 1).
\]

**Corollary 1.9.1.** Let \( d \geq 0 \) and suppose that the class \( e(d\lambda \oplus \xi \oplus m\mu) \) is trivial in \( \widetilde{KU}^0(S^{2n+1}/\mathbb{Z}_{2^k}) \). Then \( d \geq n - 2^{k-1}(m - 1) - m2^{k-2}(k - 1) \).

*Proof.* From Lemma 1.7.2 it follows that:
\[
e(d\lambda \oplus \xi \oplus m\mu) = ((x + 1) - 1)^d[(x + 1)(x + 1)^2 - 1) \cdots ((x + 1)^{2^{k-1}} - 1)]^m.
\]
If the class \( e(d\lambda \oplus \xi \oplus m\mu) \) is trivial in \( \widetilde{KU}^0(S^{2n+1}/\mathbb{Z}_{2^k}) \), it must belong to the ideal \( ((x + 1)^{2^k} - 1), x^{n+1} \). Now, the statement of the corollary follows from Proposition 1.9.3. \( \square \)

### 1.10 Proof of Theorem 1.6.2

In this section we will use the results of the previous sections to finish the proof of Theorem 1.6.2. Recall that for a map \( f: S^{2n+1} \to M \) we define a coincidence set:
\[
A(f) = \{ x \in S^{2n+1} | f(x) = f(gx) \ \forall g \in \mathbb{Z}_{2^k} \}.
\]
Let \( \lambda: S^{2n+1} \times_{\mathbb{Z}_{2^k}} \mathbb{C} \to S^{2n+1}/\mathbb{Z}_{2^k} \) be a one-dimensional complex vector bundle where the action of \( \mathbb{Z}_{2^k} \) on \( \mathbb{C} \) is given by rotations by \( \frac{2\pi}{2^k} \).

*Proof of Theorem 1.6.2.* Assume \( \dim A(f) < 2d \), then a vector bundle \( d\lambda \oplus \xi \oplus m\mu \) has a non-vanishing section by elementary dimension considerations. Thus \( e(d\lambda)e(\xi \oplus m\mu) = 0 \).

By Proposition 1.7.1 we have
\[
e(d\lambda)s(f)^*(\theta) = e(d\lambda)e(\xi \oplus m\mu) = 0.
\]
Therefore, by Proposition 1.9.3, we must have:

\[ d \geq n - 2^{k-1}(m - 1) - m2^{k-2}(k - 1). \]

It follows now that:

\[ \dim A(f) \geq (2n + 1) - [2^k(m - 1) + m2^{k-1}(k - 1) + 1]. \]
CHAPTER 2  
HUREWICZ THEOREM AND APPROXIMATION OF MAPS

2.1 Introduction

All spaces are assumed to be separable metrizable. By a map we mean a continuous function, \( I = [0; 1] \). If \( \mathcal{K} \) is a simplicial complex then by \( |\mathcal{K}| \) we mean the corresponding polyhedron. By a simplicial map we mean a map \( f: |\mathcal{K}| \rightarrow |\mathcal{L}| \) which sends simplices to simplices and is affine on them. We say that a map \( f: X \rightarrow Y \) has dimension at most \( k \) (\( \dim f \leq k \)) if and only if the dimension of each of its fibers is at most \( k \). We recall that a space \( X \) is a \( C \)-space or has property \( C \) if for any sequence \( \{\alpha_n : n \in \mathbb{N}\} \) of open covers of \( X \) there exists a sequence \( \{\mu_n : n \in \mathbb{N}\} \) of disjoint families of open sets such that each \( \mu_n \) refines \( \alpha_n \) and the union of all systems \( \mu_n \) is a cover of \( X \). Each finite-dimensional paracompact space and each countable-dimensional metrizable space has property \( C \). By a \( C \)-compactum we mean a compact \( C \)-space.

In [26] V.V. Uspenskij introduced the notion of a map admitting an approximation by \( k \)-dimensional simplicial maps. Following him we say that a map \( f: X \rightarrow Y \) admits approximation by \( k \)-dimensional simplicial maps if for every pair of open covers \( \omega_X \) of the space \( X \) and \( \omega_Y \) of the space \( Y \) there exists a commutative diagram of the following form,

\[
\begin{array}{ccc}
X & \xrightarrow{\kappa_X} & |\mathcal{K}| \\
\downarrow f & & \downarrow p \\
Y & \xrightarrow{\kappa_Y} & |\mathcal{L}|,
\end{array}
\]

where \( \kappa_X \) is an \( \omega_X \)-map, \( \kappa_Y \) is an \( \omega_Y \)-map and \( p \) is a \( k \)-dimensional simplicial map between polyhedra \( |K| \) and \( |L| \).

In that paper V.V. Uspenskij proposed the following question and conjectured that in the general case the answer to it is ”no”.

(\( Q_1 \)) Does every \( k \)-dimensional map \( f: X \rightarrow Y \) between compacta admit approximation by \( k \)-dimensional simplicial maps?
In [6] A.N. Dranishnikov and V.V. Uspenskij proved that light maps admit approximation by finite-to-one simplicial maps. In this paper we give some partial results answering the question of V.V. Uspenskij in affirmative.

**Theorem 2.1.1.** Let \( f : X \to Y \) be a \( k \)-dimensional map between \( C \)-compacta. Then for any pair of open covers \( \omega_X \) of the space \( X \) and \( \omega_Y \) of the space \( Y \) there exists a commutative diagram of the following form

\[
\begin{array}{c}
X \xrightarrow{\kappa_X} |K| \\
f \downarrow \quad \downarrow p \\
Y \xrightarrow{\kappa_Y} |L|,
\end{array}
\]

where \( \kappa_X \) is an \( \omega_X \)-map, \( \kappa_Y \) is an \( \omega_Y \)-map and \( p \) is a \( k \)-dimensional simplicial map between compact polyhedra \( |K| \) and \( |L| \). Furthermore, one can always assume that

\[
\dim |L| \leq \dim Y \quad \text{and} \quad \dim |K| \leq \dim Y + k.
\]

**Theorem 2.1.2.** \( k \)-dimensional maps between compacta admit approximation by \((k + 1)\)-dimensional simplicial maps.

**Theorem 2.1.3.** \( k \)-dimensional maps of Bing compacta (i.e. compacta with each component hereditarily indecomposable) admit approximation by \( k \)-dimensional simplicial maps.

It turned out that the question \((Q_1)\) is closely related to the next question proposed by B.A. Pasynkov in [16]. We recall that the diagonal product of two maps \( f : X \to Y \) and \( g : X \to Z \) is a map \( f \triangle g : X \to Y \times Z \) defined by \( f \triangle g(x) = (f(x), g(x)) \).

\((Q_2)\) Let \( f : X \to Y \) be a \( k \)-dimensional map between compacta. Does there exist a map \( g : X \to \mathbb{I}^k \) such that \( \dim(f \triangle g) \leq 0 \)?

In this paper we prove the following theorem which states that the questions \((Q_1)\) and \((Q_2)\) are equivalent.
Theorem 2.1.4. Let \( f: X \rightarrow Y \) be a map between compacta. Then \( f \) admits approximation by \( k \)-dimensional maps if and only if there exists a map \( g: X \rightarrow I^k \) such that \( \dim(f \triangle g) \leq 0 \).

There are a lot of papers devoted to Pasynkov’s question ([16],[17],[18],[21],[20],[8],[9],[22]). In [16] Pasynkov announced the following theorem to which the proof appeared much later in [17] and [18].

Theorem 2.1.5. Let \( f: X \rightarrow Y \) be a \( k \)-dimensional map between finite dimensional compacta. Then there exists a map \( g: X \rightarrow I^k \) such that \( \dim(f \triangle g) \leq 0 \).

In [21] Torunczyk proved the following theorem which is closely related to the theorem proved by Pasynkov.

Theorem 2.1.6. Let \( f: X \rightarrow Y \) be a \( k \)-dimensional map between finite dimensional compacta. Then there exists a \( \sigma \)-compact \( A \subset X \) such that \( \dim A \leq k - 1 \) and \( \dim f |_{X \setminus A} \leq 0 \).

One can prove that for any map \( f: X \rightarrow Y \) between compacta the statement of theorem 2.1.5 holds (for \( f \)) if and only if the statement of theorem 2.1.6 holds (for \( f \)) (the proof can be found in [8]).

We improve the argument used by Torunzyk in [21] to prove the next theorem and the implication ”\( \Leftarrow \)” of theorem 2.1.4.

Theorem 2.1.7. Let \( f: X \rightarrow Y \) be a \( k \)-dimensional map between \( C \)-compacta. Then

(i) there exists a \( \sigma \)-compact subset \( A \subset X \) such that 
\[ \dim A \leq k - 1 \text{ and } \dim f |_{X \setminus A} \leq 0. \]

(ii) there exists a map \( g: X \rightarrow I^k \) such that \( \dim(f \triangle g) \leq 0 \).

In the next corollary by \( e - \dim(X) \) we mean the extensional dimension of a compact space \( X \) introduced by A.N. Dranishnikov in [5].

Corollary 2.1.1. Let \( f: X \rightarrow Y \) be a \( k \)-dimensional map between \( C \)-compacta. Then 
\[ e - \dim(X) \leq e - \dim(Y \times I^k). \]
Proof. From [6] it follows that the extensional dimension cannot be lowered by 0-dimensional maps so the corollary is an immediate consequence of theorem 2.1.7.

One can understand the statement of the previous corollary as a generalization of the classical Hurewicz formula.

2.2 Proof of Theorem 2.1.7

Further in this section we assume that every space \( X \) is given with a fixed metric \( \rho_X \) which generates the same topology on it. By \( \rho_X(A, B) \) we mean the distance between subsets \( A \) and \( B \) in the space \( X \), namely, \( \rho_X(A, B) = \inf\{\rho_X(a, b) | a \in A, b \in B\} \). The closure of a subset \( A \) will be denoted by \([A]\).

Lemma 2.2.1. Let \( f: X \to Y \) be a map between compacta. Suppose that for any closed disjoint subsets \( B \) and \( C \) of \( X \) there exists a closed subset \( T \) of \( X \) such that \( \dim T \leq k - 1 \) and for any \( y \in Y \) the set \( T \) separates \( f^{-1}(y) \) between \( B \) and \( C \). Then there exists a \( \sigma \)-compact subset \( A \subset X \) with \( \dim A \leq k - 1 \) such that \( \dim f|_{X\setminus A} \leq 0 \).

Proof. Take a countable open base \( \mathcal{B} = \{U_\gamma | \gamma \in \Gamma\} \) on \( X \) such that the union of any finite number of sets from \( \mathcal{B} \) is again a member of \( \mathcal{B} \). Define the set \( \Lambda \subset \Gamma \times \Gamma \) by the requirement: \((\gamma, \mu) \in \Lambda \) if and only if \([U_\gamma] \cap [U_\mu] = \emptyset\). Note that \( \Lambda \) is countable. By assumption for every pair \((\gamma, \mu) \in \Lambda \) there exists a set \( T_{(\gamma, \mu)} \) of dimension at most \( k - 1 \) separating every fiber \( f^{-1}(y) \) between \([U_\gamma]\) and \([U_\mu]\). Now define \( A = \bigcup \{T_{(\gamma, \mu)} | (\gamma, \mu) \in \Lambda\} \). By definition \( A \) is \( \sigma \)-compact and, obviously, \( \dim A \leq k - 1 \). It is also easy to see that \( \dim f|_{X\setminus A} \leq 0 \). Indeed, by the additivity property of the base \( \mathcal{B} \) for every pair of disjoint closed subsets \( G \) and \( H \) of a given fiber \( f^{-1}(y) \) there exists a pair \((\gamma, \mu) \in \Lambda \) such that \( G \subset U_\gamma \) and \( H \subset U_\mu \). Then \( T_{(\gamma, \mu)} \subset A \) separates \( f^{-1}(y) \) between \( G \) and \( H \). So, \( \dim(f^{-1}(y) \setminus A) \leq 0 \). \( \square \)

Let \( \mathcal{F} = \mathbb{N}^0 \cup \bigcup \{\mathbb{N}^k : k \geq 1\} \) be the union of all finite sequences of positive integers plus empty sequence \( \mathbb{N}^0 = \{\ast\} \). For every \( i \in \mathcal{F} \) let us denote by \( |i| \) the length of the sequence \( i \) and by \((i, p)\) the sequence obtained by adding to \( i \) a positive integer \( p \).
Lemma 2.2.2. Let $f : X \to Y$ be a map between compacta. Let $B$ and $C$ be closed disjoint subsets of $X$. Suppose that for every $i \in \mathcal{F}$ there are sets $U(i)$, $V(i)$ and $F(i)$ such that:

(a) $F(i)$ is closed in $Y$, the sets $U(i)$ and $V(i)$ are open subsets of $X$ and $[U(i)] \cap [V(i)] = \emptyset$;

(b) $U(\ast) \supset B$, $V(\ast) \supset C$ and $F(\ast) = Y$;

(c) $U(i,p) \supset U(i) \cap f^{-1}(F(i,p))$ and $V(i,p) \supset V(i) \cap f^{-1}(F(i,p))$ for every $p \in \mathbb{N}$;

(d) $F(i) \subseteq \cup\{F(i,p) : p \in \mathbb{N}\}$ and $\text{diam} F(i) < \frac{1}{1+|i|}$;

(e) the set $E(i) = f^{-1}(F(i)) \setminus (U(i) \cup V(i))$ admits an open cover of order $k$ and diameter $\frac{1}{1+|i|}$;

(f) in notations of (e) the family $\{E(i,p) : p \in \mathbb{N}\}$ is discrete in $X$.

Then there exists a closed subset $T$ of $X$ such that $\dim T \leq k - 1$ and for any $y \in Y$ the set $T$ separates $f^{-1}(y)$ between $B$ and $C$.

Proof. We define the set $T$ in the following way:

$$T_n = \cup\{E(i) : |i| = n\} \text{ and } T = \cap\{T_n : n \geq 0\}.$$ 

From property (e) it follows that $\dim T \leq k - 1$. Let us show that for every $y \in Y$ the set $T$ separates $f^{-1}(y)$ between $B$ and $C$. For every $y \in Y$ there exists a sequence $\{i_n : n \in \mathbb{N}\}$ such that

$$\{y\} = F(i_1) \cap F(i_1, i_2) \cap \ldots .$$

Then $f^{-1}(y) \setminus T \subseteq U(y) \cup V(y)$ is a desired partition. Here we denote by $U(y)$ and $V(y)$ the following sets

$$U(y) = f^{-1}(y) \cap \cup\{U(i_1, ..., i_p) : p \in \mathbb{N}\},$$

$$V(y) = f^{-1}(y) \cap \cup\{V(i_1, ..., i_p) : p \in \mathbb{N}\}.$$
Lemma 2.2.3. Let $f : X \to Y$ be a $k$-dimensional map between $C$-compacta and $\epsilon$ be any positive number. Let $U$ and $V$ be open subsets of $X$ with $[U] \cap [V] = \emptyset$, and $F$ be a closed subset of $Y$ such that $f(U) \cap f(V) \supseteq F$. Then there exist families of sets $\{U_p\}, \{V_p\}$, and $\{F_p\}$ for $p \in \mathbb{N}$ such that:

1. $F_p$ is closed in $Y$, the sets $U_p$ and $V_p$ are open subsets of $X$ and $[U_p] \cap [V_p] = \emptyset$;
2. $U_p \supseteq U \cap f^{-1}(F_p), V_p \supseteq V \cap f^{-1}(F_p)$;
3. $F \subseteq \bigcup \{F_p : p \in \mathbb{N}\}$ and $\text{diam} F_p < \epsilon$;
4. the set $E_p = f^{-1}(F_p) \setminus (U_p \cup V_p)$ admits an open cover of order $k$ and diameter $\epsilon$;
5. in notations of (4) the family $\{E_p : p \in \mathbb{N}\}$ is discrete in $X$.

Proof. Let $\{W_l : l \in \mathbb{N}\}$ be a countable sequence of open disjoint sets such that each of them separates $X$ between $[U]$ and $[V]$. For every $y \in F$ let $P_l(y) \subset W_l$ be a closed $(k - 1)$-dimensional set separating $f^{-1}(y)$ between $[U]$ and $[V]$. Let $Q_l(y) \subset W_l$ be an open neighborhood of $P_l(y)$ admitting a finite open cover of size $\epsilon$ and order $k$. As the map $f$ is closed there exists a neighborhood $G_l(y)$ of $y$ in $F$ such that $f^{-1}([G_l(y)]) \cap Q_l(y)$ separates $f^{-1}([G_l(y)])$ between $[U]$ and $[V]$ and $\text{diam} G_l(y) < \epsilon$. For every $l \in \mathbb{N}$ the family $\alpha_l = \{G_l(y) : y \in F\}$ is an open cover of $F$. As $F$ is a $C$-compactum there exists a finite sequence of finite disjoint open families of sets $\{\mu_l : l \leq N\}$ such that each family $\mu_l$ refines the cover $\alpha_l$ and $\mu = \cup \{\mu_l : l \leq N\}$ is an open cover of $F$. Further, for every $G \in \mu$ there are open subsets $U(G)$ and $V(G)$ of $X$ with disjoint closures such that

$$U(G) \supseteq f^{-1}(G) \cap [U], \ V(G) \supseteq f^{-1}(G) \cap [V]$$

and if $G$ is a member of $\mu_l$ then

$$f^{-1}(G) \setminus (U(G) \cup V(G)) \subset Q_l(y)$$

for some $y \in F$. Let $\{F(G) : G \in \mu\}$ be a closed shrinking of the cover $\mu$ and let

$$E(G) = f^{-1}(F(G)) \setminus (U(G) \cup V(G))$$
for $G \in \mu$. Then for $G \in \mu_l$ and $H \in \mu_m$ we have $E(G) \cap E(H) \subset W_l \cap W_m = \emptyset$. So the family $\{E(G) : G \in \mu\}$ is discrete in $X$. Let us enumerate the members of $\mu$: $G_1, G_2, \ldots$.

To get the desired sets we set

$$F_p = F(G_p), \quad U_p = U(G_p), \quad V_p = V(G_p).$$

\[\square\]

**Lemma 2.2.4.** Let $f : X \rightarrow Y$ be a $k$-dimensional map between $C$-compacta. Then for any closed disjoint subsets $B$ and $C$ of $X$ and for any $i \in \mathcal{F}$ there exist sets $U(i)$, $V(i)$ and $F(i)$ satisfying (a)–(f) of Lemma 2.2.2.

**Proof.** We will construct the sets $U(i)$, $V(i)$ and $F(i)$ by induction on $|i|$. First set

$F(\ast) = Y$ and $U(\ast) = U$, $V(\ast) = V$ for some open subsets $U$ and $V$ of $X$ with $[U] \cap [V] = \emptyset$. Assume the sets $U(i)$, $V(i)$ and $F(i)$ are already constructed and satisfy the conditions (a)-(f) of Lemma 2.2.2. Now to get the sets $U(i, p)$, $V(i, p)$ and $F(i, p)$ for all $p \in \mathbb{N}$ apply Lemma 2.2.3 to the sets $U = U(i)$, $V = V(i)$, $F = F(i)$ and to $\epsilon = \frac{1}{1 + |i|}$. \[\square\]

**Lemma 2.2.5.** Let $f : X \rightarrow Y$ be a map between compacta admitting approximations by $k$-dimensional maps. Then for any closed disjoint subsets $B$ and $C$ of $X$ there exist sets $U(i)$, $V(i)$ and $F(i)$ satisfying (a)–(f) of Lemma 2.2.2.

**Proof.** The sets $U(i)$, $V(i)$ and $F(i)$ will be constructed by induction on $|i|$. First set

$F(\ast) = Y$ and $U(\ast) = U$, $V(\ast) = V$ for some open subsets $U$ and $V$ of $X$ with $[U] \cap [V] = \emptyset$. Assume the sets $U(i)$, $V(i)$ and $F(i)$ are already constructed and satisfy the conditions (a)-(f) of Lemma 2.2.2. Take $\epsilon = \min \{\frac{\rho(U(i), V(i))}{4}, \frac{1}{1 + |i|}\}$. By assumption, there exists a commutative diagram of the following form:

$$
\begin{array}{ccc}
X & \xrightarrow{\kappa_X} & |\mathcal{K}| \\
\downarrow f & & \downarrow p \\
Y & \xrightarrow{\kappa_Y} & |\mathcal{L}|,
\end{array}
$$

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where $\kappa_X$ and $\kappa_Y$ are maps with $\epsilon$-small fibers. Let $G = \kappa_X([U(i)])$, $H = \kappa_X([V(i)])$ and $F = \kappa_Y(F(i))$. Note that $G \cap H = \emptyset$. Let $U$ and $V$ be open subsets of $|\mathcal{K}|$ with $U \supseteq G$, $V \supseteq H$ and $[U] \cap [V] = \emptyset$. Let $\lambda_1$ be a Lebesgue number of some open covering on $|\mathcal{K}|$ whose preimage under the map $\kappa_X$ is an $\epsilon$-small covering on $X$. Let $\lambda_2$ be a number defined similarly for $|\mathcal{L}|$ and $\kappa_Y$. Let $\lambda = \min\{\lambda_1, \lambda_2\}$. Apply Lemma 2.2.3 to the sets $U$, $V$, $F$ and to $\lambda$ to produce the sets $U_p$, $V_p$ and $F_p$ for all $p \in \mathbb{N}$ satisfying conditions (1)-(5) of Lemma 2.2.3. Now set $U(i, p) = \kappa_X^{-1}(U_p)$, $V(i, p) = \kappa_X^{-1}(V_p)$ and $F(i, p) = \kappa_Y^{-1}(F_p)$. Since taking a preimage preserves intersections, unions and subtractions, the sets $U(i, p)$, $V(i, p)$ and $F(i, p)$ satisfy the conditions (a)-(f).

**Proof of theorem 2.1.7.** The statements (i) and (ii) are equivalent ([8]), so, it is sufficient to prove only (i). But (i) immediately follows from Lemmas 2.2.1, 2.2.2 and 2.2.4.

### 2.3 Proofs of the Approximation Theorems

Let $\gamma$ be an open cover on $X$. Then by $N_\gamma$ we mean the nerve of the cover $\gamma$. Let \{a_\alpha : \alpha \in A\} be some partition of unity on $X$ subordinated to the locally finite cover $\gamma$. Then the canonical map defined by the partition of unity \{a_\alpha : \alpha \in A\} is a map $\kappa : X \longrightarrow |N_\gamma|$ defined by the following formula

$$\kappa(x) = \sum_{\alpha \in A} a_\alpha(x) \cdot \alpha.$$  

If $\tau$ is some triangulation of the polyhedron $P$, then by $St(a, \tau)$ we mean the star of the vertex $a \in \tau$ with respect to triangulation $\tau$, i.e. the union of all open simplices having $a$ as a vertex.

**Proof of theorem 2.1.4.** Let $f : X \longrightarrow Y$ be a map between compacta admitting approximation by $k$-dimensional simplicial maps. By [8] to show that there exists a map $g : X \longrightarrow \mathbb{I}^k$ with $\dim(f \triangle g) \leq 0$ it is sufficient to find a $\sigma$-compact subset $A$ in $X$ of dimension at most $k - 1$ such that $\dim f|_{X \setminus A} \leq 0$. The existence of such subset $A$ follows immediately from Lemmas 2.2.1, 2.2.2 and 2.2.5.
Now suppose there exists a map $g: X \to \mathbf{I}^k$ such that $\dim(f \triangle g) \leq 0$. For every $(y, t) \in Y \times \mathbf{I}^k$ there exists a finite disjoint family of open sets $\nu_{(y, t)} = \{V_{\gamma} : \gamma \in \Gamma_{(y, t)}\}$ such that $(f \triangle g)^{-1}(y, t) \subset \cup \nu_{(y, t)}$ and $\nu_{(y, t)} \succ \omega_X$. Let $O_{(y, t)}$ be an open neighborhood of $(y, t)$ in $Y \times \mathbf{I}^k$ such that $(f \triangle g)^{-1}O_{(y, t)} \subset \cup \nu_{(y, t)}$. Let $\varsigma = \{U_{\alpha} : \alpha \in A\}$ and $\iota = \{I_{\delta} : \delta \in D\}$ be finite open covers of the spaces $Y$ and $\mathbf{I}^k$ such that:

(a) $\varsigma \succ \omega_Y$;

(b) the order of $\varsigma$ does not exceed $\dim Y + 1$;

(c) $(\varsigma \times \iota) = \{U_{\alpha} \times I_{\delta} : (\alpha, \delta) \in A \times D\} \succ \{O_{(y, t)} : (y, t) \in Y \times \mathbf{I}^k\}$.

The partition of unity $\{u_{\alpha} : \alpha \in A\}$ on $Y$ subordinated to the cover $\varsigma$ gives rise to the canonical map $\mu: Y \to |N_{\varsigma}|$. Then the map $\mu \times id: Y \times \mathbf{I}^k \to |N_{\varsigma}| \times \mathbf{I}^k$ is an $(\varsigma \times \iota)$-map.

By $\pi: |N_{\varsigma}| \times \mathbf{I}^k \to |N_{\varsigma}|$ we denote the projection. Let $\tau$ and $\theta$ be such triangulations on polyhedra $|N_{\varsigma}| \times \mathbf{I}^k$ and $|N_{\varsigma}|$ respectively such that the following conditions are satisfied:

(1') $\pi: |N_{\varsigma}| \times \mathbf{I}^k \to |N_{\varsigma}|$ is a simplicial map relative to the triangulations $\tau$ and $\theta$;

(2') $\{(\mu \times id)^{-1}(St(a, \tau)) : a \in \tau\} \succ \varsigma \times \iota$;

(3') $\{\mu^{-1}(St(b, \theta)) : b \in \theta\} \succ \varsigma$.

Let us define $\xi = \{St(a, \tau) : a \in \tau\}$ and $\zeta = \{St(b, \theta) : b \in \theta\}$. Define the partition of unity $\{w_{\alpha} : a \in \tau\}$ on $|N_{\varsigma}| \times \mathbf{I}^k$ subordinated to the cover $\xi$ by letting $w_{\alpha}(z)$ to be the barycentric coordinate of $z \in |N_{\varsigma}| \times \mathbf{I}^k$ with respect to the vertex $a \in \tau$. Analogously, define the partition of unity $\{v_{b} : b \in \theta\}$ on $|N_{\varsigma}|$ subordinated to the cover $\zeta$. Note that the projection $\pi: |N_{\varsigma}| \times \mathbf{I}^k \to |N_{\varsigma}|$ sends the stars of the vertices of the triangulation $\tau$ to the stars of the vertices of the triangulation $\theta$. That is why there is a simplicial map $\varpi: N_{\xi} \to N_{\zeta}$ between the nerves of the covers $\xi$ and $\zeta$. Moreover, the following diagram commutes.

$$
\begin{array}{ccc}
|N_{\varsigma}| \times \mathbf{I}^k & \psi \to & |N_{\varsigma}| \\
\pi \downarrow & & \downarrow \varpi \\
|N_{\varsigma}| & \phi \to & |N_{\varsigma}|.
\end{array}
$$
Here $\psi$ and $\phi$ are canonical maps defined by the partitions of unity $\{w_a : a \in \tau\}$ and $\{v_b : b \in \theta\}$ respectively. Let us remark that $\dim \varpi \leq k$. By $W_a$ we will denote the set $(\mu \times \text{id})^{-1}(St(a, \tau))$, by $\lambda$ the cover $\{W_a : a \in \tau\}$ and $\eta$ is $\mu^{-1}(\zeta)$. Further, we set $w^*_a = w_a \circ (\mu \times \text{id})$ for each $a \in \tau$ and $v^*_b = v_b \circ \mu$ for each $b \in \theta$. The partitions of unity $\{w^*_a : a \in \tau\}$ on $Y \times I^k$ and $\{v^*_b : b \in \theta\}$ on $Y$ are subordinated to the covers $\lambda$ and $\eta$ respectively. We set $K' = N_\lambda$ and $L = N_\eta$. The simplicial complexes $K'$ and $L$ are isomorphic to $N_\xi$ and $N_\zeta$ that is why the simplicial map $q : |K'| \longrightarrow |L|$ is defined and the following diagram commutes.

\[
\begin{array}{ccc}
Y \times I^k & \xrightarrow{\varphi} & |K'| \\
pr \downarrow & & \downarrow q \\
Y & \xrightarrow{\kappa_Y} & |L|
\end{array}
\]

Here $\varphi$ and $\kappa_Y$ are canonical maps defined by the partitions of unity $\{w^*_a : a \in \tau\}$ and $\{v^*_b : b \in \theta\}$ respectively. Obviously, $\dim q \leq k$.

Recall that $\varphi$ is an $\{O_{(y,t)}\}$-map. For every $a \in \tau$ pick a point $(y, t)_a$ such that $W_a \subset O_{(y,t)_a}$. Let $w^{**}_a = w^*_a \circ (f \triangle g)$ and $B_a = \Gamma_{(y,t)_a}$. Then

\[
\text{supp} \ (w^{**}_a) \subset (f \triangle g)^{-1}(W_a) \subset (f \triangle g)^{-1}(O_{(y,t)_a}).
\]

Consequently, $\cup_{(y,t)_a} \supset \text{supp}(w^{**}_a)$. As the family $\nu_{(y,t)_a}$ is disjoint, there exists a family of non-negative functions $\{b_\beta : \beta \in B_a\}$ such that $w^{**}_a = \sum_{\beta \in B_a} b_\beta$ and $\text{supp}(b_\beta) \subset V_\beta$. Let $B = \cup\{B_a : a \in \tau\}$ and $\mathcal{V} = \{V_\beta \cap (f \triangle g)^{-1}(W_a) : a \in \tau, \beta \in B_a\}$. The family $\{b_\beta : \beta \in B\}$ is a partition of unity on $X$ subordinated to the cover $\mathcal{V}$.

Let $\mathcal{K}$ be the nerve of the cover $\mathcal{V}$ and $\kappa_X : X \longrightarrow |\mathcal{K}|$ the canonical map defined by the partition of unity $\{b_\beta : \beta \in B\}$. We define a simplicial map $p' : |\mathcal{K}| \longrightarrow |\mathcal{K}'|$ by requiring that the vertex $\beta \in B_a$ goes to $a$. Clearly, the map $p'$ is finite-to-one. Indeed, no two vertices in $B_a$ are connected by an edge. That is why the restriction of $p'$ to any simplex is a homeomorphism. Finally we define the desired $k$-dimensional simplicial map
$p: |\mathcal{K}| \rightarrow |\mathcal{L}|$ as the composition $p = q \circ p'$. Moreover by (b1) we have $\dim |\mathcal{L}| \leq \dim Y$ and $\dim |\mathcal{K}| \leq \dim Y + k$ since $p$ is $k$-dimensional.

The following results of M. Levin [8] and Y. Sternfeld [20] are needed to prove theorems 2.1.2 and 2.1.3:

**Theorem 2.3.1.** Let $f : X \rightarrow Y$ be a $k$-dimensional map between compacta. Then there exists a map $g : X \rightarrow I^{k+1}$ such that $\dim (f \Delta g) \leq 0$.

**Theorem 2.3.2.** Let $f : X \rightarrow Y$ be a $k$-dimensional map of Bing compacta. Then there exists a map $g : X \rightarrow I^k$ such that $\dim (f \Delta g) \leq 0$.

**Proof of theorem 2.1.2.** The theorem is an immediate consequence of theorems 2.3.1 and 2.1.4.

**Proof of theorem 2.1.3.** The theorem is an immediate consequence of theorems 2.3.2 and 2.1.4.
CHAPTER 3
BORUSK-ULAM THEOREMS FOR MAPS WITH INFINITE FIBERS

3.1 Bula’s Property

We say that a surjective map \( f : X \to Y \) satisfies Bula’s property if and only if there exist closed disjoint subsets \( A \) and \( B, A, B \subset X \), such that \( f(A) = f(B) = Y \).

A question about existence of open maps between compacta with infinite fibers which does not satisfy Bula’s property is well-known in continuum theory and was first stated by Bula. The first example of such a map was given by:

\[
p : \prod_{i=0}^{\infty} S^{2i} \to \prod_{i=0}^{\infty} \mathbb{R}P^{2i}
\]

and was first suggested by A. Dranishnikov in [4]. The following theorem is a generalization of his construction and heavily relies on Theorem 1.2.1.

**Theorem 3.1.1.** Let \( n_i \geq p^i - 1 \) for any \( i \) and let \( \mu : (\mathbb{Z}^p_\infty) \otimes \prod_{i=1}^{\infty} S^{n_i} \) be a product of free actions of \( \mathbb{Z}_p \) on \( S^{n_i} \). Then the projection

\[
\prod_{i=1}^{\infty} S^{n_i} \to \prod_{i=1}^{\infty} S^{n_i} / (\mathbb{Z}_p)^\infty
\]

does not satisfy Bula’s property.

The proof of the previous theorem is almost straightforward and therefore omitted in this dissertation.

Interesting examples of open maps between compacta without Bula’s property were constructed in [9]. Recall the following theorem from that paper:

**Theorem 3.1.2.** Let \( M \) be an \( n \)-dimensional compact manifold with \( n > 3 \). Then there exists a surjective open monotone map on \( M \) with nontrivial fibers which does not satisfy Bula’s property.

The author’s interest in Borsuk-Ulam theorems and Bula’s property was stimulated by Hurewicz theorem for maps and the conjecture by V. Uspenskij about approximation of \( k \)-dimensional maps between compacta which were discussed in the previous chapter. The author believes that the answer to the V. Uspenskij’s conjecture is ”yes”. In other words,
there exist a $k$-dimensional map between compacta which cannot be approximated by $k$-dimensional simplicial maps of polyhedra or, equivalently, for which Hurewicz theorem for maps does not hold.

There is enough evidence to believe that such an example exists. For instance, the maps produced by Theorem 3.1.2 almost satisfy the requirements of being such examples.

**Proposition 3.1.1.** Let $f : X \to Y$ be an open surjective map such that for each $y \in Y$ we have $\dim f^{-1}(y) = 1$. Then the following statements are equivalent:

(i) $f : X \to Y$ does not satisfy Bula’s property,
(ii) for any $\varphi : X \to [0,1]$ there exists $y \in Y$ such that $\varphi(f^{-1}(y)) = pt$,
(iii) Hurewicz theorem does not hold for $f$,
(iv) The map $f$ cannot be approximated by 1-dimensional simplicial maps.

**Proof.** Suppose (i) holds and there exists a map $\varphi : X \to [0,1]$ which does not collapse any fiber of $f$ into a point. Then define a map $\bar{\varphi} : X \to [0,1]$ by a formula: $\bar{\varphi}(x) = \varphi(x)/\text{diam}(f^{-1}(f(x)))$. Then define $A = \bar{\varphi}^{-1}(0)$ and $B = \bar{\varphi}^{-1}(1)$. Obviously, $A \cap B = \emptyset$ and $f(A) = f(B) = Y$ which contradicts (i).

The implication $(ii) \Rightarrow (i)$ is obvious.

By Theorem 2.1.4 the statements (ii) and (iv) are equivalent. To see how $(i) \Leftrightarrow (ii)$ note that if the map $g : X \to [0,1]$ exists, then it does not map any fiber of $f$ to a point.

The previous proposition shows how Bula’s property is related to Hurewicz theorem for maps and V.Uspenskij’s conjecture. Our goal now would be to construct an example of a strictly 1-dimensional map between compacta which would satisfy at least one of the properties $(i) - (iv)$.

The examples of maps produced by Theorem 3.1.2, although supply evidence that the desired example exists, do not have uniform dimensionality of the fibers, i.e. in case $n = 1$ the fibers of the maps produced by Theorem 3.1.2 are not strictly 1-dimensional.
In the next sections we will outline the idea how to construct an example of an open, strictly 1-dimensional map between metric compacta without Bula’s property.

3.2 Lipschitz Compactification

In this section we will discuss compactifications of proper metric spaces with respect to a family of certain types of Lipschitz functions. Recall that a metric space is called proper if and only if every closed ball in it is compact.

It is a well-known fact in general topology that each compactification of a sufficiently good topological space can be described as the set of maximal ideals of a Banach algebra of functions. Lipschitz maps on a proper compact metric space do not form a Banach algebra, so we will consider the smallest Banach algebra containing all Lipschitz functions.

Let $X$ be a proper metric space. Denote by $C_L(X)$ the closure of the set of all bounded Lipschitz functions $f : X \to \mathbb{R}$ with $\text{Lip}(f) < \infty$. For $x \in X$ define $\psi(x) = (f(x))_{f \in C_L(X)} \in \mathbb{R}^{C_L(X)}$. It is easy to prove that $\psi : X \hookrightarrow \mathbb{R}^{C_L(X)}$ is an embedding. Define the Lipschitz compactification $LX$ of the proper metric space $X$ as

$$LX = \overline{\psi(X)}.$$

Proposition 3.2.1. Let $(X, d)$ be a proper metric space and let $\tilde{X}$ be the Lipschitz compactification of $X$. Let $U \subseteq X$ be an open subset of $X$ and let $F \subset U$ be a closed subset of $U$. Let $\tilde{U} \subset \tilde{X}$ be the unique maximal open subset of $\tilde{X}$ such that $\tilde{U} \cap X = U$ and let $\tilde{F}$ be the closure of $F$ in $\tilde{X}$. Then

$$\tilde{F} \subset \tilde{U} \iff \text{dist}(X \setminus U, F) > 0. \quad (\star)$$

Proof. ($\implies$) There exists a function $\tilde{g} : \tilde{X} \to [0; 1]$ such that $\tilde{g}(\tilde{F}) = 0$ and $\tilde{g}(\tilde{X} \setminus \tilde{U}) = 1$. As long as $g$ is continuous on $\tilde{X}$ there exist a sequence of Lipschitz functions $\{g_i(x)\}_{i \in \mathbb{N}}$ such that $g(x) = \lim_{i \to \infty} g_i(x)$.

We claim that there exist $i_0 \in \mathbb{N}$ for which $g_{i_0}(F) < \frac{1}{3}$ and $g_{i_0}(X \setminus \tilde{U}) > \frac{2}{3}$. This can be proven as follows. Assume there is no such $i_0 \in \mathbb{N}$. Then there exist a sequence $x_n \in X \setminus U$
with \( \lim_{n \to \infty} \text{dist}(x_n, F) = 0 \) and a sequence \( y_n \in F \) with \( \lim_{n \to \infty} d(x_n, y_n) = 0 \). Thus, there exist the following double inequality:

\[
\frac{1}{3} < d(g_i(x_n), g_i(y_n)) \leq \lambda d(x_n, y_n),
\]

from which it follows that \( g_i(x) \) is not a Lipschitz function.

\((-\leq=)\) Define a function \( f: X \to [0; 1] \) by a formula \( f(x) = d(x, X \setminus U) \). By assumption, there exists \( \delta > 0 \) such that \( f(F) > \delta \). Note that \( f(U) = 0 \). It is easy to check that \( f: X \to [0; 1] \) is a Lipschitz function. Therefore, there exists a function \( \tilde{f}: X \to [0; 1] \) such that \( \tilde{f}|_X = f \). Obviously, \( \tilde{f}(\bar{F}) \geq \delta \).

An easy consequence of the previous proposition is the following

**Proposition 3.2.2.** Let \( f: X \to Y \) be a continuous Lipschitz map between proper metric spaces. The map \( f \) can be extended to an open continuous map \( Lf: LX \to LY \) between Lipschitz compactifications.

**Proof.** Let \( x \in LX \) and let \( \bar{U} \subset LX \) be an open neighborhood of \( x \in LX \). We need to prove that there exists an open neighborhood \( V \subset LY \) of the point \( Lf(x) \) such that \( V \subset Lf(\bar{U}) \).

Let \( F \subset \bar{F} \subset \bar{U} \) be a subset of \( X \) such that the closure of \( F \) in \( LX \) contains \( x \in LX \). Then by Proposition 3.2.1, \( \text{dist}(F, X \setminus \bar{U}) > 0 \). Then \( \text{dist}(f(F), Y \setminus f(\bar{U})) > 0 \), because \( f \) is locally a projection. Therefore, by Proposition 3.2.1, we have

\[
\overline{f(F)} \subset f(\overline{\bar{U} \cap \bar{S}^\infty}).
\]

\(\square\)

### 3.3 The Construction

Let \( G \) be a finite group acting freely on a manifold \( M \). We say that the action of \( G \) on \( M \) has \( m \)-Borsuk-Ulam property if for every continuous map \( f: M \to \mathbb{R}^m \) there exists an orbit of \( G \) in \( M \) which is collapsed by \( f \) to a point.
Let \( \{p_i\} \) be a sequence of prime numbers such that \( \lim_{i \to \infty} p_i = \infty \). Recall that by Theorem 1.1.3 we can choose a sequence \( \{n_i\} \) of odd numbers such that a free \( \mathbb{Z}_{p_i} \)-action on \( S^{n_i} \) will have 1-Borsuk-Ulam property, but will not have 2-Borsuk-Ulam property.

**Proposition 3.3.1.** There exists a sequence of continuous Lipschitz functions \( f_i : S^{n_i} \to \mathbb{R}^2 \) such that:

(i) For any \( x \in S^{n_i} \) we have \( \text{diam} f_i(\mathbb{Z}_{p_i} \cdot x) \geq 1 \),

(ii) \( \lim_{i \to \infty} \frac{L(f_i)}{p_i} = 0 \), where \( L(f_i) \) is the Lipschitz constant of \( f_i \).

Now consider the following obvious map:

\[ \sqcup p_i : \sqcup S^{n_i} \to \sqcup \mathbb{C}P^{[n_i]} \].

This is an open Lipschitz map between proper metric spaces. By Proposition 3.2.2, we have an open map

\[ L(\sqcup p_i) : L(\sqcup S^{n_i}) \to L(\sqcup \mathbb{C}P^{[n_i]}) \]

between Lipschitz compactifications.

**Proposition 3.3.2.** Every fiber of \( L(\sqcup p_i) : L(\sqcup S^{n_i}) \to L(\sqcup \mathbb{C}P^{[n_i]}) \) is a non-degenerate connected compact space.

**Proof.** The space \( \hat{\mathbb{C}}P^\infty \) is a countable union of finite dimensional spaces. Therefore, it is a \( C \)-space. In [22] it was proved that there exists a map \( g : \hat{S}^\infty \to [0; 1] \) such that the map

\[ \hat{p} \triangle g : \hat{S}^\infty \to \hat{\mathbb{C}}P^\infty \times [0; 1] \]

is 0-dimensional. It follows from the last statement that \( g(\hat{p}^{-1}(y)) = [a, b] \) with \( a \neq b \) for every \( y \in \hat{\mathbb{C}}P^\infty \). Let \( L_y : [a, b] \to [0, 1] \) be a linear transformation of \([a, b]\) into \([0, 1]\). Define a map \( \varphi : \hat{S}^\infty \to [0, 1] \) as a composition \( \varphi = L_y \circ g \). The continuity of the map \( \varphi \) essentially follows from the fact that \( \hat{p} \) is an open map. Now, define \( A = \varphi^{-1}(0) \) and \( B = \varphi^{-1}(1) \).

The sets \( A \) and \( B \) are closed and disjoint and each one of them intersects each fiber of \( \hat{p} \).

Also, note that \( \text{dist}(A, B) > 0 \), therefore, the closures of \( A \) and \( B \) in are disjoint \( L\hat{S}^\infty \) by
Lemma 3.2.1. Now, non-triviality of each fiber of the map \( L\hat{p} \) follows from Proposition ??.

**Proposition 3.3.3.** The map \( L(\sqcup p_i) : L(\sqcup S^n_i) \to L(\sqcup \mathbb{C}P^{[\frac{n}{2}]}) \) has 1-Borsuk-Ulam property, i.e. for every continuous function \( \varphi : L(\sqcup S^n_i) \to \mathbb{I} = [0; 1] \) there exists \( y \in L(\sqcup \mathbb{C}P^{[\frac{n}{2}]}) \) such that \( \varphi(L(\sqcup p_i)^{-1}(y)) = pt. \)

**Proof.** Suppose there exists a function \( \varphi : L\hat{p} : L\hat{S}^\infty \to \mathbb{I} = [0; 1] \) such that for every \( y \in L\hat{C}^\infty \) we have \( \varphi(L\hat{p}^{-1}(y)) \neq pt. \) Then there exists \( \varepsilon > 0 \) such that \( \text{diam } \varphi(L\hat{p}^{-1}(y)) > \varepsilon \) for every \( y \in L\hat{C}^\infty. \)

Let \( \{O_\alpha| \alpha \in A\} \) be an open covering of \( L\hat{S}^\infty \) such that for any \( \alpha \in A \) and any \( x, x' \in O_\alpha \) it follows that \( |\varphi(x) - \varphi(x')| < \frac{\varepsilon}{2}. \) Let \( e^{\frac{2\pi i}{p^k}} \) be the generator of the group \( \mathbb{Z}_{p^k}. \) We can choose \( k \in \mathbb{N} \) to be large enough, so that the points \( x \in S^1 = \hat{p}^{-1}(y) \) and \( e^{\frac{2\pi i}{p^k}} \cdot x \in S^1 = \hat{p}^{-1}(y) \) are \( \{O_\alpha| \alpha \in A\}-close \) for any choice of \( y \in \hat{C}P^\infty, \) i.e. there exists \( \alpha \in A \) such that \( x, e^{\frac{2\pi i}{p^k}} \cdot x \in O_\alpha. \)

Now, choose \( n \) large enough, so that Theorem ?? guarantees existence of an orbit of \( \mathbb{Z}_{p^k} \) which will be collapsed to a point under the following composition:

\[
S^{2n-1} \xrightarrow{i} L\hat{S}^\infty \xrightarrow{\varphi} \mathbb{I}.
\]

Here \( i : S^{2n-1} \hookrightarrow L\hat{S}^\infty \) is an inclusion. As \( \varphi \circ i \) is a Lipschitz map, the image of such an orbit will have a diameter less than \( \varepsilon, \) because of our pervious assumptions. The later conclusion contradicts our initial assumption.

To produce an example of an open strictly 1-dimensional map it remains to prove Proposition 3.3.1 and the fact that the fibers of our map are are all strictly 1-dimensional. The later fact intuitively seems obvious, since a fiber in the corona of our map is being approached by circles in a Lipschitz manner. So, it seems, that the fibers in the corona should also be 1-dimensional, since dimension cannot be raised by a Lipschitz map. Once we have proved all these, we have produced the desired example which remains to have
only one downside, namely, the spaces involved in the map are non-metarizable spaces. To
get rid of this downside, one needs to use Scepin’s Spectral Theorem.

After applying Scepin’s Spectral Theorem we have produced an example of a map
which gives a positive solution to the V.Uspenskij’s conjecture and answers the question of
B.A.Pasynkov in the negative.
Set $G = \mathbb{Z}_p \times ... \times \mathbb{Z}_p$ for a fixed odd prime $p$ and $k \geq 1$ and let $M = S^{2n_1-1} \times ... \times S^{2n_k-1}$. In this section we will discuss a failed attempt to prove a Borsuk-Ulam theorem for $G$-actions on products of spheres and maps into Euclidean spaces. This case is more difficult than the case of $(\mathbb{Z}_p)^k$-actions, because the Euler class of the corresponding vector bundle in the ordinary cohomology theory turns out to be trivial. Below we will also show that the Euler class of the vector bundle, which need to be considered, in the complex $K$-theory is trivial as well. Therefore, this particular case calls for a more sophisticated cohomology theory in order to extract a Borsuk-Ulam theorem.

Let $L_i$ be a 1-dimensional complex $\mathbb{Z}_{p^i}$-module, in which $\mathbb{Z}_{p^i}$ acts by multiplication by $e^{\frac{2\pi i}{p^i}}$ and let

$$
\lambda': S^{2n_i-1} \times_{\mathbb{Z}_{p^i}} L_i \to L^{n_i}(p^i)
$$

be the associated complex vector bundle. Here $L^{n_i}(p^i)$ is the lens space $S^{2n_i-1}/G$. Denote by

$$
\pi_i: L^{n_1}(p) \times ... \times L^{n_k}(p^k) \to L^{n_i}(p^i)
$$

the projection on the $i^{th}$ factor and set $\lambda_i = \pi^*(\lambda'_i)$. Then it follows from the isomorphism

$$
\mathbb{C}[G] \cong \mathbb{C}[\mathbb{Z}_p] \otimes ... \otimes \mathbb{C}[\mathbb{Z}_{p^k}]
$$

that

$$
\xi = (M \times_{\mathbb{Z}_p} \mathbb{C}[G] \to M/G) \cong \xi_1 \otimes ... \otimes \xi_k,
$$

where $\xi_i = (M \times_{\mathbb{Z}_{p^i}} \mathbb{C}[\mathbb{Z}_{p^i}] \to L^{n_i}(p^i))$. From elementary representation theory we have isomorphism $\mathbb{C}[\mathbb{Z}_{p^i}] \cong L_i \oplus L_i^2 \oplus ... \oplus L_i^{p^i}$ and therefore

$$
\xi_i \cong \lambda_i \oplus \lambda_i^2 \oplus ... \oplus \lambda_i^{p^i}.
$$
Thus we have
\[
\xi \simeq \bigotimes_{i=1}^{k} (\lambda_i \oplus \lambda_i^{2} \oplus \ldots \oplus \lambda_i^{p_i}) \simeq \bigoplus_{(\alpha_1, \ldots, \alpha_k) \in G} (\lambda_1^{\alpha_1} \otimes \ldots \otimes \lambda_k^{\alpha_k}) \simeq \epsilon_1^C \oplus \bigoplus_{(\alpha_1, \ldots, \alpha_k) \neq 0} (\lambda_1^{\alpha_1} \otimes \ldots \otimes \lambda_k^{\alpha_k}),
\]
where \( \epsilon_1^C \) is 1-dimensional trivial complex vector bundle. It is easy to see that
\[
\eta = (M \times_I G \rightarrow M/G) = \bigoplus_{(\alpha_1, \ldots, \alpha_k) \neq 0} (\lambda_1^{\alpha_1} \otimes \ldots \otimes \lambda_k^{\alpha_k}).
\]
To prove a Borsuk-Ulam type theorem for maps from \( M \) to \( \mathbb{R}^1 \) one needs to prove that the Euler class \( e(\psi) \) of the vector bundle
\[
\psi = (M \times_I G \rightarrow M/G)
\]
is non-trivial. Obviously, \( \psi_C = \eta \), therefore, it is sufficient to prove that \( e(\psi) \neq 0 \). This probably can be done by means of some extraordinary cohomology theory. Let us see what happens in the case of complex K-theory.

First, recall that
\[
\tilde{K}_C^0(L^{n_1}(p) \times \ldots \times L^{n_k}(p^k)) \simeq \mathbb{Z}[x_1, \ldots, x_k]/I,
\]
where
\[
I = (x_1^{n_1}, \ldots, x_k^{n_k}, (x_1 + 1)^p - 1, \ldots, (x_k + 1)^{p^k} - 1).
\]
Here \( x_i = [\lambda_i - 1] \) (\( 1 \leq i \leq k \)). This facts follow easily from the existence of homeomorphisms
\[
(\lambda_i^{p_i})_{S} \simeq L^{n_i}(p^i) \ (1 \leq i \leq k),
\]
from the Gysin sequence and the Kunneth formula for K-theory. By \( (\lambda_i^{p_i})_{S} \) we mean the spherization of the vector bundle \( \lambda_i^{p_i} \).
From the structure of the formal group law for $\tilde{K}_C$ given by a formula $F_{\tilde{K}_C}(x, y) = x + y + xy$ we are able to conclude that $e(\lambda^\alpha_i) = (x_i + 1)^{\alpha_i} - 1$ and therefore

$$e(\lambda_1^{\alpha_1} \otimes \ldots \otimes \lambda_k^{\alpha_k}) = \sum_{i=1}^k \sigma_i((x_1 + 1)^{\alpha_1} - 1, \ldots, (x_k + 1)^{\alpha_k} - 1),$$

where $\sigma_i$ is the $i^{th}$ symmetric polynomial. Thus we have

$$e(\eta) = \prod_{(\alpha_1, \ldots, \alpha_k) \neq 0} \sum_{i=1}^k \sigma_i((x_1 + 1)^{\alpha_1} - 1, \ldots, (x_k + 1)^{\alpha_k} - 1) =$$

$$= \prod_{(\alpha_1, \ldots, \alpha_k) \neq 0} ((x_1 + 1)^{\alpha_1}(x_2 + 1)^{\alpha_2} \cdots (x_k + 1)^{\alpha_k} - 1).$$

To prove a Borsuk-Ulam theorem we need to find conditions under which the polynomial $e(\eta)$ does not belong to the ideal

$$I = (x_1^{\alpha_1}, \ldots, x_k^{\alpha_k}, (x_1 + 1)^p - 1, \ldots, (x_k + 1)^p - 1)$$

from above. The author is being able to prove that in fact the polynomial $e(\eta)$ almost always belong to the ideal $I$ which make the complex $K$-theory approach inefficient when trying to prove a Borsuk-Ulam theorem for these kinds of actions.
REFERENCES


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BIOGRAPHICAL SKETCH

I was born in Troitsk, Moscow Region, on September 4, 1978. After graduation from High School 5 at the age of 16, I enrolled into a program in mathematics at People’s Friendship University. In a couple of years I realized that for me to have a realistic chance of becoming a mathematician I need to get into a better school. In fall 1997, I started attending a research seminar on General Topology at Moscow State University organized by Boris Alekseevich Pasynkov. In summer 1998 Professor Pasynkov helped me transfer to the Mathematics Department of Moscow State University, where I continued studying topology under his supervision. I graduated from Moscow State University in June 2002, with a bachelors degree in Mathematics. In August 2002 I started in the PhD program at the University of Florida which I completed in May 2007.