THE 2-LIEN OF A 2-GERBE

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
UNIVERSITY OF FLORIDA
2008
To my parents.
ACKNOWLEDGMENTS

I would like to thank my advisor, Richard Crew, for guiding me through the literature, for suggesting this problem, and for patiently answering my questions over the years. I have learned a great deal from him. I am also grateful to David Groisser and Paul Robinson. Both of them helped me many times and served on my committee. Thanks also go to committee members Peter Sin and Bernard Whiting for their feedback on this project. The support of my family and friends has been invaluable to me. I thank all of them.
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Principal bundles have a well-known description in terms of nonabelian cocycles of degree 1 with values in a sheaf. A more general notion than that of a sheaf on a space $X$ is that of a lien on $X$. A lien on $X$ is an object that is locally defined by a sheaf of groups, with descent data given up to inner conjugation. Equivalences classes of gerbes with a given lien $L$ are classified by nonabelian degree 2 cocycles. In his work, Lawrence Breen has given a similar classification of 2-gerbes using nonabelian degree 3 cocycles that take values in a family of group stacks. In our work, we defined the notion of a 2-lien on a space $X$. It is an object that is given locally by a group stack, with 2-descent given up to inner equivalence. We have proved some theorems about 2-liens of 2-gerbes which correspond to well known results about liens of gerbes. Also, Deligne has shown that any strict Picard stack $\mathcal{G}$ corresponds to a 2-term complex of abelian sheaves $K^\cdot = [ K^0 \xrightarrow{d} K^1 ]$. In this case we proved that $\check{H}^3(X, \mathcal{G})$ is isomorphic to the hypercohomology group $\check{H}^3(X, K^\cdot)$. 
CHAPTER 1
INTRODUCTION

In geometry a fundamental concept is the notion of a principal $G$-bundle (also known as a $G$-torsor). If $G$ is a sheaf on a space $X$, a $G$-torsor on $X$ is a geometric realization of $G$-valued Čech 1-cocycle. The problem of defining non-abelian degree 2 sheaf cohomology leads to the concept of a gerbe. A gerbe on $X$ may be thought of naively as a “sheaf of categories” with certain gluing axioms for objects and arrows. A gerbe on $X$ is a geometric realization of a 2-cocycle on $X$ with values in a non-abelian sheaf (or more generally a lien on $X$). This theory was worked out by Giraud [15].

Brylinski [7] developed a theory of differential geometry for gerbes. By his theory one can realize classes in $H^3(M, \mathbb{Z})$ as equivalence classes of abelian gerbes via the exponential map $H^2(M, \mathbb{C}^\times) \to H^3(M, \mathbb{Z})$. Then Murray invented the notion of a bundle gerbe [21]. The theory of bundle gerbes has proven very useful to physicists; for example bundle gerbes have been used to explore anomalies in quantum field theory [1], [2]. Hitchin has used the theory of gerbes for studying mirror symmetry [17]. Brylinski has used gerbes to give an interpretation of Beilinson’s regulator maps in algebraic $K$-theory [8].

One can then ask, what kinds of objects are classified by non-abelian degree 3 sheaf cohomology? The answer is a notion which is built up from 2-categories which is due to Breen. He calls such objects 2-gerbes on $X$ [4]. Brylinski and McLaughlin have studied a certain class of 2-gerbes [9], [10], namely 2-gerbes bound by the abelian sheaf $\mathbb{C}^\times$, which give rise to classes in $H^4(M, \mathbb{Z})$ via the exponential map $H^3(M, \mathbb{C}^\times) \to H^4(M, \mathbb{Z})$. They also prove that the class of the canonical 2-gerbe associated to a principal bundle $P$ in $H^4(M, \mathbb{Z})$ is the same as the first Pontryagin class of $P$.

This provides the motivation for our project. Given a 2-gerbe on $X$ the associated 3-cocycle takes its values in a family of group stacks $G_i$. Breen then points out that one could introduce the appropriately defined notion of a 2-lien $\mathcal{L}$ on a space $X$. Then the
3-cocycle could be viewed as defining a class in a corresponding $\mathcal{L}$-valued cohomology set $H^3(X, \mathcal{L})$.

In this thesis we define the notion of a 2-lien on a space $X$ and show that each 2-gerbe gives rise to such a 2-lien. After proving some analogues of theorems about liens of gerbes, we establish some results about 2-liens that are strict Picard stacks, and an associated degree 3 hypercohomology group. We now provide a more detailed description of the contents of this thesis.

In chapter 2, we review the necessary background material that forms the basis of this area, namely nonabelian cohomology. We begin with the definition of a $G$-torsor for a sheaf of groups $G$ and explain their classification by $H^1(X, G)$. We then describe the objects that embody degree 2 cohomology: that is, gerbes. To do so, we first give the definition of a stack, a morphism between stacks and the definition of a 2-arrow between such morphisms. This data gives us a 2-category of stacks on a space $X$. After this setup, we give the definition of a gerbe and that of its associated lien. We then provide a description of the objects that embody degree 3 cohomology: that is, 2-gerbes. As with the case for gerbes, we first review the notion of 2-stacks prior to defining a 2-gerbe. The latter half of Chapter 2 is devoted to reviewing two notions from category theory that will prove essential to defining the notion of a quotient in the 2-category of stacks: these notions are those of the representability of a functor and of a 2-functor. We close by reviewing the original approach taken by Giraud [15] to introduce gerbes. It is this approach that we will adapt in order to define a 2-lien of a 2-gerbe in chapter 6.

The key to defining a 2-lien is to figure out how to define a quotient in the 2-category of stacks. To answer this question locally, we first need to define a quotient in the 2-category of (small) categories $CAT$. This is what is accomplished in chapter 3: we prove the existence of a coequalizer in the 2-category of (small) categories $CAT$. Since the definition of an equalizer is used to define a coequalizer, we first prove the representability of an equalizer in $CAT$. We then construct a category $Coeq$ corresponding to a (small)
pair of functors, and prove that this category represents the coequalizer of this pair in 

$\text{CAT}$. 

In chapter 4, we begin by giving a definition of inner equivalence for group categories 
and show that such inner equivalences for a fixed group category form a categorical group. 
We then define what it means for a group category to act on a category, and define the 
quotient by such an action using the concept of a coequalizer developed in chapter 3. 

In chapter 5, we begin by defining what it means for a group stack to act on a stack. 
Since the goal is to define the quotient by such an action, we do the work of proving 
that coequalizers are representable in the 2-category $\text{STACKS}$. This is done using the 
coequalizer construction in chapter 3, and via the universal properties of the coequalizer. 
After proving the existence of a coequalizer in $\text{STACKS}$ we are able to define a quotient 
of an action by a group stack on a stack. 

In chapter 6, we define the notion of a 2-lien on a space $X$, following the approach 
of Giraud for an ordinary lien. We then do the necessary work of showing how to any 
2-gerbe, one may associate such a 2-lien, and that this association is given up to canonical 
2-equivalence. After establishing this, we give a more down-to-earth description of the 
2-lien of a 2-gerbe: it is an object that is given locally by a group stack, with descent data 
given up to inner equivalence. 

In chapter 7, our goal is to prove some theorems about $G$-2-gerbes, whose 2-lien arises 
from a strict Picard stack $\mathcal{G}$. For this purpose, in section 1 we review Breen’s treatment 
of the 3-cocycle description of a $G$-2-gerbe [4]. In section 2 of chapter 7, we prove results 
about $G$-2-gerbes which are analogues of theorems about for liens of gerbes. In section 3, 
we wish to give a nice cohomological description of connected $G$-2-gerbes where $\mathcal{G}$ is strict 
Picard. Deligne has shown that any strict Picard stack $\mathcal{G}$ corresponds to a 2-term complex 
of abelian sheaves $K^\cdot = [ K^0 \overset{d}{\longrightarrow} K^1 ]$. We begin by reviewing Deligne’s contruction, and 
then giving the definition of a degree 3 hypercohomology group. We then prove that the
set of equivalence classes $\hat{H}^3(X, G)$ of connected gerbes with 2-lie $G$ is isomorphic to the hypercohomology group $\hat{H}^3(X, K^-)$.

We conclude with chapter 8, where we summarize our results, and indicate some directions for future work.
2.1 Torsors and $H^1$

We will rapidly review some elements of the theory of torsors/principal bundles [5], [7].

Recall that given sets $X, Y$ and $Z$ with maps $f : X \to Z$, $g : Y \to Z$, the fibered product $X \times_Z Y$ is the subset of the product $X \times Y$ consisting of $(x, y)$ such that $f(x) = g(y)$.

Let $G$ be a sheaf of groups on a space $X$.

**Definition 2.1.1.** A right $G$-torsor on $X$ is a space $\pi : P \to X$ above $X$, together with a right group action $P \times G \to P$ of $G$ on $P$ such that the induced morphism

$$P \times_X G \cong P \times_X P$$

(2–1)

$$(p, g) \mapsto (p, pg)$$

is an isomorphism. In addition, we require that there exist a family of local sections $s_i : U_i \to P$ of $\pi$, for some open cover $\mathcal{U} = (U_i)_{i \in I}$ of $X$.

The trivial $G$-torsor is $X \times G$ itself, with the action of $G$ given by multiplication. Any $G$-torsor is locally isomorphic to the trivial torsor $T_G$. A $G$-torsor is isomorphic to $T_G$ if and only if it has a global section $s : X \to P$.

Now, given a space $X$ and a sheaf $G$ of groups, we wish to consider isomorphism classes of $G$-torsors $q : P \to X$. Given an open covering $(U_i)_{i \in I}$ and a section $s_i$ of $q$ over $U_i$, we have a function $g_{ij} : U_{ij} \to G|_{U_{ij}}$ such that $s_j = s_i \cdot g_{ij}$ (recall that $G$ acts on the right on $P$). These transition functions satisfy the equality $g_{ik} = g_{ij}g_{jk}$ over $U_{ijk}$. If the section $s_i$ is replaced over $U_i$ by $s_i' = s_i \cdot h_i$, then $g_{ij}$ is replaced by $h_i^{-1}g_{ij}h_j$.

This leads to the appropriate notion of 1-cocycles and coboundaries. For a sheaf $G$ of (possibly nonabelian) groups on a space $X$, and $(U_i)_{i \in I}$ an open covering, we define a Čech 1-cocyle with values in $G$ to consist of a family $a_{ij} \in \Gamma(U_{ij}, G)$ such that the cocycle
relation
\[ a_{ik} = a_{ij}a_{jk} \]
holds. Next, two Čech 1-cocycles \( a_{ij} \) and \( a'_{ij} \) are said to be cohomologous if there exists a section \( h_i \) of \( G \) over \( U_i \) such that
\[ a'_{ij} = h_i^{-1}a_{ij}h_j \]
thus defining an equivalence relation on the set of Čech 1-cocycles.

**Definition 2.1.2.** 1. For a sheaf \( G \) of groups on the space \( X \), and an open covering \( \mathcal{U} = (U_i)_{i \in I} \) of \( X \), the first cohomology set \( \check{H}^1(\mathcal{U}, G) \) is defined as the quotient of the set of 1-cocycles with values in \( G \) by the equivalence relation: “\( a \) is cohomologous to \( \bar{a} \)”.

2. The first cohomology set \( H^1(X, G) \) is defined as the direct limit \( \lim_{\mathcal{U} \to} \check{H}^1(\mathcal{U}, G) \) where the limit is taken over the set of all open coverings of \( X \), ordered by the relation of refinement.

Note that the set \( H^1(X, G) \) has a distinguished element, the class of the trivial 1-cocycle 1. So it is a pointed set.

The definition of first cohomology makes the following result immediate.

**Proposition 2.1.3.** The set of isomorphism classes of \( G \)-torsors over \( X \) is in a natural bijection with the sheaf cohomology set \( H^1(X, G) \).

### 2.2 Fibered Categories and Stacks

We will review the definitions from category theory [4], [5] relevant to our discussion. Recall that a groupoid is a category in which every morphism is invertible.

**Definition 2.2.1.** A category fibered in groupoids above a space \( X \) consists of a family of groupoids \( \mathcal{C}_U \), for each open set \( U \) in \( X \), together with an inverse image functor
\[ f^*: \mathcal{C}_U \longrightarrow \mathcal{C}_{U_1} \]
associated to every inclusion of open sets \( f : U_1 \subset U \) (which is the identity whenever \( f = 1_U \)), and a natural transformation

\[
\phi_{f,g} : (fg)^* \to g^*f^*
\]  

(2–5)

for every pair of composable inclusions

\[
U_2 \xleftarrow{g} U_1 \xrightarrow{f} U.
\]  

(2–6)

For each triple of composable inclusions

\[
U_3 \xleftarrow{h} U_2 \xleftarrow{g} U_1 \xrightarrow{f} U
\]  

(2–7)

we require that the composite natural transformations

\[
\psi_{f,g,h} : (fgh)^* \to h^*(fg)^* \to h^*(g^*f^*)
\]  

(2–8)

and

\[
\chi_{f,g,h} : (fgh)^* \to (gh)^*f^* \to (h^*g^*)f^*
\]  

(2–9)

coincide.

We will frequently refer to the inverse image of an object \( x \in C \) by an inclusion \( i : V \to U \) as the restriction \( x|_V \) of \( x \) above \( V \).

**Definition 2.2.2.** A Cartesian functor \( F : \mathcal{C} \to \mathcal{D} \) between fibered categories consists of a family of functors \( F_U : C_U \to D_U \) indexed by the open sets \( U \), together with, for every morphism \( f : U_2 \to U_1 \), a natural transformation

\[
\varphi_f : f^* \circ F_{U_1} \to F_{U_2} \circ f^*.
\]  

(2–10)

This is required to be compatible, for any pair of composable inclusions \( f \) and \( g \), with the natural transformations \( (fg)^* \to g^*f^* \) for \( \mathcal{C} \) and for \( \mathcal{D} \). Finally, a 2-arrow

\[
\Psi : F \to G
\]

between a pair of Cartesian functors \( F \) and \( G \) from \( \mathcal{C} \) to \( \mathcal{D} \) is defined
by a family of natural transformations \( \Psi_U : F_U \Rightarrow G_U \) which are compatible with the restriction functors \( f^* \).

**Definition 2.2.3.**

- A prestack in groupoids above a space \( X \) is a fibered category in groupoids above \( X \) such that “arrows glue” i.e. for every pair of objects \( x, y \in \mathcal{C}_U \), the presheaf \( \text{Hom}_{\mathcal{C}_U}(x, y) \) is a sheaf on \( U \).
- If, in addition, “objects glue” i.e. descent is effective for objects in \( \mathcal{C} \), then the prestack is called a stack in groupoids above \( X \).
- A morphism of stacks is just a morphism of the underlying fibered categories.

By descent data we mean that we are given, for every open cover \( \mathcal{U} = (U_\alpha) \) of an open set \( U \subset X \), a family of objects \( x_\alpha \in \mathcal{C}_{U_\alpha} \), and a family of isomorphisms \( \phi_{\alpha\beta} : x_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} x_\beta|_{U_{\alpha\beta}} \) such that

\[
\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma} \tag{2–11}
\]

above \( U_{\alpha\beta\gamma} \).

The descent condition \( (x_i, \phi_{ij}, \psi_{\alpha\beta\gamma}) \) is effective if there exists an object \( x \in \mathcal{C}_U \) together with isomorphisms \( f_\alpha : x|_{U_\alpha} \cong x_\alpha \) compatible with the morphisms \( \phi_{\alpha\beta} \) i.e. the following diagram commutes:

\[
\begin{array}{ccc}
x|_{U_{\alpha\beta}} & \xrightarrow{f_\alpha} & x_\alpha|_{U_{\beta\gamma}} \\
\phi_{\alpha\beta} & & \phi_{\alpha\gamma}
\end{array}
\]

By a construction based on the corresponding construction for sheaves, given any prestack \( \mathcal{C} \) on \( X \), there corresponds an associated stack \( \mathcal{C} \), together with a cartesian functor

\[
i : \mathcal{C} \to \mathcal{C} \tag{2–13}
\]

which is universal for cartesian functors from \( \mathcal{C} \) into stacks (see [18]).
Definition 2.2.4. A stack in groupoids \( C \) on \( X \), endowed with a monoidal associative Cartesian functor \( \cdot : C \times C \to C \), with identity objects and with inverses will be called a \textbf{group stack} (or just a \textbf{gr-stack}) on \( X \). For any \( V_i \in \text{Ob}(C_U) \) (\( i=1, 2, 3, 4 \)) the associativity isomorphisms \( \alpha_{V_i V_j V_k} : (V_i \cdot V_j) \cdot V_k \mapsto V_i \cdot (V_j \cdot V_k) \) must sit within the following two commutative diagrams.

1. Pentagon Axiom.

\[
\begin{array}{ccc}
(V_1 \cdot (V_2 \cdot V_3)) \cdot V_4 & \xrightarrow{\alpha_{1,2,3,4}} & (V_1 \cdot V_2) \cdot (V_3 \cdot V_4) \\
V_1 \cdot (V_2 \cdot (V_3 \cdot V_4)) & \xrightarrow{id_1 \cdot \alpha_{2,3,4}} & V_1 \cdot (V_2 \cdot (V_3 \cdot V_4))
\end{array}
\]

\( (2-14) \)

2. Triangle Axiom.

\[
\begin{array}{ccc}
(V_1 \cdot 1) \cdot V_2 & \xrightarrow{\alpha} & V_1 \cdot (1 \cdot V_2) \\
V_1 \cdot V_2 & \xrightarrow{\rho \cdot id} & V_1 \cdot V_2
\end{array}
\]

\( (2-15) \)

where \( \rho_{V_i} : V_1 \cdot 1 \mapsto V_1 \) and \( \lambda_{V_i} : 1 \cdot V_i \mapsto V_i \).

Definition 2.2.5.  

\( \bullet \) A group category \( G \) is said to be braided when its group law is endowed with a commutivity isomorphism \( S_{X,Y} : XY \to YX \) which is functorial in \( X \) and \( Y \) and sits in a commutative square

\[
\begin{array}{ccc}
X1 & \xrightarrow{S} & 1X \\
\downarrow{m} & & \downarrow{S} \\
X & = & X
\end{array}
\]

\( (2-16) \)

and two hexagonal “associativity” diagrams.

\( \bullet \) \( G \) is said to be Picard if in addition \( S_{Y,X} \circ S_{X,Y} = 1_{XY} \) for all \( X, Y \) in \( G \).

\( \bullet \) \( G \) is said to be a strict Picard category if in addition to all the above, \( S_{X,X} = 1_X \) for all \( X \).

Definition 2.2.6. A gr-stack is said to be a strict Picard stack if it is endowed with a commutativity natural transformation \( S \) for the group law \( S_{X,Y} \) and satisfies the corresponding conditions.
2.3 Gerbes and their Liens

The following definitions are due to Giraud [15]. We follow the presentation in [4], [5] and [20].

To say that a stack $\mathcal{G}$ is locally non-empty means there exists a covering $\mathcal{U} = (U_i)$ of $X$ for which the set of objects of the category $\mathcal{G}_{U_i}$ is non-empty. The locally connectedness condition on $\mathcal{G}$ is the requirement that for any pair of objects $x$ and $y$ in $\mathcal{G}_{U_i}$, there exists an open cover $\mathcal{V} = (V_\alpha)$ of $U$ such that the set of arrows from $x|_{V_\alpha}$ to $y|_{V_\alpha}$ is non-empty for all $\alpha$.

**Definition 2.3.1.**

- A (1)-gerbe on a space $X$ is a stack in groupoids $\mathcal{G}$ on $X$ which is locally non-empty and locally connected.
- A morphism of gerbes is just a morphism of the underlying fibered categories.
- A gerbe $\mathcal{G}$ on $X$ is said to be neutral (or trivial) when the fiber category $\mathcal{G}_X$ is non-empty.

A lien on a space $X$, as first defined by Giraud in his thesis [15], is a collection $(G_i)$ of sheaves of groups corresponding to an open cover $(U_i)$ of $X$ with descent data up to inner conjugation.

In other words a lien on $X$ is an object which is defined locally by a sheaf of groups, but in a category where morphisms between groups differing by inner conjugation are identified.

We give a cocycle description of a lien. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of $X$, and suppose we have a family of sheaves $G_i$ of groups, defined on the open sets $U_i$. Denote each sheaf by $lien(G_i)$. The sheaves $lien(G_i)$ and $lien(G_j)$ are glued on the open set $U_{ij}$ by a section $\psi_{ij}$ of the quotient sheaf $Out(G_j, G_i) := Isom(G_j, G_i)/Int(G_i)$ on $U_{ij}$. Thus the lien $L$ is determined by a family of sections of $Out(G_j, G_i)$ on $U_{ij}$ satisfying the 1-cocycle condition $\psi_{ij} \circ \psi_{jk} = \psi_{ik}$ in $Out(G_k, G_i)$ and the normalization condition $\psi_{ii} = 1$ for all $i$. 

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Two liens $L$ and $L'$ on $X$, each defined locally by families of groups $(G_\alpha)$ and $(G'_\beta)$, are isomorphic whenever there exists a common refinement $\mathcal{U} = (V_i)$ of the defining covers $\mathcal{U}$ and $\mathcal{U}'$, and a family of isomorphisms $\chi_i : \text{lien}(G_i) \to \text{lien}(G'_i)$ on the open sets $V_i$, which are compatible with the gluing data. The cocycles $(\psi_{ij})$ and $(\psi'_{ij})$ are therefore related by the coboundary conditions $\psi'_{ij} = \chi_i \psi_{ij} \chi_j^{-1}$, the isomorphisms $\chi_i$ are viewed as sections on the open sets $V_i$ of the sheaf $Out(G_i, G_i)$ of outer automorphisms of $G_i$.

Let $G$ be a sheaf of groups, and let $G_i = G|_{U_i}$. Then when a lien is locally of the form $\text{lien}(G_i)$, each $\psi_{ij}$ is a section on the set $U_{ij}$ of the sheaf $Out(G)$ of outer automorphisms of $G$. Following [4] we will call such liens, which are locally isomorphic to the lien $\text{lien}(G)$, $G$-liens. It follows from this description that they are classified by the non-abelian cohomology set $H^1(X, Out(G))$.

Let $G$ be a gerbe. It is locally non-empty so there exists an open cover $(U_\alpha)$ of $X$ such that $G(U_\alpha)$ is non-empty. So choose a family of objects $x_\alpha \in G(U_\alpha)$. Now $Aut(x_\alpha)$ is a sheaf on $U_\alpha$. Now $G$ is also locally connected so for each $\alpha, \beta$ there exists an open cover of $(U^\xi_{\alpha\beta})$ of $U_{\alpha\beta}$ and for each $\xi$ an arrow $f^\xi_{\alpha\beta} : x_\beta \to x_\alpha$ in $G(U^\xi_{\alpha\beta})$. Conjugation by this arrow defines an isomorphism of sheaves of groups $\lambda^\xi_{\alpha\beta} : Aut(x_\beta)|_{U^\xi_{\alpha\beta}} \to Aut(x_\alpha)|_{U^\xi_{\alpha\beta}}$. This isomorphism $\lambda^\xi_{\alpha\beta}$ depends on the choice of the $f^\xi_{\alpha\beta}$ but different choices define the same outer isomorphism. In particular, on overlaps $U^\xi_{\alpha\beta} \cap U^\zeta_{\alpha\beta}$ the two isomorphisms $\lambda^\xi_{\alpha\beta}$ and $\lambda^\zeta_{\alpha\beta}$ define the same outer isomorphism. Thus for fixed $\alpha, \beta$ the family $\lambda^\xi_{\alpha\beta}$ defines an “outer isomorphism” of sheaves on $U_{\alpha\beta}$:

$$\lambda_{\alpha\beta} : Aut(x_\beta)|_{U_{\alpha\beta}} \to Aut(x_\alpha)|_{U_{\alpha\beta}} \quad (2-17)$$

which does not depend on the choice of $f^\xi_{\alpha\beta}$. This system of sheaves of groups $Aut(x_\alpha)$ and outer isomorphisms $\lambda_{\alpha\beta}$ is called the lien of the gerbe $G$. To any gerbe with lien $L$, we can attach a 2-cocycle that takes its values in $L$, see [20] for details.

**Definition 2.3.2.** Let $K$ be a lien on $X$. A gerbe with lien $K$ is a gerbe $G$ on $X$ together with an isomorphism of liens $\theta : \text{lien}(G) \simeq K$. Two gerbes with lien $K$ are said to be
equivalent if there is an equivalence between the underlying fibered categories. We designate by \( H^2(X, K) \) the set of equivalence classes of gerbes with lien \( K \).

**Example 2.3.3.** Let \( G \) be a bundle of groups on \( X \). The stack \( \text{Tors}(G) \) of right \( G \)-torsors on \( X \) is a gerbe on \( X \). It is (globally) non-empty since its fiber on \( X \) always contains the trivial \( G \)-torsor on \( X \). It is also locally connected since any \( G \)-torsor is locally isomorphic to the trivial \( G \)-torsor. Thus \( \text{Tors}(G) \) is in fact a neutral gerbe. Its lien is the lien represented by the group \( G \), denoted \( \text{lien}(G) \).

**Example 2.3.4.** Let \( 1 \to A \to C \to B \to 1 \) be an exact sequence of (possibly infinite dimensional) Lie groups such that the projection \( C \to B \) is a locally trivial \( A \)-bundle. Let \( p : P \to X \) be a smooth \( B \)-bundle over a manifold \( X \). Consider the problem of finding a \( C \)-bundle \( q : Q \to X \) such that the associated \( B \)-bundle \( Q/A \to X \) is isomorphic to \( P \to X \). The fibered category of local solutions to this is a gerbe on \( X \). The lien associated to this gerbe is isomorphic to the sheaf \( A_X \) of smooth \( A \)-valued functions.

**Definition 2.3.5.** Let \( P \) be a gerbe on \( X \), let \( G \) be a sheaf of groups on \( X \), and let \( \mathfrak{U} = (U_i)_{i \in I} \) be an open cover of \( X \). We say that \( P \) is a \( G \)-gerbe on \( X \) if there exists for each \( i \in I \) an object \( x_i \in \text{Ob}(P_{U_i}) \) and an isomorphism of sheaves of groups on \( U_i \)

\[
\eta_i: G|_{U_i} \longrightarrow \text{Aut}_{P_{U_i}}(x_i) \tag{2-18}
\]

where \( \text{Aut}_{P_{U_i}}(x_i) \) is the sheaf of automorphisms of the object \( x_i \) above \( U_i \).

**Definition 2.3.6.** Let \( P \) be a gerbe on \( X \), let \( G \) be a sheaf of groups on \( X \). Suppose there exists, for every object \( x \) in a fiber category \( \mathcal{P}_U \), an isomorphism of sheaves of groups

\[
\eta_x: G|_U \longrightarrow \text{Aut}_{\mathcal{P}_U}(x) \tag{2-19}
\]

and that, for any morphism \( f: x \to y \) in \( \mathcal{P}_U \), the corresponding diagram of sheaves on \( U \)

\[
\begin{array}{ccc}
G|_U \\
\downarrow \eta_x & \cong & \downarrow \eta_y \\
\text{Aut}_{\mathcal{P}_U}(x) \rightarrow^\chi \text{Aut}_{\mathcal{P}_U}(y)
\end{array}
\tag{2-20}
\]
determined by the morphism $\lambda$ associated to $f$ commutes. The gerbe $\mathcal{P}$ is then called an abelian $G$-gerbe on $X$.

The map $\lambda$ associated to $f$ is the isomorphism

$$\lambda : \text{Aut}_{\mathcal{P}_U}(x) \to \text{Aut}_{\mathcal{P}_U}(y)$$

$$u \mapsto fuf^{-1}.$$  \hspace{1cm} (2–21)

An abelian $G$-gerbe is a $G$-gerbe by definition. In fact, we now show that in this situation the sheaf $G$ must be abelian. In particular, the commutivity of the group law in $G$ can be verified locally, for sections of a sheaf $G|_U$. Let $g$ be a section of this sheaf and consider the above triangle associated to the corresponding arrow $u = \eta_x(g) : x \to x$. This is the diagram

$$\begin{array}{ccc}
G|_U & \xrightarrow{\eta_x} & G|_U \\
\downarrow & & \downarrow \\
\text{Aut}_{\mathcal{P}_U}(x) & \xrightarrow{i_u} & \text{Aut}_{\mathcal{P}_U}(x) \\
\end{array}$$

where $i_u$ denotes inner conjugation by $u$ in the sheaf $\text{Aut}_{\mathcal{P}_U}(x)$. Commutivity of this diagram implies that $i_u$ is the identity map whence the sheaf $\text{Aut}_{\mathcal{P}_U}(x)$, and therefore $G|_U$ is abelian.

We state two propositions (see [4] for proofs) to illustrate how the notion of a lien is helpful in characterizing a $G$-gerbe.

**Proposition 2.3.7.** Let $G$ be a sheaf of groups on a space $X$. A gerbe $\mathcal{G}$ on $X$ is a $G$-gerbe if and only if its lien is locally isomorphic to lien($G$).

**Proposition 2.3.8.** Let $G$ be a sheaf of abelian groups on a space $X$. A gerbe $\mathcal{G}$ on $X$ is an abelian $G$-gerbe if and only if lien($\mathcal{G}$) $\simeq$ lien($G$).

We now state a theorem of Giraud (see [15]) giving the connection between gerbes with $G$-liens where $G$ is an abelian sheaf, and cohomology.

**Theorem 2.3.9.** Let $G$ be a sheaf of abelian groups on $X$. Let $L = $ lien($G$). A gerbe $\mathcal{G}$ with lien $L$ gives rise to a degree-2 Čech cocycle with values in $L$ as follows. Choose an
open covering \((U_i)\) of \(X\) for which each \(G(U_i)\) is nonempty, and such that any two objects of \(G(U_{ij})\) are isomorphic. Choose objects \(P_i\) of \(G(U_i)\), and let \(\phi_{ij} : P_j|_{U_{ij}} \to P_i|_{U_{ij}}\) be an isomorphism between objects in the category \(G(U_{ij})\). We define a section \(h_{ijk}\) of \(L\) over \(U_{ijk}\) by

\[
\phi_{ijk} = \phi_{ik}^{-1} \circ \phi_{ij} \circ \phi_{jk} \in \text{Aut}(P_k) \simeq L
\]  

so that \((h) = (h_{ijk})\) is an \(L\)-valued Čech 2-cocycle. The corresponding class in \(H^2(X, L)\) is independent of all choices, and the assignment \(G \mapsto (h)\) gives a one-to-one correspondence between equivalence classes of gerbes with lien \(L\) and the elements of \(H^2(X, L)\).

We conclude this section with the following definition.

**Definition 2.3.10.** Let \(G\) be a gr-stack on \(X\). A right \(G\)-torsor on \(X\) is a stack \(Q\) on \(X\) together with a right action functor

\[
Q \times G \to Q
\]  

which is coherently associative and satisfies the unit condition, and for which the induced functor

\[
Q \times G \to Q \times Q
\]  

\((q, g) \mapsto (q, qg)\)

is an equivalence. In addition we require that \(Q\) be locally non-empty.

### 2.4 2-Categories, 2-Functors, 2-Natural Transformations

We will review, in an informal spirit, some definitions from the theory of 2-categories.

**Definition 2.4.1.** A 2-category \(\mathfrak{A}\) consists of the following data:

1. a collection of objects \(A, B, C, \ldots\),
2. for each ordered pair of objects \((A, B)\), a small category \(\mathfrak{A}(A, B)\),
3. for each triple \(A, B, C\) of objects, a bifunctor

\[
c_{A,B,C} : \mathfrak{A}(A, B) \times \mathfrak{A}(B, C) \to \mathfrak{A}(A, C),
\]
4. for each object $A$, a functor

$$u_A : 1 \to \mathcal{A}(A, A).$$

(2–29)

Here $1$ denotes the terminal category (with one object and one arrow) in the category of small categories.

These elements of data are required to satisfy associativity laws for composition and the requirement that $u_A$ provides a left and right identity for this composition.

**Example 2.4.2.** A basic example is the 2-category $\text{CAT}$, whose objects are small categories, arrows are functors, and 2-arrows are natural transformations.

**Definition 2.4.3.** Given two 2-categories $\mathcal{A}$ and $\mathcal{B}$, a 2-functor $F : \mathcal{A} \to \mathcal{B}$ consists in giving

1. for each object $A \in \mathcal{A}$, an object $F(A) \in \mathcal{B}$,
2. for each pair of objects $A, A'$ in $\mathcal{A}$, a functor

$$F_{A,A'} : \mathcal{A}(A, A') \to \mathcal{B}(F(A), F(A'))$$

(2–30)

(For brevity’s sake we often write $F$ instead of $F_{A,A'}$).

This data is required to satisfy the following axioms:

1. **Compatibility with composition:** given three objects $A, A', A''$ in $\mathcal{A}$, the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{A}(A, A') \times \mathcal{A}(A', A'') & \xrightarrow{c_{AA',A''}} & \mathcal{A}(A, A'') \\
F_{AA'} \times F_{A'A''} \downarrow & & \downarrow F_{AA''} \\
\mathcal{B}(F(A), F(A')) \times \mathcal{B}(F(A'), F(A'')) & \xrightarrow{c_{F(A),F(A'),F(A'')}} & \mathcal{B}(F(A), F(A''))
\end{array}$$

(2–31)
2. Unit: for every object $A \in \mathfrak{A}$ the following diagram commutes.

\[
\begin{array}{c}
1 \quad \xrightarrow{u_A} \quad \mathfrak{A}(A, A) \\
\downarrow \quad \downarrow \\
\mathfrak{B}(F(A), F(A)) \\
\phantom{1} \quad \xrightarrow{u_{F(A)}} \quad \xrightarrow{F_{AA}} \phantom{1} \\
\end{array}
\]

Definition 2.4.4. Consider two 2-categories $\mathfrak{A}$, $\mathfrak{B}$, and two 2-functors $\mathfrak{A} \xrightarrow{F} \mathfrak{B}$ between them. A 2-natural transformation $\theta : F \Rightarrow G$ consists in giving, for each object $A \in \mathfrak{A}$, an arrow $\theta_A : F(A) \to G(A)$ such that the following diagram commutes for each pair of objects $A, A'$.

\[
\begin{array}{c}
\mathfrak{A}(A, A') \quad \xrightarrow{F_{AA'}} \quad \mathfrak{B}(F(A), F(A')) \\
\downarrow \quad \downarrow \\
\mathfrak{B}(G(A), G(A')) \quad \xrightarrow{\mathfrak{B}(1_{F(A)}, \theta_{A'})} \quad \mathfrak{B}(F(A), F(A')) \\
\end{array}
\]

2.5 Fibered 2-Categories, 2-Stacks and 2-Gerbes

Definition 2.5.1. A fibered 2-category in 2-groupoids above a space $X$ consists of a family of 2-groupoids $\mathcal{C}_U$, for each open set $U$ in $X$, together with an inverse image 2-functor

\[
f^* : \mathcal{C}_U \longrightarrow \mathcal{C}_{U_1}
\]

associated to every inclusion of open sets $f : U_1 \subset U$ (which is the identity whenever $f = 1_U$), and a natural transformation

\[
\phi_{f,g} : (fg)^* \longrightarrow g^* f^*
\]

for every pair of composable inclusions

\[
U_2 \xrightarrow{g} U_1 \xrightarrow{f} U.
\]
For each triple of composable inclusions
\[ U_3 \xrightarrow{h} U_2 \xrightarrow{g} U_1 \xrightarrow{f} U \] (2–37)
we require a modification (which is a map between natural transformations)
\[ \psi_{f,g,h} : (fgh)^* \rightarrow h^*(fg)^* \rightarrow h^*(g^*f^*) \] (2–38)

between the composite natural transformations
\[ \psi_{f,g,h} : (fgh)^* \rightarrow h^*(fg)^* \rightarrow h^*(g^*f^*) \] (2–39)
and
\[ \chi_{f,g,h} : (fgh)^* \rightarrow (gh)^*f^* \rightarrow (h^*g^*)f^* . \] (2–40)

Finally, for any \( U_4 \hookrightarrow U_3 \), the two methods by which the modifications \( \alpha \) compare the composite two arrows
\[ (fghk)^* \rightarrow (ghk)^*(f)^* \rightarrow ((hk)^*g^*)f^* \rightarrow k^*h^*g^*f^* \] (2–41)
and
\[ (fghk)^* \rightarrow k^*(fgh)^* \rightarrow k^*(h^*(fg)^*) \rightarrow k^*h^*g^*f^* \] (2–42)
must coincide.

**Definition 2.5.2.** A Cartesian 2-functor \( F : \mathcal{C} \rightarrow \mathcal{D} \) between fibered 2-categories over \( X \) consists of a family of 2-functors \( F_U : \mathcal{C}_U \rightarrow \mathcal{D}_U \) indexed by the open sets \( U \), together with, for every morphism \( f : U_2 \rightarrow U_1 \), a natural transformation of 2-functors
\[ \phi_f : f^* \circ F_{U_1} \rightarrow F_{U_2} \circ f^*. \] (2–43)

For any pair of composable inclusions \( f \) and \( g \), a 3-arrow \( \alpha_{f,g} \) is also given, which compares the natural transformation from \((fg)^* \circ F_{U_1}\) to \( F_{U_3} \circ g^* \circ f^* \) defined by \( \phi_f \) and \( \phi_g \) with
that constructed from $\phi_{fg}$. This 3-arrow is required to satisfy a coherence condition relating the two induced natural transformations associated to a triplet $(f, g, h)$ of composable inclusions. This condition will not be made explicit here.

**Definition 2.5.3.**

- A 2-prestack in groupoids $\mathcal{C}$ above a space $X$ is a fibered 2-category in 2-groupoids above $X$ such that for every pair of objects $x, y \in \mathcal{C}_U$, the fibered category $\text{Arc}_{\mathcal{C}}(x, y)$ is a stack on $U$.

- If, in addition, 2-descent is effective for objects in $\mathcal{C}$, then the 2-prestack is called a 2-stack in 2-groupoids above $X$.

- A morphism of 2-stacks is just a morphism of the underlying fibered 2-categories.

By 2-descent data we mean that we are given, for an open cover $\mathcal{U} = \{U_\alpha\}$ of an open set $U \subset X$, a family of objects $x_i \in \mathcal{C}_{U_i}$, of 1-arrows $\phi_{\alpha\beta} : x_\alpha|_{U_{\alpha\beta}} \longrightarrow x_\beta|_{U_{\alpha\beta}}$ and a family of 2-arrows $\psi_{\alpha\beta\gamma} : \phi_{\alpha\gamma} \circ \phi_{\beta\gamma} \Rightarrow \phi_{\alpha\beta}$

\begin{equation}
(2-44)
\end{equation}

for which the tetrahedral diagram of 2-arrows induced by $\psi$ in $\mathcal{C}_{U_{\alpha\beta\gamma\delta}}$ commutes:

\begin{equation}
(2-45)
\end{equation}

The 2-descent condition $(x_i, \phi_{ij}, \psi_{\alpha\beta\gamma})$ is effective if there exists an object $x \in \mathcal{C}_U$ together with isomorphisms $f_\alpha : x|_{U_\alpha} \simeq x_\alpha$ compatible with the morphisms $\phi_{ij}$ and $\psi_{\alpha\beta\gamma}$.

For any 2-prestack $\mathcal{C}$ on $X$, one defines by the same sheafification method as for sheaves and 1-stacks, an “associated 2-stack” 2-functor

\begin{equation}
a : \mathcal{C} \rightarrow \mathcal{C}
\end{equation}
which is universal for Cartesian 2-functors $b : \mathcal{C} \rightarrow \mathcal{D}$ from $\mathcal{C}$ into 2-stacks. The latter property characterizes $\mathcal{C}$ up to 2-equivalence. For, suppose that a Cartesian 2-functor $b : \mathcal{C} \rightarrow \mathcal{D}$ satisfies:

1. $b$ is fiberwise fully faithful, i.e. for any pair of objects $x$ and $y$ in a fiber category $\mathcal{C}_U$, the induced map of stacks on $U$

$$\text{Ar}(x, y) \rightarrow \text{Ar}(b(x), b(y))$$

(2–47)

is an equivalence.

2. every object in $\mathcal{D}$ is locally isomorphic to one in the image of $\mathcal{C}$.

Then $\mathcal{D}$ is an associated 2-stack of $\mathcal{C}$.

**Definition 2.5.4.**

- A 2-gerbe $\mathcal{P}$ on a space $X$ is a 2-stack in 2-groupoids on $X$ which is locally non-empty, locally connected, in which 1-arrows are weakly invertible, and 2-arrows are invertible.

- A morphism of 2-gerbes is just a morphism of the underlying 2-stacks.

To say that $\mathcal{P}$ is locally non-empty means there exists a covering $\mathfrak{U} = (U_i)$ of $X$ for which the set of objects of the 2-category $\mathcal{P}_{U_i}$ is non-empty. The connectedness condition on $\mathcal{P}$ is the requirement that for any pair of objects $x$ and $y$ in the fibered 2-category $\mathcal{G}_U$, there exists an open cover $\mathfrak{V} = (V_\alpha)$ of $U$ such that the set of 1-arrows from $x|_{V_\alpha}$ to $y|_{V_\alpha}$ is non-empty for all $\alpha$. To say that 1-arrows are weakly invertible means that for any 1-arrow $f : x \rightarrow y$ in a fibered 2-category $\mathcal{P}_U$, there exists an arrow $g : y \rightarrow x$ in $\mathcal{P}_U$ which is both a left and right inverse of $f$ (upto a pair of 2-arrows $\lambda : g \circ f \Rightarrow 1_x$ and $\rho : f \circ g \Rightarrow 1_y$).

**Example 2.5.5.** (Breen) Let $L$ be a lien on $X$. We can ask when is $L$ isomorphic to a lien of the form $\text{lien}(G)$ for some gerbe $G$ on $X$? Locally the answer is always, since the lien $L$ is locally isomorphic to a lien of the form $\text{lien}(G)$ for some sheaf of groups $G$, whence it is realized by the neutral gerbe $\text{Tors}(G)$ corresponding to $G$. Globally this gives a 2-gerbe on $X$. 

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For a 2-gerbe $\mathcal{P}$, we can associate to each object $x \in \mathcal{P}_U$ a gr-stack $\mathcal{P}_x := \text{Ar}_U(x, x)$ above $U$.

**Definition 2.5.6.** Let $\mathcal{P}$ be a 2-gerbe on $X$, and let $\mathcal{G}$ be a gr-stack on $X$. We say that $\mathcal{P}$ is a $\mathcal{G}$-2-gerbe on $X$ if there exists an open cover $\mathcal{U} = (U_i)_{i \in I}$ of $X$, for each $i \in I$ an object $x_i \in \text{Ob}(\mathcal{P}_U)$ and an equivalence $\mathcal{G}_{U_i} \simeq \mathcal{G}_{x_i}$ over $U_i$.

**Definition 2.5.7.** Let $\mathcal{P}$ be a 2-gerbe on $X$, let $\mathcal{G}$ be a gr-stack on $X$. Suppose there exists, for every object $x$ in a fiber 2-category $\mathcal{P}_U$, an isomorphism of sheaves of gr-stacks

$$
\eta_x : G|_U \longrightarrow \mathcal{E}_{q\mathcal{P}_U}(x),
$$

and for any morphism $f : x \rightarrow y$ in $\mathcal{P}_U$, a 2-arrow $\eta_f : \lambda_f \circ \eta_x \Rightarrow \eta_y$

\[\eta_x \xrightarrow{\eta_f} \eta_y \xrightarrow{\lambda_f} \mathcal{E}_{q\mathcal{P}_U}(y)\]

where $\lambda_f$ is the morphism of gr-stacks

$$
\lambda : \mathcal{E}_{q\mathcal{P}_U}(x) \rightarrow \mathcal{E}_{q\mathcal{P}_U}(y)
$$

\[u \mapsto fuf^{-1}.\]

defined by $f$. The natural transformations $\eta_f$ are required to respect the group structures, and satisfy the following transitivity and normalization conditions:

1. For any pair of composable morphisms $f : x \rightarrow y$ and $g : y \rightarrow z$ in $\mathcal{P}_U$, the composite 2-arrow obtained by pasting $\eta_f$ and $\eta_g$ is equal to $\eta_{gf}$.

2. For any 2-arrow $\phi : f \Rightarrow g$ between a pair of morphisms $f, g : x \rightarrow y$ in $\mathcal{P}_U$,

$$\eta_f = \eta_g \circ \lambda_\phi,$$ where $\lambda_\phi : \lambda_f \Rightarrow \lambda_g$ is conjugation by $\phi$.

3. For every $x$ in $\mathcal{P}_U$, $\eta_1 = 1$

A 2-gerbe $\mathcal{P}$ satisfying these conditions is called an abelian $\mathcal{G}$-2-gerbe on $X$.  

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2.6 Representable Functors

We will rapidly review the notion of a representable functor. The primary source for this summary is \cite{12}.

Let $\mathcal{C}$ be a category and let \textit{SET} denote the category of sets. The \textit{category of presheaves of sets on} $\mathcal{C}$ is the functor category $\text{Hom}(\mathcal{C}^{\text{op}}, \text{SET})$. Now choose an object $X$ of $\mathcal{C}$ and define a presheaf $h_X$ by:

$$h_X(Y) = \text{Hom}_\mathcal{C}(Y, X), \quad h_X(f) = \text{Hom}_\mathcal{C}(f, X) := \circ f$$

(2–52)

for any object $Y$ of $\mathcal{C}$ and arrow $f$ in $\mathcal{C}$. This functor is called the \textit{functor represented by} $X$. This construction is functorial in $X$, i.e. gives a functor

$$h : \mathcal{C} \to \text{Hom}(\mathcal{C}^{\text{op}}, \text{SET})$$

(2–53)

defined by $h(X) = h_X$, and for $f \in \text{Hom}(X,Y)$,

$$h(f)(Z) : \text{Hom}(Z,X) \to \text{Hom}(Z,Y)$$

(2–54)

$$g \mapsto f \circ g.$$  

(2–55)

\textbf{Definition 2.6.1.} A functor $F \in \text{Hom}(\mathcal{C}^{\text{op}}, \text{SET})$ is representable if it is in the essential image of $h$, i.e. if it is isomorphic to some $h_X$. We say that $F$ is represented by (the object) $X$. Note that the isomorphism is part of the data.

If $(X, \alpha : h_X \simeq F)$ represents $F$ then we obtain a universal element $\xi = \alpha_X(id_X) \in F(X)$. The isomorphism can be recovered from $\xi$. To see this, if $f : X \to Y$ is any arrow in $\mathcal{C}$, then the diagram

$$\begin{align*}
\text{Hom}(X,X) & \xrightarrow{\alpha_X} F(X) \ , \\
\text{Hom}(f,X) & \quad \downarrow \\
\text{Hom}(Y,X) & \xrightarrow{\alpha_Y} F(Y)
\end{align*}$$

(2–56)
commutes, since \( \alpha \) is a natural transformation. But \( id_X \in \text{Hom}(X,X) \) maps under \( \text{Hom}(f,X) \) to \( f \in \text{Hom}(Y,X) \), whence by the above diagram we have

\[
\alpha_Y(f) = F(f)(\alpha_X(id_X)) = F(f)(\xi)
\]

which shows how to compute \( \alpha \) from \( \xi \). Thus we can also say \( F \) is represented by the pair \((X,\xi)\). Yoneda’s lemma states that:

**Theorem 2.6.2.** The functor \( h : C \to \text{Hom}(C^{\text{op}},\text{SET}) \) is fully faithful.

**Corollary 2.6.3.** If a functor \( F \) is representable then the representing object in \( C \) is defined uniquely up to a unique isomorphism.

In the category \( \text{SET} \) we have definitions for products, fibered products, equalizers, coequalizers, etc. We can use representable functors to import these notions from \( \text{SET} \) to any arbitrary category. To see how this is done we will examine carefully the definition of product.

**Products.** The product of two sets \( X \) and \( Y \), together with the projection maps \( p_1 : X \times Y \to X, \ p_2 : X \times Y \to Y \), is characterized up to isomorphism by a universal property: given any sets \( Z \) with maps \( q_1 : Z \to X, \ q_2 : Z \to Y \), there is a unique map \( q : Z \to X \times Y \) such that \( q_1 = p_1 \circ q \) and \( q_2 = p_2 \circ q \). In fact, if \( P \) and \( P' \) are two sets with this universal property, then there exist maps \( f : P \to Q, \ g : Q \to P \) and uniqueness implies \( g \circ f = id_P \) and \( f \circ g = id_Q \). Thus the product of two sets can be defined as the object with this universal property. This description of the product of the sets \( X \) and \( Y \) mentions only objects and morphisms; it says nothing about ordered pairs or any thing specifically about the objects under consideration. So we can import this definition to any arbitrary category. In particular, if \( C \) is any random category and \( X, \ Y \) are objects of \( C \), we can define the product of \( X \) and \( Y \) as being an object \( P \) of \( C \) with morphisms \( p_1 : P \to X, \ p_2 : P \to Y \) which is universal, i.e. if \( Z \) is any other object of \( C \) with maps \( q_1 : Z \to X, \ q_2 : Z \to Y \), there is a unique map \( q : Z \to P \) such that \( q_1 = p_1 \circ q \) and
\[ q_2 = p_2 \circ q. \] In general such a universal object may not exist. In the category \( \text{SET} \) it does: it is the set consisting of the ordered pairs \((x, y)\) for \(x \in X, y \in Y\).

Universal properties (such as products) can be expressed by saying that a certain presheaf is representable.

**Definition 2.6.4.** If for every pair of objects \( X \) and \( Y \) of a category \( \mathcal{C} \), the presheaf \( h_X \times h_Y \):

\[ Z \mapsto \text{Hom}(Z, X) \times \text{Hom}(Z, Y) \]  (2–58)

is representable, then we say that products are representable in \( \mathcal{C} \). The object that represents this functor is unique up to unique isomorphism whence we say that the product of \( X \) and \( Y \) is the object that represents this functor.

To see why this definition makes sense, say an object \( P \) represents the product of \( X \) and \( Y \). By definition this means we have an isomorphism \( h_P \simeq h_X \times h_Y \), i.e. an isomorphism

\[ \alpha_Z : \text{Hom}(Z, P) \rightarrow \text{Hom}(Z, X) \times \text{Hom}(Z, Y) \]  (2–59)

for each \( Z \in \text{Ob}(\mathcal{C}) \). Then plugging \( P \) in for \( Z \) above, we get that the universal element \( \alpha_Z(id_P) \) is a pair of morphisms \( p_1 : P \rightarrow X, p_2 : P \rightarrow Y \) with the property that given any pair of morphisms \( q_1 : Z \rightarrow X, q_2 : Z \rightarrow Y \), both factor through a unique morphism \( Z \rightarrow P \), see diagram below.

\[ \begin{align*}
q \in \text{Hom}(Z, P) & \xrightarrow{\sim} \text{Hom}(Z, X) \times \text{Hom}(Z, Y) \\
\text{Hom}(q, P) & \downarrow \\
\text{id}_P \in \text{Hom}(P, P) & \xrightarrow{\sim} \text{Hom}(P, X) \times \text{Hom}(P, Y)
\end{align*} \]  (2–60)

The definition for arbitrary products is an obvious extension of the above definition.

We can make similar definitions for fibered products, equalizers, and coequalizers. First we list these definitions for the category \( \text{SET} \). Given sets \( X, Y \) and \( Z \) with maps...
Given two maps $f, g : X \to Y$ of sets, the equalizer $Eq(f, g)$ of $f$ and $g$ is the set $\{x \in X | f(x) = g(x)\}$. If $i : Eq(f, g) \to X$ is the inclusion, then $f \circ i = g \circ i$, and $Eq(f, g)$ is universal for this property. The definition of the coequalizer is just dual to that of the equalizer.

We now make this definitions for any arbitrary category $C$.

**Definition 2.6.5.** 1. Given arrows $f : X \to Z$, $g : Y \to Z$ in $C$, we define the fibered product $X \times_Z Y$ of $X$ and $Y$ to be the object $P$ which represents the functor

$$Q \mapsto Hom(Q, X) \times_{Hom(Q, Z)} Hom(Q, Y).$$

(2–61)

If the object $P$ exists then we have a functorial isomorphism

$$\alpha_Q : Hom(Q, P) \simeq Hom(Q, X) \times_{Hom(Q, Z)} Hom(Q, Y).$$

(2–62)

The universal object $\alpha_P(id_P)$ is a pair of morphisms $p_1 : P \to X$, $p_2 : P \to Y$.

Since it is universal, for any pair of morphisms $q_1 : Q \to X$, $q_2 : Q \to Y$ such that $f \circ q_1 = g \circ q_2$, there is a unique arrow $h : Q \to P$ such that $q_1 = p_1 \circ f$ and $q_2 = p_2 \circ f$.

2. Given arrows $f, g : X \to Y$, we define their equalizer to be the object $K$ which represents the functor

$$Z \mapsto Eq(Z \leftarrow X \begin{array}{c} f \\ g \end{array} \rightarrow Y).$$

(2–63)

i.e.,

$$Z \mapsto Eq(\begin{array}{c} Hom(Z, X) \xrightarrow{Hom(Z, f)} Hom(Z, Y) \\ Hom(Z, g) \end{array}).$$

(2–64)

If $K$ exists then there is a functorial isomorphism

$$Hom(Z, K) \simeq Eq(Hom(Z, f), Hom(Z, g)).$$

(2–65)
Thus there is a canonical morphism $i : K \to X$ such that $f \circ i = g \circ i$, and any morphism $h : Z \to X$ such that $f \circ h = g \circ h$ factors through a unique morphism $Z \to K$.

3. Given arrows $f, g : X \to Y$, we define their coequalizer to be the object $K$ which represents the functor

$$Z \mapsto \text{Eq}(\text{Hom}(X, Z) \xrightarrow{\text{Hom}(f,Z)} \text{Hom}(Y, Z)).$$

(2–66)

If $K$ exists then there is a functorial isomorphism

$$\text{Hom}(Z, K) \simeq \text{Eq}(\text{Hom}(f, Z), \text{Hom}(g, Z)).$$

(2–67)

Thus there is a canonical map $p : Y \to K$ such that $p \circ f = p \circ g$, and any morphism $h : Y \to Z$ such that $h \circ f = h \circ g$ factors through a unique morphism $K \to Z$.

### 2.7 2-Representability

We will now briefly discuss the notion of 2-representability. These definitions can be found in [16]. Let $\mathcal{C}$ be a 2-category. Recall that $\mathcal{C}^{\text{op}}$ is obtained from $\mathcal{C}$ by “inverting the direction” of the 1-cells of $\mathcal{C}$. i.e. if $[X, Y]$ is the collection of 1-cells in $\mathcal{C}$ from $X$ to $Y$ then $[X, Y]$ in $\mathcal{C}^{\text{op}}$ is $[Y, X]$. We will define 2-representability for 2-functors $f : \mathcal{C}^{\text{op}} \to \text{CAT}$.

Let $F(\mathcal{C})$ be the 2-category of 2-functors from $\mathcal{C}^{\text{op}}$ to $\text{CAT}$. We define the following strict 2-functor:

$$h : \mathcal{C} \to \text{CAT}$$

(2–68)

$$h_X(\cdot) = \text{Hom}_\mathcal{C}(\cdot, X)$$

(2–69)

and which is defined similarly on 1-cells and 2-cells. Let $f$ be any 2-functor i.e. $f \in \text{Ob}(F(\mathcal{C}))$. Then for all $X \in \text{Ob}(\mathcal{C})$, we can define the functor

$$h_X, : \text{Hom}_{F(\mathcal{C})}(h_X, f) \to f(X).$$

(2–70)
For $u \in Ob(Hom_{F(C)}(h_X, f))$ we associate $h_{X,f}(u) = u(id_X) \in Ob(f(X))$ and to $\phi : u \to u'$, where $\phi \in Mor(Hom_{F(C)}(h_X, f))$ we associate

$$h_{X,f}(\phi) = \phi(id_X) : u(id_X) \to u'(id_X). \tag{2-71}$$

The functor $h_{X,f}$ is an equivalence of categories with the quasi-inverse given by $k_{X,f} : f(X) \to Hom_{F(C)}(h_X, f)$ where for all $\xi \in Ob(f(X))$, $k_{X,f}(\xi) = u_\xi$ is the morphism of 2-functors $u_\xi : h_X \to f$, $u_\xi(\_ ) = f(\_)(\xi)$ and for $\phi : \xi \to \xi' \in Mor(f(X))$,

$$k_{X,f}(\phi) = f(\_)(\phi) : u_\xi \to u_{\xi'}. \tag{2-72}$$

In particular, letting $f = h_X$, we get the following proposition.

**Proposition 2.7.1.** The 2-functor $h : C \to F(C)$ is 2-faithful.

**Definition 2.7.2.** We say that a 2-functor $f \in Ob(F(C)$ is 2-representable if $f$ is equivalent to a 2-functor of the form $h_X$. We say that the solution to the universal 2-problem determined by $f$ is the pair $(X, u)$ where $X \in Ob(C)$ and $u : h_X \simeq f$ is an equivalence of 2-functors. We say that the pair $(X, \xi)$, where $\xi = u(id_X) \in Ob(f(X))$ gives a representation of $f$.

See [16] for the proofs of the following propositions.

**Proposition 2.7.3.** Let $f$ be a representable 2-functor in $F(C)$ and let $(X, \xi)$ and $(X', \xi')$ be two representations of $f$. Then there exists a pair $(\lambda, \epsilon)$ which gives an equivalence in $C$: $\lambda : X \simeq X'$ and a 2-isomorphism $\epsilon : f(\lambda)(\xi') \simeq \xi$. Further, for all other such pairs $(\lambda_1, \epsilon_1)$ there exists a unique 2-isomorphism $\alpha : \lambda \simeq \lambda_1$ such that $\epsilon_1 \circ (f(\lambda)(\xi')) = \epsilon$.

**Proposition 2.7.4.** Let $C$ be a 2-category and let $\mathcal{F}(C)$ denote the 2-category of 2-functors from $C^{op}$ to $CAT$. Let $F$ be a 2-functor which is representable in $\mathcal{F}(C)$, and suppose $(X, \xi)$ and $(X', \xi')$ are two given representations of $F$. Then there exists a pair $(\lambda, \epsilon)$ such that $\lambda : X \overset{\sim}{\longrightarrow} X'$ is an equivalence in $C$ and $\epsilon : F(\lambda)(\xi') \overset{\sim}{\longrightarrow} \xi'$ is a 2-isomorphism. Further, for every other such pair $(\lambda_1, \epsilon_1)$ there exists a unique 2-isomorphism $\alpha : \lambda \overset{\sim}{\longrightarrow} \lambda'$ such that $\epsilon_1 \circ (F(\lambda)(\xi')) = \epsilon$. 

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2.8 Giraud’s approach to Liens of Gerbes

We now describe Giraud’s original definition of a lien given in [15]. It is this approach that we will generalize to define the 2-lien of a 2-gerbe.

Let \( X \) be a topological space, and let \( F \) and \( G \) be sheaves of groups on \( X \). Consider the sheaf \( \text{Isom}(F, G) \) (the sheaf of isomorphisms of sheaves of groups from \( F \) to \( G \)). Let \( \text{Int}(G) \) denote the sheaf of inner automorphisms of the sheaf \( G \). Define the quotient sheaf

\[
\text{Out}(F, G) := \text{Isom}(F, G) / \text{Int}(G)
\]  

where \( \text{Int}(G) \) acts on \( \text{Isom}(F, G) \) from the right via composition. Now, for every open \( U \subset X \), let \( LI(X)(U) \) denote the category whose objects are the sheaves of groups on \( U \), and whose arrows between two objects \( A \) and \( B \) are the global sections of the sheaf \( \text{Out}(A, B)(U) \) i.e. \( \text{Hom}_{LI(X)(U)}(A, B) = \Gamma(U, \text{Out}(A, B)) \). The composition law is defined after passing to the quotient sheaf \( \text{Out} \), and is well-defined (see the discussion of a lien of a gerbe in Section 2.3).

Now if \( V \hookrightarrow U \) is an inclusion of open sets in \( X \), then the formation of the quotient \( \text{Out}(A, B) \) commutes with the restriction of \( U \) to \( V \). So we can define a fibered category \( LI(X) \). Let \( \text{Grp}(X) \) denote the fibered category of sheaves of groups on \( X \). Then by construction we have a morphism of fibered categories

\[
\text{Grp}(X) \rightarrow LI(X).
\]  

Now if \( A \) and \( B \) are any two objects of the category \( LI(X)(U) \), then the presheaf \( \text{Hom}_{LI(X)(U)}(A, B) \) of arrows from \( A \) to \( B \) is identified with the sheaf \( \text{Out}(A, B) \). Thus \( LI(X) \) is a prestack. Thus upon applying the stackification functor, we obtain the associated stack \( LIEN(X) \). Composing the stackification functor \([15]\) with the morphism (1) yields a morphism of stacks

\[
lien(X) : \text{Grp}(X) \rightarrow LIEN(X).
\]
Definition 2.8.1.  
1. The stack $LIEN(X)$ is called the stack of liens on $X$.

2. We call a global section of this stack a lien over $X$.

For every $U \subset X$, we say that a lien of $U$ is an object of the category $LIEN(X|U)$ which is the fiber over $U$ of the stack of liens. We call $LIEN(X|U)$ the category of liens over $U$. To generalize this definition of a lien from topological spaces to topoi, we just replace “global section” in Definition 2.8.1.2 with “a cartesian section” of the stack $LIEN(X)$, as done by Giraud in [15].

Now let $\mathcal{G}$ be a stack on $X$. Recall that $Grp(X)$ denotes the stack of sheaves of groups on $X$. We have a cartesian functor

$$AUT: \mathcal{G} \to Grp(X) \quad (2-76)$$

$$AUT(\mathcal{G})(x) = Aut_U(x) \quad (2-77)$$

where $x \in Ob(\mathcal{G}(U))$. Composing this morphism with the morphism $lien(X): Grp(X) \to LIEN(X)$ gives a morphism of stacks

$$liau(\mathcal{G}): \mathcal{G} \to LIEN(X) \quad (2-78)$$

which to every object $x \in \mathcal{G}(U)$ associates the lien

$$liau(\mathcal{G})(x) = lien(Aut_U(x)) \quad (2-79)$$

(called the lien represented by the sheaf of $U$-automorphisms of $x$) and to every $U$-isomorphism $\alpha: x \to y$ of $\mathcal{G}$ associates the morphism

$$liau(\mathcal{G})(\alpha) = lien(Int(\alpha)) \quad (2-80)$$

(where $Int(\alpha): Aut_U(x) \to Aut_U(y)$ defined by $Int(\alpha)(a) = \alpha a \alpha^{-1}$ is a morphism of sheaves of groups over $X$).
By the definition of the stack of liens, if \( m, n : x \to y \) are two \( U \)-isomorphisms of \( \mathcal{G} \), we have
\[
\text{liau}(\mathcal{G})(m) = \text{liau}(\mathcal{G})(n)
\] (2–81)
since the functor \( \text{lien} \) transforms every inner automorphism to an identity arrow.

Now let \( m : \mathcal{F} \to \mathcal{G} \) be a morphism of stacks. Then we have a morphism of morphisms of stacks (i.e a 2-arrow):
\[
\text{liau}(m) : \text{liau}(\mathcal{F}) \Rightarrow \text{liau}(\mathcal{G}) \cdot m
\] (2–82)
where,
\[
\text{liau}(m) = \text{lien}(X) \circ \text{AUT}(m).
\] (2–83)
For every \( x \in \text{Ob}(\mathcal{F}(U)) \), the functor \( m \) induces a morphism of sheaves of groups
\[
\mu_x : \text{Aut}_U(x) \to \text{Aut}_U(m(x))
\] (2–84)
and by definition of the map \( \text{liau}(m) \), the morphism \( \text{liau}(m(x)) \) is the morphism of liens represented by \( \mu_x \):
\[
m_x = \text{lien}(\mu_x), \quad m_x : \text{lien}(\text{Aut}_U(x)) \to \text{lien}(\text{Aut}_U(m(x))).
\] (2–85)

**Definition 2.8.2.** Let \( \mathcal{F} \) be a stack over \( X \) and let \( L \) be a lien over \( X \) i.e. it is a global section \( L : X \to \text{LIEN}(X) \) of the stack of liens. An action of \( L \) on \( \mathcal{F} \) is a morphism of morphisms of stacks:
\[
a : L \circ f \Rightarrow \text{liau}(\mathcal{F})
\] (2–86)
where \( f : \mathcal{F} \to X \) is the projection onto the Zariski site of \( X \)
The Zariski site of \( X \) is the category whose objects are the open sets of \( X \), and the morphisms are the inclusion maps. By definition, \( a \) is a morphism of (cartesian) functors i.e. a family \( a(x) : L(U) \to \text{lien}(\text{Aut}_U(x)), x \in \text{Ob}(\mathcal{F}(U)), U \subset X \), of morphisms of liens on \( U \) that satisfies the conditions of compatibility with the restriction to open sets, and with the composition of morphisms.

**Definition 2.8.3.** Let \((L, a)\) be a lien operating on a stack \( \mathcal{F} \) and let \( u : L' \to L \) be a morphism of liens over \( X \). Then the action induced by \( a \) and \( u \) is the morphism \( b \) defined by
\[
b : L' \cdot f \to \text{liau}(\mathcal{F}),
\]
\[
b = a \cdot (u \circ f \circ L)
\]
(2–88) (where \( f : \mathcal{F} \to X \) is the projection onto the Zariski site of \( X \)). So \( b \) is the action such that, for every \( x \in \text{Ob}(\mathcal{F}(U)) \), \( b(x) \) is the composition
\[
L'(U) \xrightarrow{u(U)} L(U) \xrightarrow{a(x)} \text{liau}(\mathcal{F})(x).
\]
(2–90)

**Proposition 2.8.4.** Let \( \mathcal{F} \) be a gerbe on \( X \).

1. Let \((L, a)\) be a lien operating on \( \mathcal{F} \). The following are equivalent:
   
   (a) \( a \) is an isomorphism.

   (b) for every lien \( L' \) on \( X \) and every action \( b \) of \( L' \) on \( \mathcal{F} \) there exists a unique morphism of liens \( u : L' \to L \) such that \( b \) is the action induced by \( a \) and \( u \).

2. There exists a lien \((L, a)\) operating on \( \mathcal{F} \) that satisfies the conditions of (1).

**Definition 2.8.5.** We say that the lien of a gerbe \( \mathcal{F} \) is a lien operating on \( \mathcal{F} \) and satisfying the conditions of the above proposition.

Condition 1(b) characterizes the lien of a gerbe \( \mathcal{F} \) up to canonical equivalence. So we are justified in saying “the” lien of a gerbe \( \mathcal{F} \). By abuse of notation, the map \( a \) is often not mentioned. If \( L \) is the lien of a gerbe \( \mathcal{F} \), we say \( \mathcal{F} \) is bound by \( L \). Conversely, if \( L \) is a lien on \( X \), then an \( L \)-gerbe is a pair \((\mathcal{F}, a)\) where \( \mathcal{F} \) is a gerbe and \((L, a)\) is a lien of \( \mathcal{F} \).
The lien of a gerbe $\mathcal{F}$ was in fact made explicit before: it is a lien $L$ and a family of isomorphisms of liens over $U$,

$$a(x) : L(U) \sim \rightarrow \text{lien}(\text{Aut}_U(x))$$ (2–91)

where $x \in \mathcal{F}(U)$, $U \subset X$. This family is required to be compatible with the restriction of open sets and also satisfy the condition that if $i : x \rightarrow y$ is a $U$-isomorphism of $\mathcal{F}$, then the morphism of sheaves of groups $\text{Int}(i) : \text{Aut}_U(x) \rightarrow \text{Aut}_U(y)$ represents the identity morphism of $L(U)$.

Proof of Proposition 2.8.4. We follow Giraud’s proof in [15]. Since $\mathcal{F}$ is a gerbe, the projection $f : \mathcal{F} \rightarrow X_{zar}$ to the Zariski site is fully faithful, hence the map $\text{Hom}(L', L) \rightarrow \text{Hom}(L'f, Lf)$ given by $u \mapsto u \circ f$ is bijective. This forces $b = a(u \circ f)$ whence we have that 1(a) implies 1(b). Next, we show that for every gerbe $\mathcal{F}$, there exists a lien $L$ and an isomorphism $a : Lf \sim \rightarrow \text{liau}(\mathcal{F})$. Then (2) follows trivially, and so does $1(b) \Rightarrow 1(a)$ since 1(b) determines $L$ up to unique isomorphism. Let $I$ be the image category of $\text{liau}(\mathcal{F})$. Clearly this is a fibered category of $\text{LIEN}(X)$ and the functor induced by $\text{liau}(\mathcal{F})$,

$$L' : \mathcal{F} \rightarrow I$$ (2–92)

is cartesian. Moreover, the projection $I \rightarrow X_{zar}$ is both fully faithful and essentially surjective (this follows trivially from the fact that $\mathcal{F} \rightarrow X$ is fully faithful and because $\text{liau}(m) = \text{liau}(n)$ for any two $U$-isomorphisms $m, n : x \rightarrow y$ of $\mathcal{F}$). By the universal property of the associated stack, it follows that $X$ is the stack associated to $I$, and that there exists a cartesian section $L_0 : X \rightarrow I$ of $I$ and an isomorphism $a_0 : L_0f \sim \rightarrow L'$. Then setting $L = iL_0$ and $a = i \circ a_0$ (where $i : I \rightarrow \text{LIEN}(X)$ is the inclusion) the conclusion follows.

Corollary 2.8.6. Let $m : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of gerbes and let $(L, a)$ and $(M, b)$ denote the liens of $\mathcal{F}$ and $\mathcal{G}$ respectively. Then there exists a unique morphism of liens $u : L \rightarrow M$ such that $m$ is a $u$-morphism i.e. $\text{liau}(m) \cdot a = (b \circ m)(u \circ f)$. 38
CHAPTER 3
EQUALIZERS AND COEQUALIZERS

Recall that a lien is defined by assigning a family of sheaves of groups \((G_i)\) to the open sets \((U_i)\) that cover \(X\), and these sheaves are glued on the overlaps \(U_{ij}\) by a section of the quotient sheaf \(\text{Out}(G_j, G_i) = \text{Isom}(G_j, G_i)/\text{Inn}(G_i)\).

Our project involves appropriately defining the action of a \(gr\)-stack on a stack, and the quotient by such an action. To do this we first need to define these notions for group categories.

### 3.1 Equalizers

We begin by recalling the definition of equalizer for the category \(\text{SET}\).

Given two maps \(f, g : X \to Y\) of sets, the equalizer \(\text{Eq}(f, g)\) of \(f\) and \(g\) is the set \(\{x \in X | f(x) = g(x)\}\).

For any arbitrary category \(\mathcal{C}\), given arrows \(f, g : X \to Y\), we define their equalizer to be the object \(K\) which represents the functor

\[
Z \mapsto \text{Eq}( \text{Hom}(Z, X), \text{Hom}(Z, Y) ) .
\]

i.e.,

\[
Z \mapsto \text{Eq}( \text{Hom}(Z, X) \xrightarrow{\text{Hom}(Z,f)} \text{Hom}(Z,Y) ) .
\]

If \(K\) exists then there is a functorial isomorphism

\[
\text{Hom}(Z, K) \simeq \text{Eq}(\text{Hom}(Z, f), \text{Hom}(Z, g)) .
\]

(3–2)

Thus there is a canonical morphism \(i : K \to X\) such that \(f \circ i = g \circ i\), and any morphism \(h : Z \to X\) such that \(f \circ h = g \circ h\) factors through a unique morphism \(Z \to K\).

Equalizers are representable in the category \(\text{SET}\) by the object \(\text{Eq}(f, g)\). In fact we have a functorial isomorphism

\[
\text{Hom}(Z, \text{Eq}(f, g)) \simeq \text{Eq}(\text{Hom}(Z, f), \text{Hom}(Z, g)) .
\]

(3–3)
To see why this is true, let \( h : Z \to E(q,f,g) \) be any arrow in \( \text{Hom}(Z, Eq(f,g)) \). Now \( Eq(f,g) = \{ x \in X | f(x) = g(x) \} \) so \( Eq(f,g) \) is a subset of \( X \). Thus \( h : Z \to Eq(f,g) \) is an arrow from \( Z \) to \( X \). Further for \( z \in Z \) we have \( (f \circ h)(z) = f(h(z)) = g(h(z)) = (g \circ h)(z) \) since \( h(z) \in Eq(f,g) \). Thus \( h \) is an element of \( Eq(\text{Hom}(Z,f), \text{Hom}(Z,g)) \). Thus we can define \( \varphi : \text{Hom}(Z, Eq(f,g)) \to Eq(\text{Hom}(Z,f), \text{Hom}(Z,g)) \) to be the identity map, which is clearly an isomorphism.

### 3.2 Representability of Equalizers in \( CAT \)

Let \( X, Y \in \text{Ob}(CAT) \) and let \( f, g : X \to Y \) denote two functors.

**Definition 3.2.1.** The category \( E(q,f,g) \) of \( f \) and \( g \) is the category defined as follows:

1. The objects of \( E(q,f,g) \) are pairs \((x, \alpha)\) where each \( x \) is an object of the category \( X \) and \( \alpha : f(x) \cong g(x) \).

2. Morphisms \((x, \alpha) \to (y, \beta)\) are those arrows \( h : x \to y \) in \( X \) such that the diagram below commutes.

\[
\begin{array}{ccc}
  f(x) & \xrightarrow{\alpha} & g(x) \\
  f(h) \downarrow & & \downarrow g(h) \\
  f(y) & \xrightarrow{\beta} & g(y)
\end{array}
\]  

(3–4)

Observe that we have an arrow \( \psi : E(q,f,g) \to X \) that sends each object \((x, \alpha)\) to \( x \in X \), and is the inclusion on morphisms. We now define equalizers in an arbitrary 2-category.

**Definition 3.2.2.** Let \( C \) be a 2-category. We say that equalizers are representable in \( C \) if for every pair of objects \( X, Y \) in \( C \) and every pair of 1-cells \( F, G : X \to Y \) in \( C \) the 2-functor

\[
E_q(F,G) : C \to CAT,
\]

\[
Z \mapsto Eq( \text{Hom}(Z,X) \xrightarrow{\text{Hom}(F,Z)} \text{Hom}(Z,Y) )
\]

(3–5)

is 2-representable.

**Proposition 3.2.3.** Equalizers are representable in \( CAT \).
In fact for every pair of objects $X, Y$ in $C$ and every pair of 1-cells $f, g : X \to Y$ in $C$, there is a 2-functorial equivalence of categories:

$$
\varphi : \text{Hom}(Z, \mathcal{E}q(f, g)) \xrightarrow{\sim} \mathcal{E}q(\text{Hom}(Z, f), \text{Hom}(Z, g)) \quad (3-6)
$$

where,

$$
\mathcal{E}q(\text{Hom}(Z, f), \text{Hom}(Z, g)) := \mathcal{E}q[ \text{Hom}(Z, X) \xrightarrow{f_*} \text{Hom}(Z, Y) ].
$$

Proof. Let $f_*$ and $g_*$ denote $\text{Hom}(Z, f)$ and $\text{Hom}(Z, g)$ respectively. Define $\varphi : \text{Hom}(Z, \mathcal{E}q(f, g)) \to \mathcal{E}q( \text{Hom}(Z, X) \xrightarrow{f_*} \text{Hom}(Z, Y) )$ as follows:

- **On objects:** Given $F : Z \to \mathcal{E}q(f, g)$, denote the functor $F$ as follows: $F(Z) = (F_0(Z), f(F_0(Z)) \xrightarrow{g_*(F_0(Z))} g(F_0(Z)))$. Thus $F_0 : Z \to X$ and $F_1 : f \circ F_0 \xrightarrow{\sim} g \circ F_0$ are functors. Then define $\varphi(F) := \psi \circ F_0$ where $\psi : \mathcal{E}q(f, g) \to X$ is the map defined above. Then $\varphi(F)$ is in $\text{Hom}(Z, X)$. Also $f_* (\varphi(F)) = f(\psi(F_0)) \xrightarrow{\psi(F)} g(\psi(F_0)) = g_*(\varphi(F))$. Thus $(\varphi(F), F_1)$ is an object of $\mathcal{E}q( \text{Hom}(Z, X) \xrightarrow{f_*} \text{Hom}(Z, Y) )$.

- **On arrows:** Suppose $F$ and $G$ are two functors in $\text{Hom}(Z, \mathcal{E}q(f, g))$, and let $k : F \to G$ be a morphism of functors. Then $\varphi(F) := \psi \circ F_0$ and $\varphi(G) := \psi \circ G_0$ are in $\mathcal{E}q( \text{Hom}(Z, X) \xrightarrow{f_*} \text{Hom}(Z, Y) )$. By definition of $k$, if $x$ and $y$ are objects of $Z$ and $\lambda : x \to y$ is an arrow in $Z$ then the diagram:

$$
\begin{array}{ccc}
F(x) & \xrightarrow{k_\lambda} & G(x) \\
F(\lambda) \downarrow & & \downarrow G(\lambda) \\
F(y) & \xrightarrow{k_y} & G(y)
\end{array}
$$

(3-7)
commutes. So define $\varphi(K) := \psi \circ k$ (recall that $\psi : \mathcal{E}q(f, g) \to X$ is the inclusion on morphisms). Thus the diagram

$$
\begin{array}{ccc}
\varphi F(x) & \xrightarrow{\varphi k_x} & \varphi G(x) \\
\downarrow \varphi F(\lambda) & & \downarrow \varphi G(\lambda) \\
\varphi F(y) & \xrightarrow{\varphi k_y} & \varphi G(y)
\end{array}
$$

(3–8)

commutes in $\text{Hom}(Z, X)$. But $\varphi(F)$ and $\varphi(G)$ are in $\mathcal{E}q(\text{Hom}(Z, X) \xrightarrow{f_*\over g_*} \text{Hom}(Z, Y))$ so given a morphism of functors in $\text{Hom}(Z, X)$ we have in the category $\mathcal{E}q(\text{Hom}(Z, X) \xrightarrow{f_*\over g_*} \text{Hom}(Z, Y))$ a commutative cube:

$$
\begin{array}{ccc}
f_* (\varphi F(x)) & \xrightarrow{f_* (\varphi k)} & f_* (\varphi G(x)) \\
\downarrow \varphi F(\lambda) & & \downarrow \varphi G(\lambda) \\
g_* (\varphi F(x)) & \xrightarrow{g_* (\varphi k)} & g_* (\varphi G(x)) \\
\downarrow \varphi F(\lambda) & & \downarrow \varphi G(\lambda) \\
g_* (\varphi F(y)) & \xrightarrow{g_* (\varphi k)} & g_* (\varphi G(y)) \\
\downarrow \varphi F(\lambda) & & \downarrow \varphi G(\lambda) \\
g_* (\varphi F(y)) & \xrightarrow{g_* (\varphi k)} & g_* (\varphi G(y))
\end{array}
$$

(3–9)

Thus $\varphi : \text{Hom}(Z, \mathcal{E}q(f, g)) \to \mathcal{E}q(\text{Hom}(Z, X) \xrightarrow{f_*\over g_*} \text{Hom}(Z, Y))$ is a well-defined morphism.

We now show that $\varphi$ is in fact an equivalence of categories.

- $\varphi$ is faithful: Consider $\text{Hom}_{\text{Hom}(Z, \mathcal{E}q(f, g))}(F, G) \to \text{Hom}_{\mathcal{E}q(f_*\over g_*)}(\varphi(F), \varphi(G))$ where $\mathcal{E}q(f_*\over g_*) := \mathcal{E}q(\text{Hom}(Z, X) \xrightarrow{f_*\over g_*} \text{Hom}(Z, Y))$. If $k : F \to G$ is in $\text{Hom}(F, G)$
then recall that $\varphi(k) := \psi \circ k$. Further recall the definition of $\psi : Eq(f,g) \to X$. It takes an object $(x, \alpha)$ to $x$ and it is the inclusion on arrows. So if $\varphi(k) = \varphi(k')$ then $\psi \circ k = \psi \circ k'$ which implies that $k = k'$ since $\psi$ is the inclusion on arrows. So $\varphi$ is faithful.

- $\varphi$ is full: Now suppose $k : \varphi(F) \to \varphi(G)$ is an arrow. Then $k_x : (\psi \circ F)(x) \to (\psi \circ G)(x)$ is an arrow for each $x$. But $\psi$ is a functor from $Eq(f,g)$ to $X$ whence $(\psi \circ F_0)(x) \to (\psi \circ G_0)(x) = \psi(F_0(x) \xrightarrow{\alpha} G_0(x))$ for some $\alpha_x$. Let $\alpha : F \to G$ be defined by $\alpha = \alpha_x$ for each $x$ in $Ob(Z)$. Then $\varphi(\alpha) = \psi \circ \alpha_x = k_x$ so $\varphi(\alpha) = k$. Thus $\varphi$ is full.

- $\varphi$ is essentially surjective: Let $(G, \alpha)$ be an object of $Eq(f_*, g_*)$. Then $G$ is an arrow from $Z$ to $X$ and $\alpha$ is a morphism of functors i.e. $\alpha : f_*(G) \xrightarrow{\sim} g_*(G)$ i.e. for $k : G \to G'$ in $Hom(Z, X)$ have a commutative diagram

$$
\begin{array}{ccc}
f_*(G) & \xrightarrow{\alpha} & g_*(G) \\
\downarrow f_*(k) & & \downarrow g_*(k) \\
f_*(G') & \xrightarrow{\alpha\circ g_0} & g_*(G')
\end{array}
$$

Consider $F : Z \to Eq(f,g)$ defined as follows:

1. On objects: for $x$ in $Ob(Z)$, $F(x) := (G(x), \alpha_{G,x})$,

2. On arrows: for $\lambda : x \to y$ in $Z$, $F(\lambda) := G(\lambda)$.

Then for any $x$ in $Ob(Z)$, $\varphi(F(x)) = \psi \circ F_0(x) = \psi(F_0(x)) = \psi((G_0(x), \alpha_{G,x})) = G(x)$. Further $\varphi(F) = (G, \beta)$ where $\beta$ is an isomorphism of functors $f_* \xrightarrow{\sim} g_*$ as is $\alpha$ so $\alpha \xrightarrow{\sim} \beta$. Thus $\varphi(F) \xrightarrow{\sim} (G, \alpha)$ whence $\varphi$ is essentially surjective.

Thus $\varphi$ is an equivalence of categories.

### 3.3 Coequalizers

The concept of coequalizer is a useful way of formulating quotients.

Let $\mathcal{C}$ be a category.
Definition 3.3.1. Given arrows $f, g : X \to Y$, we define their coequalizer to be the object $K$ which represents the functor

$$Z \mapsto \text{Eq}(\text{Hom}(X, Z) \frac{\text{Hom}(f, Z)}{\text{Hom}(g, Z)} \text{Hom}(Y, Z)).$$

(3–11)

If $K$ exists then there is a functorial isomorphism

$$\text{Hom}(Z, K) \simeq \text{Eq}(\text{Hom}(f, Z), \text{Hom}(g, Z)).$$

(3–12)

Thus there is a canonical map $p : Y \to K$ such that $p \circ f = p \circ g$, and any morphism $h : Y \to Z$ such that $h \circ f = h \circ g$ factors through a unique morphism $K \to Z$.

Definition 3.3.2. Given two maps $f, g : X \to Y$ of sets, let $\text{Coeq}(f, g)$ of $f$ and $g$ be the projection $p : Y \to Y/E$ on the quotient of $Y$ by the smallest equivalence relation $E \subset Y \times Y$ which contains all pairs $(f(x), g(x))$ for $x \in X$.

Theorem 3.3.3. Coequalizers are representable in the category $\text{SET}$ by the object $\text{Coeq}(f, g)$.

The proof is similar to that given for equalizers above, and we do not include it here.

If $F : G \times A \to A$ is the action of a group $G$ on a set $A$ then the quotient by the action is just the coequalizer of the arrows $G \times A \xrightarrow{p_2} A$. It is this formulation of quotients which lends itself to generalization i.e. we use this approach to define a quotient when a group category acts on a category.

3.4 Representability of Coequalizers in $\text{CAT}$

We have the definition of an equalizer in the 2-category $\text{CAT}$. We will now use this definition to establish that coequalizers exist in $\text{CAT}$.

First we define coequalizers in an arbitrary 2-category.

Definition 3.4.1. Let $\mathcal{C}$ be a 2-category. We say coequalizers are representable in $\mathcal{C}$ if for every pair of objects $X, Y$ of $\mathcal{C}$ and every pair of 1-cells $F, G : X \to Y$ in $\mathcal{C}$ the 2-functor

$$C_{eq}(F, G) : \mathcal{C} \to \text{CAT},$$

is representable.
is 2-representable. This definition is dual to the one given for the equalizer $\text{Eq}(F,G)$.

**Definition 3.4.2.** Given two functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$, there is an induced functor $H : \mathcal{C} \to \mathcal{D} \times \mathcal{D}$, $x \mapsto (F(x), G(x))$. We say that $F$ and $G$ are a small pair of functors if the fibers of $H$ are small categories i.e. for all $(a, b) \in \text{Ob} (\mathcal{D} \times \mathcal{D})$, $H^{-1}(a, b)$ is a set.

We will devote the rest of this section to proving that the coequalizer of a small pair of functors is representable in $\text{CAT}$. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $F$ and $G$ denote a small pair of functors from $\mathcal{C}$ to $\mathcal{D}$. Define a category $\text{Coeq}(F,G)$ as follows. The objects of $\text{Coeq}(F,G)$ will be objects of $\mathcal{D}$. To define the arrows in $\text{Coeq}(F,G)$ we will first define a set of “prearrows”. We will then define the arrows in $\text{Coeq}(F,G)$ to be the quotient of this set of prearrows by a certain equivalence relation.

We define the set of prearrows $\text{PreAr}(F,G)$ to be the disjoint union of the following two sets:

1. $\text{Ar}(\mathcal{D})$

2. The set $S$ consisting of, for every $x \in \mathcal{C}$, a formal isomorphism $F(x) \xrightarrow{\sim} G(x)$ and its inverse: that is, an arrow of the form $G(x) \xrightarrow{\sim} F(x)$. The source and target of an element of $\text{Ar}(\mathcal{D})$ is just its usual source and target in $\mathcal{D}$.

A member of $S$ of the form $F(x) \xrightarrow{\sim}_{\alpha} G(x)$ has source $F(x) \in \text{Ob}(\mathcal{D})$ and target $G(x) \in \text{Ob}(\mathcal{D})$, and an element of the form $G(x) \xrightarrow{\sim}_{\alpha^{-1}} F(x)$ has source $G(x) \in \text{Ob}(\mathcal{D})$ and target $F(x) \in \text{Ob}(\mathcal{D})$. So within $S$, after one fixes a source $s$ and a target $t$, the arrows between $s$ and $t$ form a set.

**Definition 3.4.3.** A prearrow is a finite sequence $(s_1, s_2, \ldots, s_n)$ where $s_i \in \text{Ar}(\mathcal{D}) \cup S$ and $\text{target}(s_i)=\text{source}(s_{i-1})$ for all $i$. Let $\text{PreAr}(F,G)$ be the set of prearrows on $\text{Ar}(\mathcal{D}) \cup S$.

Next we introduce a binary operation on $\text{PreAr}(F,G)$. Given two prearrows $(r_1, r_2, \ldots, r_m)$ and $(s_1, s_2, \ldots, s_n)$, if the target $(s_1)=\text{source}(r_m)$ then their product is
defined as:

\[
(r_1, r_2, \cdots, r_m) \ast (s_1, s_2, \cdots, s_n) = (r_1, r_2, \cdots, r_m, s_1, s_2, \cdots, s_n)
\]  \hspace{1cm} (3-14)

Observe that by this definition \((s_1, s_2, \cdots, s_n) = s_1 \ast s_2 \ast \cdots \ast s_n\). We will henceforth refer to the above product as \textit{composition}. Then there is a smallest equivalence relation \(\sim\) on \(PreAr(F, G)\) such that

1. For any \(s \in S\), if \(x = \text{source}(s)\) and \(y = \text{target}(s)\), then

\[
s \ast \text{id}_x \sim \text{id}_y \ast s \sim s.
\]  \hspace{1cm} (3-15)

2. For any composable \(\alpha, \beta \in Ar(D)\), \((\alpha \ast \beta, \alpha \circ \beta)\) is in the relation i.e. \(\alpha \ast \beta \sim \alpha \circ \beta\).

3. for any \(a, b \in \text{Ob}(C)\) and \(t : a \to b\) in \(Ar(C)\)

\[
F(a) \xrightarrow{F(t)} F(b) \sim G(b) \sim F(a) \xrightarrow{G(t)} G(b).
\]  \hspace{1cm} (3-16)

4. If \(\alpha, \beta \in Ar(D) \cup S\) and \(\text{source}(\alpha) = \text{target}(\beta)\) then \((s_1, \alpha) \ast (\beta, s_2) \sim (s_1, (\alpha \circ \beta), s_2)\) for \(s_i \in PreAr(F, G)\). Further, if \(\alpha_1, \alpha_2\) and \(\beta_1, \beta_2\) are elements of \(PreAr(F, G)\) such that \(\alpha_1 \sim \beta_1\) and \(\alpha_2 \sim \beta_2\) and suppose \(\alpha_2 \circ \alpha_1\) and \(\beta_2 \circ \beta_1\) is defined, then \(\alpha_2 \circ \alpha_1 \sim \beta_2 \circ \beta_1\).

Define \(\text{Coeq}(F, G)\) as follows: \(\text{Ob}(\text{Coeq}(F, G)) = \text{Ob}(D)\) and \(\text{Ar}(\text{Coeq}(F, G)) = PreAr(F, G)/\sim\).

**Proposition 3.4.4.** \(\text{Coeq}(F, G)\) satisfies the axioms of a category.

**Proof.** Let \([f] \in \text{Hom}_{\text{Coeq}(F,G)}(A, B)\) and \([g] \in \text{Hom}_{\text{Coeq}(F,G)}(B, C)\). Define \([g] \circ [f] := [g \circ f]\).

To see that this is well defined, suppose \(f_1, f_2 \in [f]\) and \(g_1, g_2 \in [g]\). Then \(f_1 \sim f_2\) and \(g_1 \sim g_2\), which implies that \(g_1 \circ f_1 \sim g_2 \circ f_2\) whence \([g_1 \circ f_1] = [g_2 \circ f_2]\). So we have a composition law. Similarly, the associative law holds since it holds on the level of prearrows. For suppose \([f] \in \text{Hom}_{\text{Coeq}(F,G)}(A, B)\), \([g] \in \text{Hom}_{\text{Coeq}(F,G)}(B, C)\), and \([h] \in \text{Hom}_{\text{Coeq}(F,G)}(C, D)\). Then \([h] \circ ([g] \circ [f]) = [h] \circ ([g \circ f]) = [h \circ (g \circ f)] = [(h \circ g) \circ f] = ([h \circ g]) \circ [f] = ([h] \circ [g]) \circ [f]\) where the second equality follows since \(h \circ (g \circ f)\) is in
\( \text{PreAr}(F, G) \). Thus we have associativity. Finally, consider \( 1_A \in \text{Hom}_{\text{Coeq}(F, G)}(A, A) \).

If \( \alpha \sim 1_A \) in \( \text{Coeq}(F, G) \) then \( 1_A = 1_A \circ \alpha = \alpha \) which implies that \( [1_A] = 1_A \) in \( \text{Coeq}(F, G) \). So for \( [f] \in \text{Hom}_{\text{Coeq}(F, G)}(A, B) \) and \( [g] \in \text{Hom}_{\text{Coeq}(F, G)}(B, C) \) we have that \( [1_B] \circ [f] = [1_B \circ f] = [f] \) and \( [g] \circ [1_B] = [g \circ 1_B] = [g] \) so the identity axiom is true.

By construction of \( \text{Coeq}(F, G) \) we have an arrow \( \mathcal{D} \xrightarrow{\lambda} \text{Coeq}(F, G) \) which is the identity on objects and sends arrows in \( \mathcal{D} \) to their corresponding class in \( \text{Coeq}(F, G) \).

**Theorem 3.4.5.** The coequalizer of a small pair of functors is representable in \( \text{CAT} \). In fact the diagram

\[
\text{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\lambda} \text{Coeq}(F, G)
\]

gives an equivalence

\[
\text{Hom}(\text{Coeq}(F, G), \mathcal{E}) \xrightarrow{\sim} \text{Eq}(\text{Hom}(\mathcal{D}, \mathcal{E}) \xrightarrow{F^*} \text{Hom}(\mathcal{C}, \mathcal{E})) \tag{3-17}
\]

i.e. the functor category \( \text{Hom}(\text{Coeq}(F, G), \mathcal{E}) \) is equivalent to a category \( \mathcal{B} \) whose objects consist of functors \( f : \mathcal{D} \to \mathcal{E} \) together with isomorphisms \( f \circ F \xrightarrow{\sim} f \circ G \) and whose arrows are morphisms \( f \to g \) of functors such that for each object \( x \) in the category \( \mathcal{C} \) we have a commutative diagram:

\[
f F(x) \xrightarrow{\sim} f G(x) \\
\downarrow \quad \downarrow \\
g F(x) \xrightarrow{\sim} g G(x)
\tag{3-18}
\]

**Proof.** Define \( \varphi : \text{Hom}(\text{Coeq}(F, G), \mathcal{E}) \to \mathcal{B} \) as follows: if \( J : \text{Coeq}(F, G) \to \mathcal{E} \) is a functor, define \( \varphi(J) = J \circ \lambda \). If \( f : \mathcal{D} \to \mathcal{E} \) denotes this composition (i.e if \( f := J \circ \lambda \)) then an explicit description of \( f \) on objects and arrows is as follows:

- **Objects** \( (\mathcal{D}) \): for \( x \in \text{Ob}(\mathcal{D}) \), \( f(x) = J(x) \).
- **Arrows** \( (\mathcal{D}) \): if \( \alpha : x \to y \) is an arrow in \( \mathcal{D} \), send this arrow to \( J(x) \xrightarrow{f(\alpha)} J(y) \).

We show that \( \varphi \) is fully faithful. Let \( \alpha, \beta : J \to K \) where \( J, K : \text{Coeq}(F, G) \to \mathcal{E} \) are functors. Let \( f = \varphi(J) \) and \( g = \varphi(K) \). Then for \( \alpha \in \text{Hom}_{\text{Hom}(\text{Coeq}(F, G), \mathcal{E})}(J, K) \), its image in \( \text{Hom}_B(f, g) \) is a map \( \gamma : f \to g \) which is a morphism of functors. So by definition, \( \gamma \) is determined when we give maps \( \gamma_x : f(x) \to g(x) \) for each \( x \in \text{Ob}(\mathcal{D}) \), compatible with the morphisms in \( \mathcal{D} \). But \( \text{Ob}(\mathcal{D}) = \text{Ob}(\text{Coeq}(F, G)) \) and since \( f(x) = J(x) \) and \( g(x) = K(x) \)
for all \( x \in \text{Ob}(\mathcal{D}) \), \( \gamma_x \) is defined to be \( \alpha_x \). Thus if \( \alpha, \beta \in \text{Hom}_{\text{Hom}(\text{Coeq}(F,G),\mathcal{E})}(J,K) \) have the same image \( \gamma \) in \( \text{Hom}_B(f,g) \) then \( \alpha_x = \beta_x \) for all \( x \in \text{Ob}(\text{Coeq}(F,G)) \) whence \( \alpha = \beta \).

Now let \( \gamma : f \to g \) be a morphism in \( B \) i.e. \( \gamma \in \text{Hom}_B(f,g) \). Define \( \alpha : J \to K \) by \( \alpha_x = \gamma_x \). This makes sense since \( \text{Ob}(\mathcal{D}) = \text{Ob}(\text{Coeq}(F,G)) \). For \( x, y \in \text{Ob}(\text{Coeq}(F,G)) \) suppose \([u] : x \to y\) is an arrow in \( \text{Coeq}(F,G) \). Let \( u \) represent the class of \([u]\). Then \( u \) is a composition of arrows in \( \mathcal{D} \) and isomorphisms \( F(x) \xrightarrow{\sim} G(x) \) for \( x \in \text{Ob}(\mathcal{C}) \).

Case 1: Suppose \( u \in \text{Ar}(\mathcal{D}) \). Then since \( \gamma \) is a morphism of functors, the diagram

\[
\begin{array}{ccc}
  f(x) & \xrightarrow{\gamma} & g(x) \\
  f(u) & \downarrow & g(u) \\
  f(y) & \xrightarrow{\gamma} & g(y)
\end{array}
\]

commutes, whence the diagram

\[
\begin{array}{ccc}
  f(x) = J(x) & \longrightarrow & K(x) = g(x) \\
  f(u) = J([u]) & \downarrow & K([u]) = g(u) \\
  f(y) = J(y) & \longrightarrow & K(y) = g(y)
\end{array}
\]

commutes.

Case 2: Suppose \( u : x \to y \) where \( x = F(a) \) and \( y = G(a) \) for \( a \in \mathcal{C} \). Then by definition of \( \gamma \) the diagram

\[
\begin{array}{ccc}
  fF(a) & \xrightarrow{\sim} & fG(a) \\
  \gamma_{F(a)} & \downarrow & \gamma_{G(a)} \\
  gF(a) & \xrightarrow{\sim} & gG(a)
\end{array}
\]

commutes so

\[
\begin{array}{ccc}
  JF(a) & \xrightarrow{\sim} & KF(a) \\
  \alpha_{F(a)} = \gamma_{F(a)} & \downarrow & \alpha_{G(a)} = \gamma_{G(a)} \\
  JG(a) & \xrightarrow{\sim} & KG(a)
\end{array}
\]

commutes since the vertical arrows in the above two diagrams are the same by definition of the map \( \varphi \).
Since for any $x, y \in Ob(Coeq(F, G))$, $u : x \to y$ is a composition of the types of arrows in Cases 1 and 2, we get that the diagram

$$
\begin{align*}
J(x) & \xrightarrow{J([u])} J(y) \\
\alpha_x & \downarrow \downarrow \alpha_y \\
K(x) & \xrightarrow{K([u])} K(y)
\end{align*}
$$

(3–23)

commutes for all $[u] \in Ar(Coeq(F, G))$. Thus we have a bijection between $Hom_B(f, g)$ and $Hom_{Hom(Coeq(F, G), E)}(J, K)$: that is, $\varphi$ is fully faithful.

We now show that $\varphi$ is essentially surjective. Let $f : D \to E$ with isomorphisms $f \circ F \sim f \circ G$ be an object of $B$. We wish to define $J : Coeq(F, G) \to E$ such that $\varphi(J) \sim f$. Define $J : Coeq(F, G) \to E$ by $J(x) = f(x)$ on objects. Let $[u]$ be an arrow in $Coeq(F, G)$. Let $u$ represent the class of $[u]$. Then $u$ is just a composition of arrows in $D$ and formal isomorphisms. We define $J$ on morphisms by sending arrows $x \xrightarrow{i} y$ in $D$ to $f(x)(i) \xrightarrow{f} f(y)$, and sending formal isomorphisms $F(a) \sim G(a)$ to $fF(a) \sim fG(a)$. To see that this is well defined, consider for any arrow $t : x \to y$ in $Ar(C)$ the arrows $u_1$ and $u_2$ defined by:

$$
u_1 : F(x) \xrightarrow{\sim} G(x) \xrightarrow{G(t)} G(y)\tag{3–24}
$$

and

$$
u_2 : F(x) \xrightarrow{F(t)} F(y) \xrightarrow{\sim} G(y)\tag{3–25}
$$

Then $u_1 \sim u_2$. Then we have

$$
J(u_1) : fF(x) \xrightarrow{\sim} fG(x) \xrightarrow{fG(t)} fG(y)\tag{3–26}
$$

and

$$
J(u_2) : fF(x) \xrightarrow{fF(t)} fF(y) \xrightarrow{\sim} fG(y)\tag{3–27}
$$

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are the same arrows in $\mathcal{D}$ because in $Coeq(F, G)$ the diagram

\[
\begin{align*}
F(x) & \xrightarrow{\sim} G(x) \\
F(t) & \downarrow \quad \quad \quad \downarrow G(t) \\
F(y) & \xrightarrow{\sim} G(y)
\end{align*}
\]

commutes whence the diagram

\[
\begin{align*}
fF(x) & \xrightarrow{\sim} fG(x) \\
fF(t) & \downarrow \quad \quad \quad \downarrow fG(t) \\
fF(y) & \xrightarrow{\sim} fG(y)
\end{align*}
\]

also commutes. Thus $J(u_1) = J(u_2)$. Since $u$ is a composition of arrows of the above type get that $F(u)$ is well defined. We claim $\varphi(J) \xrightarrow{\sim} f$. Clearly $\varphi(J) = f$ on objects. If $u : x \to y$ is in $Ar(\mathcal{D})$ then $\varphi(J)$ sends this arrow to $J(x) \xrightarrow{J(u)} J(y)$ which is isomorphic to $f(x) \xrightarrow{f(u)} f(y)$. Thus $\varphi$ is essentially surjective.
CHAPTER 4
GROUP CATEGORIES

4.1 Inner Equivalences of Group Categories

We define the notion of inner equivalences for group categories. We mention that L. Breen [6] has also given a similar description of this notion.

Let $\mathcal{C}$ be a group category. Let $Y, Z \in \text{Ob}(\mathcal{C})$ such that $ZY \simeq I \simeq YZ$. Define $F_{YZ} : \mathcal{C} \to \mathcal{C}$ as follows:

- On objects: for $A \in \text{Ob}(\mathcal{C})$ let $F_{YZ}(A) := (ZA)Y$.
- On arrows: given an arrow $f : A \to B$ in $\mathcal{C}$, functoriality of multiplication gives an arrow $F(f) : F(A) \to F(B)$ such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\text{\scriptsize $F_{YZ}$} & & \text{\scriptsize $F_{YZ}$} \\
\downarrow & & \downarrow \\
F(A) & \xrightarrow{F_{YZ}(f)} & F(B)
\end{array}
\]  

commutes. So $F_{YZ}$ is a functor. Further, for $A, B \in \text{Ob}(\mathcal{C})$ we have:

\[
F_{YZ}(AB) = (Z(AB))Y
\]

So $F_{YZ}$ respects the group structure of $\mathcal{C}$. For $A, B, C$ in $\text{Ob}(\mathcal{C})$ we have:
The horizontal arrows are the associative and identity laws.

- Arrows $i_1$ and $i_2$ are the axioms for a monoidal category.
- The source of the arrows $i_1$ and $i_2$ is the definition of $F_{YZ}[A(BC)]$.
- The triangle commutes because of the associative axiom.
- The second and fifth square commute because of the general associative law.
- The first and third squares are obtained by repeated application of the functoriality of the associative isomorphism, where each application gives a commutative square.
- The source of arrow $i_4$ (and the target of $i_3$) is the definition of $F(AB)F(C)$.
- The source of arrow $j_4$ (and the target of $j_3$) is the definition of $F(A)(F(BC))$. 
Thus we have a diagram

\[
\begin{array}{c}
F(A(BC)) \xrightarrow{a} F(A)F(BC) \xrightarrow{b} F(A)(F(B)F(C)) \\
\downarrow{c} & \downarrow{f} \\
F((AB)C) & \xrightarrow{d} F(AB)F(C) \xrightarrow{e} (F(A)F(B))F(C)
\end{array}
\]

Since

\[
\begin{align*}
a &= j_3 \circ j_2 \circ j_1 \circ c, \\
b &= j_6 \circ j_5 \circ j_4, \\
c &= F(\text{assoc.}), \\
d &= i_3 \circ i_2 \circ i_1, \\
e &= i_6 \circ i_5 \circ i_4, \\
f &= \text{associative law},
\end{align*}
\]

the above commutative diagram shows that \( b \circ a \simeq f \circ e \circ d \circ c \).

Next we define an isomorphism \( F_{YZ} \circ F_{ZY} \simeq Id_c \):

\[
\begin{align*}
(F_{YZ} \circ F_{ZY})(A) & \simeq F_{YZ}(F_{ZY}(A)) \\
& \simeq F_{YZ}((YA)Z) \\
& \simeq (Z((YA)Z))Y \\
& \simeq ((ZY)A(ZY)) \\
& \simeq (IAI) \\
& \simeq A
\end{align*}
\]

where the fourth arrow above is just the associativity isomorphism. Similarly we define an isomorphism \( F_{ZY} \circ F_{YZ} \simeq Id_c \). Thus \( F_{YZ} \) gives an equivalence of group categories.

Let \( Inn(C) \) denote the collection of equivalences of \( C \) which are isomorphic to ones of the form \( F_{YZ} \). Define arrows between them to be morphisms of functors, and let the composition law be the usual vertical composition of natural transformations, which is again a morphism in \( Inn(C) \). Then \( Inn(C) \) is a groupoid. For \( G, H \in \text{Ob}(Inn(C)) \), if
\( G \simeq F_{YZ} \) and \( H \simeq F_{Y'Z'} \) then we define an isomorphism between \( H \circ G \) and some \( F_{IJ} \) as follows:

\[
(H \circ G)(A) = H(G(A)) \\
\simeq H(F_{YZ}(A)) \\
= H((ZA)Y) \\
\simeq F_{Y'Z'}((ZA)Y) \\
= (Z'(ZA)Y))Y' \\
\simeq ((Z'Z)A(YY')) \\
\simeq F_{(YY')(Z'Z)}(A)
\]

where the second last arrow is just the associative law. Thus the composition of functors and horizontal composition of natural transformations defines a tensor functor:

\[
\otimes : Inn(C) \times Inn(C) \to Inn(C).
\]

We let \( I = I_{Inn(C)} \) and an inverse for an object \( G \xrightarrow{\sim} F_{YZ} \) is obtained by taking the quasi-inverse \( F_{ZY} \). Thus \( Inn(C) \) is a categorical group. We call \( Inn(C) \) the group category of inner equivalences of \( C \). Note that it is a full subcategory of \( Eq(C) \).

### 4.2 Action of a Group Category on a Category

If \( F : G \times A \to A \) is the action of a group \( G \) on a set \( A \) then the quotient by the action is just the coequalizer of the arrows \( G \times A \rightrightarrows A \). We use this approach to define a quotient when a group category acts on a category.

We will use notation similar to that of [22] but our definitions are more general.

**Definition 4.2.1.** Let \( \mathcal{G} \) be a group category and \( \mathcal{C} \) be a category. Let \( m \) denote multiplication in \( \mathcal{G} \) and \( I \) its unit object. An action of \( \mathcal{G} \) on \( \mathcal{C} \) is a triple \((F, \alpha, \beta)\) where
$F : G \times C \to C$ is a bifunctor that sits in the following two pasting diagrams:

$$
\begin{align*}
G \times G \times C & \xrightarrow{m \times 1_{dC}} G \times C \\
\downarrow{F} & & \downarrow{F} \\
G \times C & \xrightarrow{\alpha} C
\end{align*}
(4-5)
$$

$$
\begin{align*}
G \times C & \xrightarrow{F} C \\
\downarrow{m \times 1_{dC}} & & \downarrow{1_{dC}} \\
G \times C & \xrightarrow{\beta} C
\end{align*}
(4-6)
$$

where $\alpha : F \circ (1_{dG} \times F) \Rightarrow F \circ (m \times 1_{dC})$ and $\beta : F \circ (I \times 1_{dC}) \Rightarrow 1_{dC}$ are 2-isomorphisms.

Further $\alpha$ and $\beta$ must be compatible with associativity in $G$: that is, they sit in the pasting diagrams shown below.

$$
\begin{align*}
G \times G \times G \times C & \xrightarrow{1_{d} \times 1_{d} \times F} G \times G \times C \\
\downarrow{m \times 1_{d} \times 1_{d}} & & \downarrow{1_{d} \times 1_{d} \times m} \\
G \times G \times C & \xrightarrow{\alpha} G \times C \\
\downarrow{1_{d} \times F} & & \downarrow{1_{d} \times \alpha} \\
G \times C & \xrightarrow{\alpha} C \\
\downarrow{F} & & \downarrow{F} \\
C & \xrightarrow{\alpha} C
\end{align*}
(4-7)
$$
Note that “asc” above refers to the associative law. The 2-arrow $\gamma$ requires explanation.

Note that the existence of the 2-isomorphisms $\alpha$ and $\beta$ means that we have isomorphisms in $\mathcal{C}$, natural in $(g, h, x)$,

$$\alpha : g \cdot (h \cdot x) \sim (gh) \cdot x \quad (4-9)$$

and

$$\beta : 1 \cdot x \sim x. \quad (4-10)$$

In particular there exists an isomorphism in $\mathcal{C}$ natural in $g$, 1, and $a$, obtained from the following composition

$$\begin{align*}
(g \cdot 1 \cdot a) & \overset{\sim}{\alpha^{-1}} (g \cdot (1 \cdot a)) \\
 & \overset{\sim}{\beta} g \cdot a \\
 & \overset{\sim}{\alpha} 1 \cdot (g \cdot a) \\
 & \overset{\sim}{\beta^{-1}} (1 \cdot g) \cdot a
\end{align*} \quad (4-11)$$

which yields a 2-isomorphism $\gamma$ that sits in the diagram.
Definition 4.2.2. Let $\mathcal{G}$ be a group category acting via the maps $(F, \alpha, \beta)$ on a category $\mathcal{C}$. We define the quotient of $\mathcal{C}$ by $\mathcal{G}$ to be the category that represents the coequalizer of the diagram $\mathcal{G} \times \mathcal{C} \xrightarrow{p_2} \mathcal{C}$ in the 2-category $\text{CAT}$. Denote this quotient by $\mathcal{C}/\mathcal{G}$.

Definition 4.2.3. Let $\mathcal{G}$ and $\mathcal{H}$ be group categories. Let $\text{Eq}(\mathcal{H}, \mathcal{G})$ denote the group category of equivalences of group categories from $\mathcal{H}$ to $\mathcal{G}$. Recall that $\text{Inn}(\mathcal{G})$ is the group category of inner equivalences of $\mathcal{G}$. $\text{Inn}(\mathcal{G})$ acts from the right on $\text{Eq}(\mathcal{H}, \mathcal{G})$ by composition of morphisms: that is, given an equivalence $u : \mathcal{H} \to \mathcal{G}$ and an inner equivalence $F$, have $F \cdot u = u \circ F$. We define a group category

$$\text{Out}(\mathcal{H}, \mathcal{G}) := \text{Eq}(\mathcal{H}, \mathcal{G})/\text{Inn}(\mathcal{G}).$$
5.1 Action of a Group Stack on a Stack

Definition 5.1.1. Let \( \mathcal{G} \) be a gr-stack on \( X \) and \( \mathcal{C} \) be a stack on \( X \). An action of \( \mathcal{G} \) on \( \mathcal{C} \) is a triple \((F, \alpha, \beta)\) where \( F : \mathcal{G} \times \mathcal{C} \rightarrow \mathcal{C} \) is a morphism of stacks that sits in the following two pasting diagrams in the 2-category \( \text{Stack}_X \) of stacks on \( X \):

\[
\begin{align*}
\mathcal{G} \times \mathcal{G} \times \mathcal{C} & \xrightarrow{m \times \text{Id}_C} \mathcal{G} \times \mathcal{C} \\
\text{Id}_G \times F & \xrightarrow{\alpha} F \\
\mathcal{G} \times \mathcal{C} & \xrightarrow{F} \mathcal{C} \\
\text{Id} \times \text{Id}_C & \xrightarrow{\beta} \text{Id}_C
\end{align*}
\]

where \( m \) is the monoidal functor associated with the gr-stack \( \mathcal{G} \), \( I \) its identity object, and \( \alpha : F \circ (\text{Id}_G \times F) \Rightarrow F \circ (m \times \text{Id}_C) \) and \( \beta : F \circ (I \times \text{Id}_C) \Rightarrow \text{Id}_C \) are 2-isomorphisms.
Further $\alpha$ and $\beta$ must be compatible with associativity in $G$: that is, they sit in the pasting diagrams shown, where $I$ refers to the identity object in $G$. The 2-arrow $\gamma$ exists for the same reason as in the case for group categories.

5.2 Coequalizers in STACKS

Let STACKS denote the 2-category of stacks on $X$.

**Definition 5.2.1.** 1. Given two morphisms of fibered categories $\xymatrix{ C \ar[r]^F & D \ar[l]_G }$, there is an induced morphism $H : C \to D \times D$ where for each open $U \subset X$ and $x \in \text{Ob}(C_U)$, $x \mapsto (F_U(x), G_U(x))$. We say that $F$ and $G$ are a small pair of morphisms of fibered categories if the fibers of $H_U$ are small categories i.e. for all $(a, b) \in \text{Ob}(D_U \times D_U)$, $H_U^{-1}(a, b)$ is a set.

2. If $\mathcal{C}$ and $\mathcal{D}$ are prestacks (resp. stacks), and $F$ and $G$ are morphisms of prestacks (resp. stacks), we say that $F$ and $G$ are a small pair of morphisms of prestacks (resp. stacks) if they form a small pair of morphisms of the underlying fibered categories.
Our goal is to prove that the coequalizer of a small pair of morphisms of stacks is representable in STACKS. We will begin with the case of FIBER – the 2-category of fibered categories.

Let \( C \) and \( D \) be fibered categories, and let \( F \) and \( G \) denote a small pair of morphisms from \( C \) to \( D \) i.e. we have a diagram \( C \xrightarrow{F} \xrightarrow{G} D \).

Let \( \text{Coeq}(F, G) \) denote the following fibered category. To each open \( U \subset X \), \( \text{Coeq}(F, G)(U) := \text{Coeq}(F_U, G_U) \). Let \( V \hookrightarrow U \) be an inclusion of open sets in the space \( X \).

To define the restriction functors \( f^* \) for \( \text{Coeq}(F, G) \) consider the following diagram:

\[
\begin{align*}
\text{C}(U) \xrightarrow{F_U} \text{D}(U) & \xrightarrow{\varphi_U} \text{Coeq}(F, G)(U) \\
\xrightarrow{c^*} & \xrightarrow{d^*} \xrightarrow{f^*} \text{C}(V) \xrightarrow{F_V} \text{D}(V) \xrightarrow{\varphi_V} \text{Coeq}(F, G)(V)
\end{align*}
\]

(5–5)

where \( c^* \) and \( d^* \) denote the restriction functors of the stacks \( C \) and \( D \) respectively, \( \varphi_U \) and \( \varphi_V \) are the canonical maps into the coequalizer. Also, as part of the data, we have 2-arrows \( \alpha : F_V \circ c^* \Rightarrow d^* \circ F_U \) and \( \beta : G_V \circ c^* \Rightarrow d^* \circ G_U \). The dotted arrow \( f^* \) is the map we need to construct. Now, from the data of a coequalizer, there is an isomorphism

\[
\varphi_V \circ F_V \xrightarrow{\sim} \varphi_V \circ G_V.
\]

(5–6)

So

\[
\varphi_V \circ F_V \circ c^* \xrightarrow{\sim} \varphi_V \circ G_V \circ c^*. 
\]

(5–7)

But horizontal composition with \( \alpha \) yields

\[
\varphi_V \circ d^* \circ F_U \xrightarrow{\sim} \varphi_V \circ G_V \circ c^*
\]

(5–8)

and horizontal composition with \( \beta \) yields

\[
\varphi_V \circ d^* \circ F_U \xrightarrow{\sim} \varphi_V \circ d^* \circ G_U.
\]

(5–9)
Therefore there exists an isomorphism

\[ \varphi_V \circ d^* \circ F_U \cong \varphi_V \circ d^* \circ G_U \]  \hspace{1cm} (5–10)

Then by the universal property of coequalizers, there exists an arrow

\[ f^* : Coeq(F, G)(U) \to Coeq(F, G)(V) \]  \hspace{1cm} (5–11)

That \( f^* \) is unique up to unique isomorphism follows from the following result of Hakim in [16].

**Proposition 5.2.2.** Let \( \mathcal{C} \) be a 2-category and let \( \mathcal{F}(\mathcal{C}) \) denote the 2-category of 2-functors from \( \mathcal{C}^{op} \) to \( \text{CAT} \). Let \( F \) be a 2-functor which is representable in \( \mathcal{F}(\mathcal{C}) \), and suppose \((X, \xi)\) and \((X', \xi')\) are two given representations of \( F \). Then there exists a pair \((\lambda, \epsilon)\) such that \( \lambda : X \cong X' \) is an equivalence in \( \mathcal{C} \) and \( \epsilon : F(\lambda)(\xi') \cong \xi' \) is a 2-isomorphism. Further, for every other such pair \((\lambda_1, \epsilon_1)\) there exists a unique 2-isomorphism \( \alpha : \lambda \cong \lambda' \) such that \( \epsilon_1 \circ (F(\lambda)(\xi')) = \epsilon \).

Thus the natural transformation corresponding to every pair of composable inclusions of open sets is the unique 2-arrow which exists from the above result, and thus ensures that for each triple of composable inclusions of open sets the corresponding composite natural transformations coincide.

By construction of \( Coeq(F, G) \) we have an arrow \( \mathcal{D} \to \to \text{Coeq}(F, G) \) which, over each open set \( U \), is the identity on objects and sends arrows in \( \mathcal{D}_U \) to their corresponding class in \( \text{Coeq}(F, G)(U) \).

**Theorem 5.2.3.** The coequalizer of a small pair of morphisms of fibered categories is representable in \( \text{FIBER} \). In fact, the diagram

\[ \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\varphi} \text{Coeq}(F, G) \]  \hspace{1cm} (5–12)
gives an equivalence
\[ \text{Hom}(\text{Coeq}(F, G), \mathcal{E}) \xrightarrow{\sim} \text{Eq}(\text{Hom}(\mathcal{D}, \mathcal{E}) \xrightarrow{F^*} \text{Hom}(\mathcal{C}, \mathcal{E})). \] (5–13)

Proof. Define \( \lambda : \text{Hom}(\text{Coeq}(F, G), \mathcal{E}) \longrightarrow \text{Eq}(\text{Hom}(\mathcal{D}, \mathcal{E}) \xrightarrow{F^*} \text{Hom}(\mathcal{C}, \mathcal{E})) \) as follows: if \( J : \text{Coeq}(F, G) \rightarrow \mathcal{E} \) is an arrow, let \( \lambda(J) = J \circ \varphi \). By Theorem 3.14, for each open set \( U \subset X \) this yields an equivalence
\[ \text{Hom}(\text{Coeq}(F, G)(U), \mathcal{E}(U)) \xrightarrow{\sim} \text{Eq}(\text{Hom}(\mathcal{D}(U), \mathcal{E}(U)) \xrightarrow{F^*} \text{Hom}(\mathcal{C}(U), \mathcal{E}(U))) \] (5–14)
of categories. Since \( J : \text{Coeq}(F, G) \rightarrow \mathcal{E} \) is a map of fibered categories so it is compatible with the restriction functors of \( \text{Coeq}(F, G) \) and \( \mathcal{E} \). Similarly \( \lambda(J) : \mathcal{D} \rightarrow \mathcal{E} \) is a map of fibered categories so it is compatible with the restriction functors of \( \mathcal{D} \) and \( \mathcal{E} \). Thus
\[ \text{Hom}(\text{Coeq}(F, G), \mathcal{E}) \xrightarrow{\lambda} \text{Eq}(\text{Hom}(\mathcal{D}, \mathcal{E}) \xrightarrow{F^*} \text{Hom}(\mathcal{C}, \mathcal{E})) \] (5–15)
is an equivalence.

Now let \( \mathcal{C} \) and \( \mathcal{D} \) be prestacks, and let \( \xymatrix{ \mathcal{C} \ar[r]^F & \mathcal{D} } \) be a small pair of morphisms between them. We wish to show that a coequalizer of such a small pair is representable in \text{PRESTACKS}.

Let \( \text{Fib}(F, G) \) be the fibered category coequalizer of \( F \) and \( G \). Construct a prestack \( \mathcal{H} \) as follows: for each open set \( U \subset X \) and \( x, y \) in \( \text{Ob}(\text{Fib}(F, G)(U)) \) let \( \text{Ob}(\mathcal{H}(U)) = \text{Ob}(\text{Fib}(F, G)(U)) \), and \( \text{Hom}_{\mathcal{H}(U)}(x, y) \) be the sheafification of the presheaf \( \text{Hom}_{\text{Fib}(F, G)(U)}(x, y) \).

**Proposition 5.2.4.** The fibered category \( \mathcal{H} \) constructed above is a prestack.

**Proof.** We need to specify the restriction functors and check that they have the required compatibility on triple inclusions of open sets. So let \( V \subset U \) be an inclusion of open sets
in $X$. Consider the diagram

$$\begin{align*}
\text{Hom}_{\text{Fib}(F,G)(U)}(x, y) & \xrightarrow{\tau_U} \text{Hom}_{\mathcal{H}}(x, y) \\
\downarrow f^* & \\
\text{Hom}_{\text{Fib}(F,G)(V)}(x, y) & \xrightarrow{\tau_V} \text{Hom}_{\mathcal{H}}(x, y)
\end{align*}$$

(5–16)

where $\tau_U$, $\tau_V$ denote the sheafification maps and $f^*$ is the restriction functor from $\text{Fib}(F,G)(U) \to \text{Fib}(F,G)(V)$. Then $\tau_V \circ f^*$ is an arrow from the presheaf $\text{Hom}_{\text{Fib}(F,G)(U)}(x, y)$ to the sheaf $\text{Hom}_{\mathcal{H}}(x, y)$. So by the universal property of sheafification there exists a unique arrow $h^* : \text{Hom}_{\mathcal{H}(U)}(x, y) \to \text{Hom}_{\mathcal{H}(V)}(x, y)$ such that the diagram

$$\begin{align*}
\text{Hom}_{\text{Fib}(F,G)(U)}(x, y) & \xrightarrow{\tau_U} \text{Hom}_{\mathcal{H}}(x, y) \\
\downarrow f^* & \\
\text{Hom}_{\text{Fib}(F,G)(V)}(x, y) & \xrightarrow{\tau_V} \text{Hom}_{\mathcal{H}}(x, y)
\end{align*}$$

(5–17)

commutes. The uniqueness of $h^*$ gives the required compatibility for triple inclusions of open sets.

Let $\tau : \text{Fib}(F,G) \to \mathcal{H}$ denote the morphism of fibered categories that, over each open set $U$, sends the objects of $\text{Fib}(F,G)(U)$ to the objects of $\mathcal{H}(U)$, and each presheaf $\text{Hom}_{\text{Fib}(F,G)(U)}(x, y)$ to its associated sheaf. So there is a diagram:

$$\begin{align*}
\mathcal{C} & \xrightarrow{\varphi} \text{Fib}(F,G) \\
\xrightarrow{\tau} & \mathcal{H}
\end{align*}$$

(5–18)

Let $\omega : \mathcal{D} \to \mathcal{H}$ be the arrow $\tau \circ \varphi$.

**Theorem 5.2.5.** The coequalizer of a small pair of morphisms of prestacks is representable in PRESTACKS. In fact, the diagram

$$\begin{align*}
\mathcal{C} & \xrightarrow{\varphi} \mathcal{D} \\
\xrightarrow{\omega} & \mathcal{H}
\end{align*}$$

(5–19)

gives an equivalence

$$\text{Hom}(\mathcal{H}, \mathcal{J}) \sim \text{Eq}(\text{Hom}(\mathcal{D}, \mathcal{J}) \xrightarrow{F^*} \text{Hom}(\mathcal{C}, \mathcal{J}))$$

(5–20)
Proof. Define $\lambda : \text{Hom}(\mathcal{H}, \mathcal{J}) \to \text{Eq}(\text{Hom}(\mathcal{D}, \mathcal{J}) \xrightarrow{F^*} \text{Hom}(\mathcal{C}, \mathcal{J}))$ as follows: if $j : \mathcal{H} \to \mathcal{J}$ is an arrow, then let $\lambda(j) := j \circ \omega$. We first show $\lambda$ is essentially surjective.

Let $\gamma : \mathcal{D} \to \mathcal{J}$ be a given morphism of prestacks together with an isomorphism $\gamma \circ F \cong \gamma \circ G$. We need to construct an arrow $\eta : \mathcal{H} \to \mathcal{J}$, such that $\lambda(\eta) \cong \gamma$, i.e. such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow{\gamma} & \searrow{\varphi} & \downarrow{\tau} \\
\mathcal{J} & \leftarrow & \text{Fib}(F, G) & \cong & \mathcal{H} \\
\end{array}
$$

(5–21)

Now, by the universal property of coequalizers in $\text{FIBER}$ there exists an arrow $\delta : \text{Fib}(F, G) \to \mathcal{J}$, unique up to unique isomorphism, such that the diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow{\gamma} & \searrow{\varphi} & \downarrow{\tau} \\
\mathcal{J} & \leftarrow & \text{Fib}(F, G) & \rightarrow & \mathcal{H} \\
\end{array}
$$

(5–22)

commutes. For each open set $U \subset X$ there is a diagram:

$$
\begin{array}{ccc}
\text{Hom}_{\text{Fib}(F,G)(U)}(x, y) & \xrightarrow{\tau_U} & \text{Hom}_{\mathcal{H}(U)}(x, y) \\
\downarrow{\delta_U} & & \downarrow{\delta_U} \\
\text{Hom}_{\mathcal{J}(U)}(x, y) & & \\
\end{array}
$$

(5–23)

where the vertical arrow $\delta_U$ is a map from a presheaf to a sheaf. Then by the universal property of sheafification, there exists a unique arrow $\eta_U : \text{Hom}_{\mathcal{H}(U)}(x, y) \to \text{Hom}_{\mathcal{J}(U)}(x, y)$ such that the diagram:

$$
\begin{array}{ccc}
\text{Hom}_{\text{Fib}(F,G)(U)}(x, y) & \xrightarrow{\tau_U} & \text{Hom}_{\mathcal{H}(U)}(x, y) \\
\downarrow{\delta_U} & & \downarrow{\eta_U} \\
\text{Hom}_{\mathcal{J}(U)}(x, y) & & \\
\end{array}
$$

(5–24)

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commutes. Thus we obtain a morphism \( \eta : H \to J \) such that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{F} & D \\
G & \searrow & \downarrow \varphi \\
& J & \xrightarrow{\tau} H
\end{array}
\]

commutes and \( \lambda(\eta) \sim \gamma \). So \( \lambda \) is essentially surjective.

To show that \( \lambda \) is fully faithful, we need to establish a bijection between the sets \( \text{Hom}_{\text{Hom}(H, J)}(a, b) \) and \( \text{Hom}_{B}(\lambda(a), \lambda(b)) \), where \( B \) is the equalizer category \( \text{Eq}(\text{Hom}(D, J)) \xrightarrow{F^*} \text{Hom}(C, J) \). But the elements of both these sets are morphisms of prestacks. Recall that a morphism of prestacks is just a morphism of the underlying fibered categories. A morphism of prestacks consists of a family of natural transformations (each corresponding to each open subset \( U \) of \( X \)) which are compatible with the restriction functors of the source and target fibered categories. Further, natural transformations are determined solely by their action on the objects of a category. Since the objects of \( \text{Fib}(F, G)(U) \) and \( \mathcal{H}(U) \) are the same over each open set \( U \), we have that

\[
\text{Hom}_{\text{Hom}(H, J)}(a, b) \sim \text{Hom}_{\text{Hom}(\text{Fib}(F, G), J)}(a, b). \tag{5–26}
\]

Since the map

\[
\varphi : \text{Hom}_{\text{Hom}(\text{Fib}(F, G), J)}(a, b) \longrightarrow \text{Eq}(\text{Hom}(D, J) \xrightarrow{F^*} \text{Hom}(C, J)) \tag{5–27}
\]

is fully faithful by the previous theorem, it follows that \( \lambda \) is fully faithful as well.

We now come to the case of stacks. Now let \( C \) and \( D \) be stacks, and let \( C \xrightarrow{F} D \) be a small pair of morphisms between them. Form the prestack coequalizer

\[
C \xrightarrow{F} D \xrightarrow{\varphi} \text{Pre}(F, G) \quad \text{of } F \text{ and } G. \quad \text{Let } \text{St}(F, G) \text{ be the associated stack and } \alpha : \text{Pre}(F, G) \to \text{St}(F, G) \text{ be the canonical cartesian functor. Thus there is a diagram:}
\]

\[
C \xrightarrow{F} D \xrightarrow{\alpha \circ \varphi} \text{St}(F, G). \tag{5–28}
\]
Let \( \omega : \mathcal{D} \to St(F, G) \) be the arrow \( \alpha \circ \varphi \).

**Theorem 5.2.6.** The coequalizer of a small pair of morphisms of stacks is representable in \( \text{STACKS} \). In fact, the diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\xrightarrow{G} & & \xrightarrow{\omega} St(F, G)
\end{array}
\] (5–29)

gives an equivalence

\[
\text{Hom}(St(F, G), \mathcal{J}) \xrightarrow{\sim} \text{Eq}(\text{Hom}(\mathcal{D}, \mathcal{J}) \xrightarrow{F^*} \text{Hom}(\mathcal{C}, \mathcal{J})).
\] (5–30)

**Proof.** Define \( \lambda : \text{Hom}(St(F, G), \mathcal{J}) \longrightarrow \text{Eq}(\text{Hom}(\mathcal{D}, \mathcal{J}) \xrightarrow{F^*} \text{Hom}(\mathcal{C}, \mathcal{J})) \) as follows: if \( j : St(F, G) \to \mathcal{J} \) is an arrow, then let \( \lambda(j) := j \circ \omega \). We show \( \lambda \) is essentially surjective.

Let \( \mathcal{J} \) be any stack and \( \beta : \mathcal{D} \to \mathcal{J} \) be a given morphism of stacks together with an isomorphism \( \beta \circ F \xrightarrow{\sim} \beta \circ G \). Consider the diagram:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\xrightarrow{G} & \xrightarrow{\varphi} & \xrightarrow{\alpha} St(F, G) \\
& \xrightarrow{\beta} & \mathcal{J}
\end{array}
\] (5–31)

By the universal property of coequalizers in \( \text{PRESTACKS} \) there exists an arrow \( \gamma : Pre(F, G) \to \mathcal{J} \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\xrightarrow{G} & \xrightarrow{\varphi} & \xrightarrow{\alpha} St(F, G) \\
& \xrightarrow{\beta} & \mathcal{J} \\
& \xrightarrow{\gamma} &
\end{array}
\] (5–32)

commutes. Since the associated stack cartesian functor is fully faithful and universal for cartesian functors from \( \text{Pre}(F, G) \) into \( \text{STACKS} \) ([18] Lemma 3.2), there is a morphism of stacks \( \eta : St(F, G) \to \mathcal{J} \), which is unique up to unique isomorphism such that the diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\xrightarrow{G} & \xrightarrow{\varphi} & \xrightarrow{\alpha} St(F, G) \\
& \xrightarrow{\beta} & \mathcal{J} \\
& \xrightarrow{\eta} &
\end{array}
\] (5–33)
commutes. Thus \( \lambda(\eta) \sim \gamma \) whence \( \lambda \) is essentially surjective.

We now show \( \lambda \) is faithful. Let \( \varphi_1, \varphi_2 : St(F,G) \to J \) denote morphisms of stacks. Then \( \lambda(\varphi_1), \lambda(\varphi_2) : D \to J \). Suppose \( \delta, \beta \) are in \( \text{Hom}_{\text{Hom}(St(F,G),J)}(\varphi_1, \varphi_2) \) with \( \lambda(\delta) = \lambda(\beta) \), where \( \lambda(\delta) \) and \( \lambda(\beta) \) are in \( \text{Hom}_{\text{Eq}}(\lambda(\delta), \lambda(\beta)) \) (here \( \text{Eq} \) denotes the category \( \text{Eq}(\text{Hom}(D,J) \xrightarrow{F^*} \text{Hom}(C,J)) \)). We need to show that \( \delta = \beta \). Now \( \lambda(\delta) = \lambda(\beta) \) means that for each open set \( U \) and each object \( a \) in \( D(U) \), the arrow

\[
\lambda(\varphi_1(a)) \xrightarrow{\lambda(\delta)_U} \lambda(\varphi_2)(a)
\]  

(5–34)

is the same arrow as

\[
\lambda(\varphi_1(a)) \xrightarrow{\lambda(\beta)_U} \lambda(\varphi_2)(a).
\]  

(5–35)

But the definition of \( \lambda \) is \( \lambda(-) = - \circ \omega \). So the arrow

\[
(\varphi_1 \circ \omega)(a) \xrightarrow{\lambda(\delta)_U} (\varphi_2 \circ \omega)(a)
\]  

(5–36)

is the same as the arrow

\[
(\varphi_1 \circ \omega)(a) \xrightarrow{\lambda(\beta)_U} (\varphi_2 \circ \omega)(a)
\]  

i.e. the arrow

\[
\varphi_1(\omega(a)) \xrightarrow{\delta_U} \varphi_2(\omega(a))
\]  

(5–37)

is the same as the arrow

\[
\varphi_1(\omega(a)) \xrightarrow{\beta_U} \varphi_2(\omega(a)).
\]  

(5–38)

Thus \( \delta_U \) and \( \beta_U \) agree over objects in \( St(F,G)(U) \) which are in the image of \( \omega \). Now, recall that \( \omega = \alpha \circ \varphi \) where \( \varphi : D \to \text{Pre}(F,G) \) is the prestack coequalizer, and \( \alpha : \text{Pre}(F,G) \to St(F,G) \) is the canonical stackification functor. The maps \( \varphi \) and \( \alpha \) both have the property that every object in their target is locally contained in their essential image, and so this property is carried by their composition \( \omega \). In other words, every object in \( St(F,G)(U) \) is locally contained in the essential image of \( \omega \). So let \( b \) be an object in \( St(F,G)(U) \) and suppose \( (U_i) \) is an open cover of \( U \) where \( \omega(a_i) \sim b|_{U_i} \) for some object
Then over each $U_i$ there exist commutative diagrams:

\[
\begin{array}{ccc}
\varphi_1(\omega(a_i)) & \overset{\lambda(\delta)}{\longrightarrow} & \varphi_2(\omega(a_i)) \\
\varphi_1(t_i) & \downarrow & \varphi_2(t_i) \\
\varphi_1(b) & \underset{\delta}{\longrightarrow} & \varphi_2(b)
\end{array}
\]  

(5–39)

and

\[
\begin{array}{ccc}
\varphi_1(\omega(a_i)) & \overset{\lambda(\beta)}{\longrightarrow} & \varphi_2(\omega(a_i)) \\
\varphi_1(t) & \downarrow & \varphi_2(t) \\
\varphi_1(b) & \underset{\beta}{\longrightarrow} & \varphi_2(b)
\end{array}
\]  

(5–40)

By hypothesis, the two vertical and the top horizontal arrows are the same for both diagrams whence the lower horizontal arrows are the same in both diagrams i.e. $\delta$ and $\beta$ agree over each $U_i$. Since $\text{Ar}(\text{St}(F,G)(U))$ is a sheaf, $\delta = \beta$ over $U$. Since this is true for each open set $U$, get that $\delta = \beta$. Thus $\lambda$ is faithful.

Finally, we show that $\lambda$ is full. Again, let $\varphi_1, \varphi_2 : \text{St}(F,G) \to J$ denote morphisms of stacks. Then $\lambda(\varphi_1), \lambda(\varphi_2) : \mathcal{D} \to \mathcal{J}$. Suppose $\delta$ is an arrow in $\text{Hom}_{\text{Eq}}(\lambda(\delta), \lambda(\beta))$. We need to show that there exists an arrow $\overline{\delta}$ in $\text{Hom}_{\text{Hom}([\text{St}(F,G), J])}(\varphi_1, \varphi_2)$ such that $\lambda(\overline{\delta}) = \delta$. By definition, $\delta$ consists of, for each open set $U$, a natural transformation $\delta_U : \lambda(\varphi_1)_U \to \lambda(\varphi_2)_U$. Then for an object $a$ in $\mathcal{D}(U),$

\[
\delta_U : \lambda(\varphi_1)_U(a) \to \lambda(\varphi_2)_U(a)
\]  

(5–41)

so

\[
(\varphi_1 \circ \omega)(a) \overset{\delta_U}{\longrightarrow} (\varphi_2 \circ \omega)(a)
\]  

(5–42)

over $U$, so

\[
\varphi_1(\omega(a)) \overset{\delta_U}{\longrightarrow} \varphi_2(\omega(a))
\]  

(5–43)

where $\omega(a)$ is an object in $\text{St}(F,G)(U)$. Let $\overline{b}$ be any object in $\text{St}(F,G)(U)$. Since every object in $\text{St}(F,G)(U)$ is locally contained in the essential image of $\omega$ (see explanation in paragraph above), $\overline{b}$ is locally in the essential image of $\omega$ i.e. there exists an open cover.
(\text{U}_i) \text{ of } U \text{ and objects } b_i \text{ in } \mathcal{D}(\text{U}_i) \text{ with arrows } \omega(b_i)|_{\text{U}_i} \xrightarrow{\sim} \tilde{b}|_{\text{U}_i}. \text{ Then over each } \text{U}_i \text{ we have arrows}

\varphi_1(\omega(b_i)) \xrightarrow{\delta_{\text{U}_i}} \varphi_2(\omega(b_i)). \hspace{1cm} (5\text{-}44)

Further, on overlaps \text{U}_{ij},

\omega(b_i)|_{\text{U}_{ij}} \xrightarrow{\sim} (\tilde{b}|_{\text{U}_{ij}}) \xrightarrow{\sim} (\tilde{b}|_{\text{U}_{ij}}) \xrightarrow{\sim} \omega(b_j)|_{\text{U}_{ij}}. \hspace{1cm} (5\text{-}45)

Consider the diagram

\varphi_1(\omega(b_i)) \xrightarrow{\delta_{\text{U}_i}} \varphi_2(\omega(b_i)) \xrightarrow{\varphi_1(t_i)} \varphi_1(\tilde{b}) \xrightarrow{\varphi_2(t_i)} \varphi_2(\tilde{b}). \hspace{1cm} (5\text{-}46)

Let \delta_{\text{U}_i} : \varphi_1(\tilde{b}) \to \varphi_2(\tilde{b}) \text{ be the unique arrow that makes the above diagram into a commutative square. Since } \text{Ar}(\text{St}(F, G)(U)) \text{ is a sheaf, the collection of arrows } \{\delta_{\text{U}_i} : \varphi_1(\tilde{b}) \to \varphi_2(\tilde{b})\} \text{ glue over the various } \text{U}_i \text{ to give an arrow over } U:

\delta : \varphi_1(b) \to \varphi_2(b) \hspace{1cm} (5\text{-}47)

such that \lambda(\delta) = \delta. \text{ Thus } \lambda \text{ is full.}

**Definition 5.2.7.** Let \mathcal{G} be a group category acting on a category \mathcal{C} via a map F. Consider the induced functor \mathcal{H} : \mathcal{G} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C} which on objects is given by \((g, c) \mapsto (g \cdot c, c)\).

We say that F is a small action if the action \mathcal{F} : \mathcal{G} \times \mathcal{C} \to \mathcal{C} and the projection \(p_2 : \mathcal{G} \times \mathcal{C} \to \mathcal{C}\) are a small pair i.e. for any pair of objects \((c_1, c_2)\) in \(\mathcal{C} \times \mathcal{C}\), the subcategory of \(g\) in \(\text{Ob}(\mathcal{G})\) such that \(g \cdot c_1 = c_2\) is a small category.

**Proposition 5.2.8.** Let \mathcal{G} be a group category acting on a groupoid \mathcal{C}. Then the action of \mathcal{G} on \mathcal{C} is small if and only if the stabilizer \(\text{Stab}_\mathcal{G}(c)\) is a set for each object \(c\) of \(\mathcal{C}\).

**Proof.** Let \(c_1, c_2\) be any two objects of \(\mathcal{C}\) and \(g, h\) be two objects of \(\mathcal{G}\) such that \(g \cdot c_1 = c_2\) and \(h \cdot c_1 = c_2\). Then if \(k\) is any object of \(\mathcal{G}\) such that \(hk \xrightarrow{\sim} 1\) then \(kg\) is in \(\text{Stab}_\mathcal{G}(c_1)\).
Definition 5.2.9. Let $G$ be a gr-stack acting on a stack $C$ via a morphism $F$. We say that $F$ is a small action if $F$ is a locally small action i.e. for any open subset $U$ of $X$ the action of $G$ on $C$ over $U$ is small.

Proposition 5.2.10. Let $G$ be a gr-stack acting on a stack $C$. Then the action of $G$ on $C$ is small if and only if for each open set $U \subset X$, and each object $c$ of $C(U)$, the stabilizer $\text{Stab}_{G(U)}(c)$ is a set.

Proof. Since $C$ is a stack, for any open set $U \subset X$, $C(U)$ is a groupoid. The result now follows from the previous proposition.

Definition 5.2.11. Let $G$ be a gr-stack acting via the maps $(F, \alpha, \beta)$ on a stack $C$.

We define a quotient of $C$ by $G$ to be a coequalizer $(C/G, \alpha : C \to C/G)$ of the pair

\[
\begin{array}{ccc}
\mathcal{H} \times G & \xrightarrow{p_2} & G \\
\downarrow F & & \\
G & &
\end{array}
\]

when it is representable in the 2-category $\text{STACKS}$. Note that by the preceding theorem if the action is small then this coequalizer is always representable.
CHAPTER 6
THE 2-LIEN OF A 2-GERBE

6.1 Definition of a 2-Lien

We now define the notion of a 2-lien on a space $X$.

Definition 6.1.1. An inner equivalence $\varphi$ of a gr-stack $\mathcal{G}$ is an equivalence of $\mathcal{G}$ which is locally isomorphic to an inner equivalence of categorical groups. In other words $\varphi$ is an inner equivalence of $\mathcal{G}$ if for every point $x \in X$ there exists a neighborhood $U$ about $x$ such that the restriction of $\varphi$ to $U$ is an inner equivalence of categorical groups. This is equivalent to saying that there exists an open cover $(U_i)$ of the space $X$ such that $\varphi_{|U_i} : G(U_i) \rightarrow G(U_i)$ is an inner equivalence of categorical groups. Thus $\varphi$ is a cartesian functor such that each $\varphi_{|U_i}$ is an object of the group category $\text{Inn}(G(U_i))$.

Let $X$ be a topological space, and let $\mathcal{F}$ and $\mathcal{G}$ be gr-stacks on $X$. Let $Eq(\mathcal{F}, \mathcal{G})$ denote the fibered category of equivalences of gr-stacks from $\mathcal{F}$ to $\mathcal{G}$. Thus for each open $U \subset X$, $Eq(\mathcal{F}, \mathcal{G})(U)$ is defined to be $Eq(\mathcal{F}|_U, \mathcal{G}|_U)$ i.e. the category of equivalences from the restricted gr-stack $\mathcal{F}|_U$ to the restricted gr-stack $\mathcal{G}|_U$. By Corollary II.2.1.5 of [15] this is a stack on $X$.

Let $\text{Inn}(\mathcal{G})$ denote the fibered category of inner equivalences of the stack $\mathcal{G}$. Thus for each open $U \subset X$, $\text{Inn}(\mathcal{G})(U)$ is the group category $\text{Inn}(\mathcal{G}|_U)$ i.e. the group category of inner equivalences of the restricted gr-stack $\mathcal{G}|_U$. The restriction functors and corresponding natural transformations are the same as the ones defined in the stack $Eq(\mathcal{G}, \mathcal{G})$. Recall that an equivalence is inner if and only if it is so locally. Thus since the conditions defining $\text{Inn}(\mathcal{G})$ as a sub-fibered category of $Eq(\mathcal{G}, \mathcal{G})$ are local, $\text{Inn}(\mathcal{G})$ is a stack on $X$. Let $Out(\mathcal{F}, \mathcal{G})$ denote a quotient stack $Eq(\mathcal{F}, \mathcal{G})/\text{Inn}(\mathcal{G})$.

We now define a fibered 2-category $LI_2(X)$ as follows. For every open $U \subset X$, let $LI_2(X)(U)$ denote the 2-category whose objects are the gr-stacks on $U$, and whose arrows between two objects $A$ and $B$ are the global sections of the stack $Out(A, B)(U)$ i.e. $\text{Hom}_{LI_2(X)(U)}(A, B) = \Gamma(U, Out(A, B))$. The composition law is the same as the one in
the quotient stack $Out$. If $V \hookrightarrow U$ is an inclusion of open sets in $X$, then the formation of the quotient $Out(A, B)$ commutes with the restriction of $U$ to $V$ (see the discussion after Definition 5.2).

Let $Gr(X)$ denote the fibered 2-category of $gr$-stacks on $X$. Then by construction we have a morphism of fibered 2-categories

$$Gr(X) \to LI_2(X).$$

(6–1)

Now if $A$ and $B$ are any two objects of the 2-category $LI_2(X)(U)$, then the presheaf $Hom_{LI_2(X)(U)}(A, B)$ of arrows from $A$ to $B$ is identified with the stack $Out(A, B)$. Thus $LI_2(X)$ is a pre-2-stack. Thus upon applying the 2-stackification functor, we obtain the associated 2-stack $LIEN_2(X)$. Since $Gr(X)$ is already a 2-stack, composing the 2-stackification functor with the morphism (3) yields a morphism of 2-stacks

$$lien_2(X) : Gr(X) \to LIEN_2(X).$$

(6–2)

**Definition 6.1.2.**

1. The 2-stack $LIEN_2(X)$ is called the 2-stack of 2-liens on $X$.

2. We call a global section of the 2-stack $LIEN_2(X)$ a 2-lien over $X$.

Two 2-liens $L$ and $L'$ on $X$, each defined locally by families of $gr$-stacks $(G_\alpha)$ and $(G'_\beta)$, are equivalent whenever there exists a common refinement $\mathfrak{U} = (V_i)$ of the defining covers $\mathfrak{U}$ and $\mathfrak{U}'$, and a family of equivalences $\chi_i : lien(G_i) \to lien(G'_i)$ on the open sets $V_i$, which are compatible with the gluing data. To generalize this definition of a 2-lien from topological spaces to topoi, we just replace “global section” in the above definition with “a 2-cartesian section” of the 2-stack $LIEN_2(X)$, as done by Giraud for 1-gerbes in [15].

6.2 The 2-Lien of a 2-Gerbe

Let $\mathcal{G}$ be a 2-stack on $X$. By [4], over any open set $U \subset X$, any 1-arrow $f$ in $\mathcal{G}(U)$ has a quasi-inverse, defined up to a unique 2-arrow. Further, Breen shows that local inverses for 1-arrows always descend to global ones, so for any object $x \in \mathcal{G}(U)$, the presheaf in groupoids $Eq(x)$ of self-equivalences of $x$ is a $gr$-stack on $X$, after specific inverses for the
1-arrows have been chosen. In the following we will assume that this has been done, and denote by $\alpha^{-1}$ the chosen inverse of a 1-arrow $\alpha$ in $\mathcal{G}$. Recall that $Gr(X)$ denotes the 2-stack of $gr$-stacks on $X$. Then there is a Cartesian 2-functor

$$EQV : \mathcal{G} \rightarrow Gr(X)$$

$$EQV(\mathcal{G})(x) = \mathcal{E}_{qU}(x)$$

where $x \in Ob(\mathcal{G}(U))$ and $\mathcal{E}_{qU}(x)$ denotes the $gr$-stack of self-equivalences of $x$. Composing this morphism with the morphism

$$lien_2(X) : Gr(X) \rightarrow LIEN_2(X)$$

gives a morphism of 2-stacks

$$liau_2(\mathcal{G}) : \mathcal{G} \rightarrow LIEN_2(X)$$

which to every object $x \in \mathcal{G}(U)$ associates the 2-lien

$$liau_2(\mathcal{G})(x) = lien_2(\mathcal{E}_{qU}(x))$$

(called the 2-lien represented by the stack of equivalences of $x$ over $U$) and to every equivalence $\alpha : x \rightarrow y$ of $\mathcal{G}(U)$ associates the morphism

$$liau_2(\mathcal{G})(\alpha) = lien_2(Inn(\alpha))$$

in $LIEN_2(X)(U)$ where

$$Inn(\alpha) : \mathcal{E}_{qU}(x) \rightarrow \mathcal{E}_{qU}(y)$$

defined by

$$Inn(\alpha)(a) = a\alpha a^{-1}$$

is a morphism of $gr$-stacks over $X$, where

$$\alpha \circ \alpha^{-1} \simeq Id \simeq \alpha^{-1} \circ \alpha.$$
We check that composition of morphisms is well defined. Let $\alpha$ and $\beta$ denote a pair of composable 1-arrows in $G(U)$, and let $\gamma$ denote the chosen inverse of $\beta \circ \alpha$ in $G(U)$. Then
\[ \text{Inn}(\beta \circ \alpha)(a) = (\beta \circ \alpha)a\gamma. \]
Now $(\beta \circ \alpha) \circ (\alpha^{-1} \circ \beta^{-1}) \simeq \text{Id}$ so there exists a unique invertible 2-arrow $\eta : \alpha^{-1} \circ \beta^{-1} \Rightarrow \gamma$. Thus, $\text{Inn}(\beta) \circ \text{Inn}(\alpha)(a)$
\[ = \text{Inn}(\beta)(\alpha a \alpha^{-1}) \]
\[ = \beta(\alpha a \alpha^{-1}) \beta^{-1} \]
\[ = (\beta \circ \alpha) a (\alpha^{-1} \circ \beta^{-1}) \]
\[ \Rightarrow (\beta \circ \alpha)a\gamma \]
\[ = \text{Inn}(\beta \circ \alpha)(a). \]

We also need to check that different choices of inverses for $\alpha : x \to y$ where $x, y \in G(U)$, give us canonically equivalent categories $\text{Inn}(\alpha)$. So suppose $\beta, \beta' : y \to x$ are two inverses for $\alpha$. By [4], page 62, there exists a unique 2-arrow $\eta : \beta \Rightarrow \beta'$. So for any arrow $g : y \to y$, there is a unique commutative square
\[ \begin{array}{ccc}
\beta(y) & \xrightarrow{\eta} & \beta'(y) \\
\beta(g) & \downarrow & \beta'(g) \\
\beta(y) & \xrightarrow{\eta} & \beta'(y)
\end{array} \]
(6–12)
Then left composition by an equivalence $a \in \text{Eq}(x)$ of $x$ yields another unique commutative square
\[ \begin{array}{ccc}
a\beta(y) & \xrightarrow{I_{a\eta}} & a\beta'(y) \\
a\beta(g) & \downarrow & a\beta'(g) \\
a\beta(y) & \xrightarrow{I_{a\eta}} & a\beta'(y)
\end{array} \]
(6–13)
Then left composition by the 1-arrow $\alpha : x \to y$ yields unique commutative squares

\[
\begin{array}{c}
\alpha a\beta(y) \\
\downarrow
\end{array}
\xymatrix@C=1pc{
\alpha a\beta(y) \ar[r]^{I_\alpha I_a \eta} & \alpha a\beta'(y)
}
\]

for every $g : y \to y$. Thus $\nu := I_\alpha I_a \eta$ defines a unique 2-isomorphism between $\alpha a\beta$ and $\alpha a\beta'$, that is, we get a canonical equivalence of categories $\text{Inn}(\alpha)$ with choice of inverse $\beta$ and $\text{Inn}(\alpha)$ with choice of inverse of inverse $\beta'$.

By the definition of the 2-stack of 2-liens, if $m, n : x \to y$ are two equivalences of $G(U)$, we have

\[
liau_2(G)(m) \sim liau_2(G)(n)
\]

since the 2-functor $\text{lien}_2$ gives a natural isomorphism between every inner equivalence and the identity arrow.

If $m : \mathcal{F} \to \mathcal{G}$ is a morphism of 2-stacks, the composition

\[
\begin{array}{c}
\mathcal{F} \\
\xymatrix@C=1pc{
\mathcal{F} \ar[r]^m & \mathcal{G} & liau_2(\mathcal{G})
}
\end{array}
\]

factors through a morphism of morphisms of 2-stacks:

\[
liau_2(m) : liau_2(\mathcal{F}) \to liau_2(\mathcal{G}).
\]

For every $x \in \text{Ob}(\mathcal{F}(U))$, the 2-functor $m$ induces a morphism of $gr$-stacks

\[
\mu_x : \mathcal{E}_U(x) \to \mathcal{E}_U(m(x))
\]

and by definition of the map $liau_2(m)$, the morphism $liau(m(x))$ is the morphism of 2-liens represented by $\mu_x$:

\[
m_x = lien_2(\mu_x), \quad m_x : lien_2(\mathcal{E}_U(x)) \to lien_2(\mathcal{E}_U(m(x))).
\]
Definition 6.2.1. Let $\mathcal{F}$ be a 2-stack over $X$ and let $L$ be a 2-lien over $X$: that is, a global section $L : X \to \text{LIEN}(X)$ of the 2-stack of 2-liens. Let $f : \mathcal{F} \to X$ denote the projection map onto the Zariski site of $X$. An action of $L$ on $\mathcal{F}$ consists of a 2-arrow

$$ a : L \circ f \Rightarrow \text{liau}(\mathcal{F}) \quad (6–20) $$

that is,

$$
\begin{array}{ccc}
  & & L
  \\
  f & \downarrow a & \Rightarrow
  \\
  \mathcal{F} & \xrightarrow{\text{liau}(\mathcal{F})} & \text{LIEN}(X)
\end{array}
$$

and a 2-arrow $a_f$. The 2-arrow $a$ is a morphism of cartesian 2-functors: that is, a family of arrows

$$ a(x) : L(U) \to \text{lien}_2(\mathcal{E}q_U(x)) \quad (6–22) $$

where $x$ is in $\text{Ob}(\mathcal{F}(U))$, and $U \subset X$. For each inclusion $f : V \hookrightarrow U$ of open sets, the 2-arrow $a_f$ sits in a diagram:

$$
\begin{array}{ccc}
  L(U) & \xrightarrow{a(x)} & \text{lien}(\mathcal{E}q_U(x)) \\
  \downarrow & & \downarrow \\
  L(V) & \xrightarrow{a(y)} & \text{lien}(\mathcal{E}q_V(y))
\end{array}
$$

In addition $a_f$ satisfies the following conditions:

1. for a double inclusion $W^c \xleftarrow{g} V^c \xrightarrow{f} U$ of open sets, there is a 3-arrow (i.e. a modification)

$$
\alpha_{f,g} : a_g \circ a_f \Rightarrow a_{gf} .
$$

(6–24)

2. for a triple inclusion $Z^c \xleftarrow{h} W^c \xrightarrow{g} V^c \xrightarrow{f} U$ of open sets, the two methods by which the modifications $\alpha$ compare the composite 2-arrows

$$ a_{hgf} \Rightarrow a_h \circ a_{gf} \Rightarrow a_h \circ (a_g \circ a_f) $$

(6–25)
and
\[ a_{h \circ g \circ f} \Rightarrow a_{h \circ g} \circ a_f \Rightarrow (a_h \circ a_g) \circ a_f \] (6–26)
are identical.

**Definition 6.2.2.** Let \((L, a)\) be a 2-lien operating on a 2-stack \(F\) and let \(u : L' \to L\) be a morphism of 2-liens over \(X\). Then the action induced by \(a\) and \(u\) is the morphism \(b\) defined by
\[ b : L' \cdot f \to \text{liau}_2(F), \] (6–27)
\[ b = a \cdot (u \circ f \circ L) \]
(where \(f : F \to X\) is the projection onto the Zariski site of \(X\)). So \(b\) is the action such that, for every \(x \in \text{Ob}(\mathcal{F}(U))\), \(b(x)\) is the composition
\[ L'(U) \xrightarrow{u(U)} L(U) \xrightarrow{a(x)} \text{liau}(\mathcal{F})(x). \] (6–28)

**Proposition 6.2.3.** Let \(F\) be a 2-gerbe on \(X\).

1. Let \((L, a)\) be a 2-lien operating on \(F\). The following are equivalent:
   
   (a) \(a\) is a 2-equivalence.
   
   (b) for every 2-lien \(L'\) on \(X\) and every action \(b\) of \(L'\) on \(F\) there exists a unique morphism of 2-liens \(u : L' \to L\) (up to 2-equivalence) such that \(b\) is equivalent to the action induced by \(a\) and \(u\).

2. There exists a 2-lien \((L, a)\) operating on \(F\) that satisfies the conditions of (1).

The proof of this proposition is given below, following the proof of Proposition 6.2.5.

**Definition 6.2.4.** A 2-lien operating on a 2-gerbe \(F\) and satisfying the conditions of the above proposition will be called a 2-lien associated to \(F\).

Condition 1(b) characterizes the 2-lien of a 2-gerbe \(F\) up to a canonical 2-equivalence. Thus we are justified in saying “the” 2-lien of a 2-gerbe \(F\). By abuse of notation, the map \(a\) is often not mentioned. If \(L\) is the 2-lien of a 2-gerbe \(F\), we say \(F\) is bound by \(L\).
Conversely, if \( L \) is a 2-lien on \( X \), then an \( L \)-2-gerbe is a pair \( (\mathcal{F}, a) \) where \( \mathcal{F} \) is a 2-gerbe and \( (L, a) \) is a 2-lien of \( \mathcal{F} \).

More explicitly, the 2-lien of a 2-gerbe \( \mathcal{F} \) is a 2-lien \( L \) and a family of equivalences of 2-liens over \( U \),

\[
a(x) : L(U) \xrightarrow{\sim} \lien_2(\mathcal{E}_{qU}(x))
\]

(6–29)

where \( x \in \mathcal{F}(U), U \subset X \). This family is required to be compatible with the restriction of open sets and also satisfy the condition that if \( i : x \to y \) is a \( U \)-equivalence of \( \mathcal{F} \), then the morphism of \( gr \)-stacks \( \text{Inn}(i) : \mathcal{E}_{qU}(x) \to \mathcal{E}_{qU}(y) \) is isomorphic to the identity morphism of \( L(U) \).

**Proposition 6.2.5.** For every 2-gerbe \( \mathcal{F} \), there exists a 2-lien \( L \) and an equivalence \( a : Lf \xrightarrow{\sim} \text{liau}_2(\mathcal{F}) \).

**Proof.** Let \( I \) be the image category of \( \text{liau}_2(\mathcal{F}) \). Clearly this is a fibered 2-category of \( \text{LIEN}_2(X) \) and the 2-functor induced by \( \text{liau}_2(\mathcal{F}) \),

\[
L' : \mathcal{F} \to I
\]

(6–30)

is 2-cartesian. Moreover, the projection \( I \to X_{zar} \) is both fully 2-faithful and essentially surjective (this follows trivially from the fact that \( \mathcal{F} \to X \) is fully 2-faithful and because \( \text{liau}_2(m) \xrightarrow{\sim} \text{liau}_2(n) \) for any two equivalences \( m, n : x \to y \) of \( \mathcal{F} \) over an open set \( U \). By the universal property of the associated 2-stack, it follows that \( X \) is the 2-stack associated to \( I \), and that there exists a global section \( L_0 : X \to I \) of \( I \) and an equivalence \( a_0 : L_0f \xrightarrow{\sim} L' \).

**Proof of Proposition 6.2.3.** Since \( \mathcal{F} \) is a 2-gerbe, the projection \( f : \mathcal{F} \to X_{zar} \) is fully 2-faithful. Hence, if \( (L, a) \) and \( (L', b) \) are two 2-liens acting on \( \mathcal{F} \), the map \( \text{Hom}(L', L) \to \text{Hom}(L'f, Lf) \) given by \( u \mapsto u \circ f \) is an equivalence of group categories. This entails that \( b \xrightarrow{\sim} a(u \circ f) \) whence we have that 1(a) implies 1(b). By the above proposition (2) follows trivially, and so does 1(b) \( \Rightarrow \) 1(a) since 1(b) determines \( L \) up to unique
equivalence. Then setting \( L = iL_0 \) and \( a = i \circ a_0 \) (where \( i : I \to LIEN_2(X) \) is the inclusion) the conclusion follows.

**Corollary 6.2.6.** Let \( m : F \to G \) be a morphism of 2-gerbes and let \((L, a)\) and \((M, b)\) denote the 2-liens of \( F \) and \( G \) respectively. Then there exists a unique morphism of 2-liens \( u : L \to M \) (up to 2-equivalence) such that \( m \) is a \( u \)-morphism, i.e. \( liau_2(m) \cdot a \sim (b \circ m)(u \circ f) \).

Following the convention for 1-gerbes, if a 2-lien is defined by a family of gr-stacks \( (\mathcal{G}_i) \), we will call this family \( lien_2(\mathcal{G}_i) \).

### 6.3 Cocycle Description of the 2-Lien of a 2-Gerbe

Let \( \mathcal{G} \) be a 2-gerbe on \( X \) and \( U \) an open subset of \( X \). For an object \( a \) in \( \mathcal{G}(U) \), the prestack in groupoids \( \mathcal{E}q(a) \) of self equivalences of \( a \) is endowed with a (strictly associative) monoidal structure determined by a composition of 1-arrows. Since \( \mathcal{G} \) is a 2-gerbe, the 1-arrows are weakly invertible so this monoidal structure is group-like. In fact, local inverses for 1-arrows always descend to global ones (see [4], pg. 64-65) and so \( \mathcal{E}q(a) \) is a gr-stack on \( U \) once specific inverses for the 1-arrows have been chosen. In the following we will again assume that this has been done, and denote by \( \alpha^{-1} \) the chosen inverse of a 1-arrow \( \alpha \) in \( \mathcal{G} \) (see pages 72-73).

Thus given a 2-gerbe \( \mathcal{G} \) on \( X \) fix inverses for all the 1-arrows of \( \mathcal{G} \). Choose an open cover \( X = (U_\alpha) \), and for each \( \alpha \) choose an object \( a_\alpha \) in \( \mathcal{G}(U_\alpha) \). For each \( \alpha \) and \( \beta \), choose an open cover \( U_{\alpha \beta} = (U_{\xi \alpha \beta}) \) and for each \( \xi \) a 1-arrow \( f_{\alpha \beta}^\xi : a_\beta \to a_\alpha \) in \( \mathcal{G}(U_{\alpha \beta}) \). Then we can define an equivalence of gr-stacks:

\[
\lambda_{\alpha \beta}^\xi : \mathcal{E}q(a_\beta)|_{U_{\alpha \beta}^\xi} \to \mathcal{E}q(a_\alpha)|_{U_{\alpha \beta}^\xi}
\]

\[
\tau \mapsto (f_{\alpha \beta}^\xi)^{-1} \circ \tau \circ f_{\alpha \beta}^\xi.
\]

This equivalence \( \lambda_{\alpha \beta}^\xi \) depends on the choice of \( f_{\alpha \beta}^\xi \). But consider two different choices \( \lambda_{\alpha \beta}^{\xi_1} \) and \( \lambda_{\alpha \beta}^{\xi_2} \) corresponding to \( f_{\alpha \beta}^{\xi_1}, f_{\alpha \beta}^{\xi_2} : a_\beta \to a_\alpha \) in \( \mathcal{G}(U_{\alpha \beta}) \) i.e. suppose \( \lambda_{\alpha \beta}^{\xi_1} := (f_{\alpha \beta}^{\xi_1})^{-1} \circ \tau \circ f_{\alpha \beta}^{\xi_1} \) and \( \lambda_{\alpha \beta}^{\xi_2} := (f_{\alpha \beta}^{\xi_2})^{-1} \circ \tau \circ f_{\alpha \beta}^{\xi_2} \).
and \( \lambda_{\alpha\beta}^{\xi_2} := (f_{\alpha\beta}^{\xi_2})^{-1} \circ \tau \circ f_{\alpha\beta}^{\xi_2} \). Then there exists a 2-arrow:

\[
\lambda_{\alpha\beta}^{\xi_2} : (f_{\alpha\beta}^{\xi_2})^{-1} \circ f_{\alpha\beta}^{\xi_1} \circ [(f_{\alpha\beta}^{\xi_1})^{-1} \circ \tau \circ f_{\alpha\beta}^{\xi_1}] \circ (f_{\alpha\beta}^{\xi_1})^{-1} \circ f_{\alpha\beta}^{\xi_2} \tag{6–32}
\]

i.e.

\[
\lambda_{\alpha\beta}^{\xi_2} \Rightarrow [(f_{\alpha\beta}^{\xi_2})^{-1} \circ f_{\alpha\beta}^{\xi_1}] \circ \lambda_{\alpha\beta}^{\xi_1} \circ [(f_{\alpha\beta}^{\xi_2})^{-1} \circ f_{\alpha\beta}^{\xi_1}]^{-1} \tag{6–33}
\]

where there exists an invertible 2-arrow

\[
[(f_{\alpha\beta}^{\xi_2})^{-1} \circ f_{\alpha\beta}^{\xi_1}] \circ [(f_{\alpha\beta}^{\xi_2})^{-1} \circ f_{\alpha\beta}^{\xi_1}]^{-1} \Rightarrow \text{Id.} \tag{6–34}
\]

Thus \( \lambda_{\alpha\beta}^{\xi_1} \) and \( \lambda_{\alpha\beta}^{\xi_2} \) differ by an inner equivalence. In particular they define the same section of \( \text{Out}(\mathcal{E}q(a_{\beta}^{\xi}), \mathcal{E}q(a_{\alpha}^{\xi})) \). So for fixed \( \alpha \) and \( \beta \), the family \( \{\lambda_{\alpha\beta}^{\xi}\} \) defines an “outer” equivalence of \( gr \)-stacks on \( U_{\alpha\beta} \),

\[
\lambda_{\alpha\beta} : \mathcal{E}q(a_{\beta})|_{U_{\alpha\beta}} \rightarrow \mathcal{E}q(a_{\alpha})|_{U_{\alpha\beta}} \tag{6–35}
\]

which does not depend on the choice of the \( f_{\alpha\beta}^{\xi} \). This system of \( gr \)-stacks \( \mathcal{E}q(a_{\alpha}) \) and outer equivalences \( \lambda_{\alpha\beta} \) will be called a cocycle representation of the 2-lien of the gerbe \( \mathcal{G} \).
CHAPTER 7
2-LIENS, PICARD STACKS AND H^3

7.1 2-Gerbes and 3-Cocycles

The construction of a 3-cocycle associated to a given 2-gerbe P is lengthy and complicated. We will give a very brief outline of this construction, before specializing to the situation of strict Picard stacks. We follow the treatment in [4] and [5]. The reader may consult these sources for the details.

Let P be a G-2-gerbe on a given space X. We will assume that P is connected: that is, for each pair of objects x, y in some fiber 2-category PU, the set of arrows in PU from x to y is non-empty. Given an open cover (Ui) of X, we choose objects xi ∈ PU and paths φij : xj → xi in the 2-groupoid PU, together with quasi-inverses ψij : xi → xj. To this we associate the following data:

1. An object λij in Eq(Gj|Uij, Gi|Uij):
   \[ \lambda_{ij} : G_j|_{U_{ij}} \to G_i|_{U_{ij}} \]  
   \[ g \mapsto \phi_{ij} \circ g \circ \psi_{ij}, \]  
   determined by φij and its inverse.

2. An arrow gijk (determined by the paths φij and their inverses) viewed as an object in the fiber of the stack Gi over the open set Uijk:

3. An arrow mijk (a natural transformation induced by the 2-arrow mijk in the diagram above) in the category Eq(Gj, Gi)Uijk:
where \( i_g \) is the inner conjugation functor defined by an object \( g \) in the gr-stack \( G \), and for \( \gamma \in G_k \),

\[
\tilde{m}_{ijk}(\gamma) : i_{g_{ijk}} \circ \lambda_{ij}(\gamma) \rightarrow \lambda_{ij} \circ \lambda_{jk}(\gamma)
\]  

in \( G_i \). By right multiplication by the inverse object of its source, this arrow corresponds to an arrow:

\[
\{\tilde{m}_{ijk}, \gamma\} : 1_x \rightarrow \lambda_{ij} \circ \lambda_{jk}(\gamma) \circ (i_{g_{ijk}} \circ \lambda_{ik}(\gamma))^{-1}
\]  

in \( G_i \) sourced at the identity.

4. An arrow \( \nu_{ijkl} : 1_x \Rightarrow g_{ijkl} \circ g_{ikl} \circ (g_{ijl})^{-1} \circ \lambda_{ij}(g_{jkl})^{-1} \) in the category \( G_i|U_{ijkl} \) determined by the data listed above. It can also be viewed as a 2-arrow that sits in the diagram below.

\[
\begin{align*}
g_{ijkl} & : x_i \rightarrow g_{ijl} \circ \lambda_{ij}(g_{jkl}) \\
\lambda_{ij}(g_{jkl}) & : g_{ijkl} \rightarrow \lambda_{ij} \circ g_{ijkl} \\
\nu_{ijkl} & : \lambda_{ij}(g_{jkl}) \rightarrow g_{ijkl} \\
\end{align*}
\]  

Further, the following identity is valid for \( \nu_{ijkl} \):

\[
\tilde{m}_{ijk}(g_{ijkl} \tilde{m}_{ikl}) \nu_{ijkl} = (\lambda_{ij}(\tilde{m}_{jkl}))(\lambda_{ij}(g_{ikl}) \tilde{m}_{ijkl})
\]  

where \( g_m \) denotes the left composition of \( m \) by \( g \). In addition, \( \nu_{ijkl}|U_{ijklm} \) satisfies the following cocycle condition in \( G_i|U_{ijklm} \):

\[
\lambda_{ij}(\nu_{jklm})(\lambda_{ij}(g_{jkl}) \nu_{ijklm}) \nu_{ijkl} = (\lambda_{ij} \circ \lambda_{jk}(g_{klm}) \nu_{ijklm}) \{\tilde{m}_{ijk}, g_{klm}\}(g_{jkl} \nu_{iklm})
\]  

where \( g_\nu \) denotes conjugation of an arrow \( \nu \) by \( g \).
By Breen ([4], [5]), such a quadruple of elements \((\nu_{ijkl}, \tilde{m}_{ijk}, g_{ijk}, \lambda_{ij})\) is called a \(G\)-valued nonabelian 3-cocycle on the space \(X\).

We now discuss the coboundary relations. We again review the relevant notions from [4] and [5], before specializing to the strict Picard case.

Two nonabelian 3-cocycles \((\nu_{ijkl}, \tilde{m}_{ijk}, g_{ijk}, \lambda_{ij})\) and \((\nu'_{ijkl}, \tilde{\mu}_{ijk}, \gamma_{ijk}, \lambda'_{ij})\) are cohomologous if there exist:

1. A family of objects \((\rho_i)_*\) in \(Eq(G U_i)\) and a family of objects \(h_{ij}\) in \(G U_{ij}\).
2. A family of arrows \(\tilde{\chi}_{ij}\) in the category \(G U_{ij}\):

\[
\tilde{\chi}_{ij} : (\rho_i)_* \circ \lambda'_{ij} \Rightarrow \iota_{h_{ij}} \circ \lambda_{ij} \circ (\rho_i)_* \tag{7-10}
\]

and a family of morphisms \(a_{ijk}\):

\[
a_{ijk} : (\rho_i)_*(\gamma_{ijk}) \circ h_{ik} \Rightarrow h_{ij} \circ \lambda_{ij}(h_{jk}) \circ g_{ijk} \tag{7-11}
\]

in \(G U_{ijk}\), for which the following two identities are satisfied:

\[
\begin{align*}
\lambda_{ij}(\tilde{\chi}_{jk}) \tilde{\chi}_{ij} (\rho_i)_* (\tilde{\mu}_{ijk}) &= (h_{ij}\lambda_{ij}(h_{jk})\tilde{m}_{ijk})_{a_{ijk}} (\rho_i)_*(\gamma_{ijk}) \tilde{\chi}_{ik} \\
h_{ij}\lambda_{ij}(h_{jk}) \{m_{ijk}, h_{kl}\} a_{ijk} (\rho_i)_*(\gamma_{ijk}) a_{ikl} (\rho_i)_*(\nu'_{ijkl}) &= \lambda_{ij}(h_{jk}) \lambda_{ij}(h_{kl}) (\nu_{ijkl})_{a_{ijkl}} (\tilde{\chi}_{ij} \gamma_{jkl}) (\rho_i)_*(\lambda'_{ijkl}) a_{ijl}
\end{align*}
\]

7.2 2-Liens and Strict Picard Stacks

In this section we prove some theorems which are analogs of the results for 1-gerbes stated in Chapter 2. Recall the following definitions from Chapter 2.

**Definition 7.2.1.** A group category \(G\) is said to be a strict Picard category when its group law is endowed with a commutivity isomorphism \(S_{X,Y} : XY \rightarrow YX\) which is functorial in
and $Y$ and sits in a commutative square

\[
\begin{array}{ccc}
X_1 & \overset{s}{\rightarrow} & 1X \\
\downarrow{m} & & \downarrow{s} \\
X & \overset{\sim}{\rightarrow} & X \\
\end{array}
\] (7–12)

and two hexagonal “associativity” diagrams. In addition $S_{Y,X} \circ S_{X,Y} = 1_{XY}$ for all $X, Y$ in $G$ and $S_{X,X} = 1_X$ for all $X$. A gr-stack $\mathcal{G}$ is said to be a strict Picard stack if it is endowed with a commutativity natural transformation $S$ for the group law $S_{X,Y}$ that induces for each open $U \subset X$ the structure of a strict Picard category on $\mathcal{G}(U)$.

Note that in this section, by “Picard stack”, we always mean a strict (commutative) Picard stack. At this point the reader may wish to review the definition of a $G$-2-gerbe from Chapter 2.

**Proposition 7.2.2.** Let $\mathcal{G}$ be a gr-stack on $X$. A 2-gerbe $P$ on $X$ is a $G$-2-gerbe if and only if its 2-lien is locally equivalent to $\text{lien}_2(\mathcal{G})$.

**Proof.** If $P$ is a $G$-2-gerbe, $P$ is locally equivalent to $\text{Tors}(\mathcal{G})$. Thus its 2-lien is locally equivalent to $\text{lien}_2(\text{Tors}(\mathcal{G})) = \text{lien}_2(\mathcal{G})$. Conversely, let $P$ be a 2-gerbe whose 2-lien is equivalent, when restricted to some open cover $\mathcal{U}$ of $X$ to $\text{lien}_2(\mathcal{G})$ for some given gr-stack $\mathcal{G}$. We can then choose a second open cover $\mathcal{U}' = (U_i)$ of $X$ for which there exists a family of objects $x_i \in \mathcal{G}_{U_i}$. The 2-gerbe $P$ is then locally of the form $\text{Tors}(\mathcal{G}_i)$ for the tautological labeling of $P$ defined by $\mathcal{G}_i = \mathcal{E}q(x_i)$. It follows that $\text{lien}_2(P)|_{U_i}$ is equivalent to $\text{lien}_2(\mathcal{G}_i)$, so that, on the elements $V_\alpha$ of a common refinement $\mathcal{U}''$ of $\mathcal{U}$ and $\mathcal{U}'$, we have equivalences of 2-liens $\text{lien}_2(\mathcal{G})|_{V_\alpha} \rightarrow \text{lien}_2(\mathcal{G}_i)|_{V_\alpha}$. Such an equivalence is defined by sections $[\eta]_\alpha$ on the open sets $V_\alpha$ of the stack $\text{Out}(\mathcal{G}, \mathcal{G}_i)$. It follows from the definition of this stack that these sections lift to sections $\eta_\beta : \mathcal{G}|_{W_\beta} \rightarrow \mathcal{G}_i|_{W_\beta}$ of $\mathcal{E}q(\mathcal{G}, \mathcal{G}_i)$ on the open sets $W_\beta$ of an appropriate refinement $\mathcal{U}'''$ of $\mathcal{U}''$. Since $\mathcal{G}_i = \mathcal{E}q(x_i)$, the collection of maps $(\eta_\beta)$ give the $G$-2-gerbe structure on $P$.

**Proposition 7.2.3.** Let $\mathcal{G}$ be a Picard stack on $X$. A gerbe $P$ on $X$ is an abelian $G$-2-gerbe if and only if $\text{lien}_2(P)$ is equivalent to $\text{lien}_2(\mathcal{G})$.
Proof. Suppose $P$ is an abelian $G$-2-gerbe. Then from the diagram in the definition of an abelian $G$-2-gerbe the terms the terms $\lambda_{ij}$, $\tilde{m}_{ijk}$, and $\{\tilde{m}_{ijk}, g_{klm}\}$ of the 3-cocycle associated to an abelian $G$-2-gerbe $P$ are all trivial so that its 2-lien is globally equivalent to $\text{lien}_2(G)$. Conversely suppose $P$ is a 2-gerbe with 2-lien equivalent to $\text{lien}_2(G)$ for some Picard stack $G$. For any object $x \in G_U$ we can construct as in the proof of the above proposition, an equivalence of 2-liens

$$\text{lien}_2(Eq(x)) \to \text{lien}_2(P|_U) \to \text{lien}_2(G|_U)$$

(7–13)

which is described by an outer equivalence $Eq(x) \to G|_U$. Since $G$ is now Picard, in the diagram

$\begin{array}{ccc}
\eta_x & \rightarrow & G|_U \\
\downarrow \phi_x & \searrow & \downarrow \phi_y \\
Eq_{P_U}(x) & \to & Eq_{P_U}(y)
\end{array}$

(7–14)

of the definition of an abelian $G$-2-gerbe, the 2-arrow $\eta_f$ defines an equivalence between such an outer equivalence and an ordinary equivalence of gr-stacks $Eq(x) \to G|_U$. The required commutativity of the above diagram is equivalent to the commutativity of the corresponding diagram of 2-liens, since it involves Picard categories. This in turn follows by applying the 2-lien 2-functor to the diagram

$\begin{array}{ccc}
\phi_x & \rightarrow & G \\
\downarrow \phi_x & \searrow & \downarrow \phi_y \\
Tors(Eq(x)) & \to & Tors(Eq(y))
\end{array}$

(7–15)

where

$$\phi_x : G \to Tors(Eq(x))$$

(7–16)

$$g \mapsto Eq(x, g)$$

(7–17)

gives an equivalence between $G$ and a 2-gerbe of torsors.
7.3 Strict Picard Stacks and $H^3$

**Definition 7.3.1.** We designate by $H^3(X, L)$ the set of equivalence classes of 2-gerbes with 2-lien $L$, and by $\check{H}^3(X, L) \subset H^3(X, L)$ the set of equivalence classes of connected 2-gerbes with 2-lien $L$.

Deligne [13] has proved that any strict Picard stack $\mathcal{G}$ is 2-equivalent to a 2-term complex of abelian sheaves $K^\cdot = [K^0 \xrightarrow{d} K^1]$. We will prove that $\check{H}^3(X, \mathcal{G})$ is isomorphic to the hypercohomology group $\check{H}^3(X, K^\cdot)$.

We begin by describing Deligne’s construction for specific cases in one direction; that is, we show how one can associate a strict Picard category (respectively stack) to a given 2-term complex of abelian groups (respectively sheaves). We follow the treatment in [11] and [13].

Let $K^\cdot$ denote a 2-term complex $K^0 \xrightarrow{d} K^1$ of abelian groups. Then $\mathcal{H}$ is the strict Picard category whose objects are the elements of $K^1$, for any two objects $x, y$, a morphism between them is an element $z \in K^0$ such that $dz = x - y$ in $K^1$, composition of morphisms is addition in $K^0$, and the functor $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is given by addition of objects and morphisms. The associativity and commutativity constraints are the identities. If $K^\cdot$ denotes a 2-term complex $K^0 \xrightarrow{d} K^1$ of abelian sheaves, then $\mathcal{H}_K$ is the prestack given by the following information: for an open set $U \subset X$, $\mathcal{H}_K(U)$ is the group category whose objects are the elements of $K^1(U)$, for any two objects $x, y$, a morphism between them is an element $z \in K^0(U)$ such that $dz = x - y$ in $K^1(U)$, composition of morphisms is addition in $K^0(U)$, and the functor $\mathcal{H}_K(U) \times \mathcal{H}_K(U) \to \mathcal{H}_K(U)$ is given by addition of objects and morphisms in $K^0$ and $K^1$. Again, the associativity and commutativity constraints are the identities. Then by [13], the stackification $ch(K)$ of the prestack $\mathcal{H}_K$ is called the stack associated to the complex $K^\cdot$, where $ch(K^\cdot)$ is Picard. More precisely, if $C(X)$ denotes the 2-category of 2-term complexes of abelian sheaves on $X$, where morphisms in $C(X)$ are morphisms of complexes, and 2-isomorphism are homotopies between morphisms, then we have the following result.
Lemma 7.3.2. ([13], 1.4.13.1) For every Picard stack $P$ there exists a complex $K^\cdot \in C(X)$ such that $P \simeq \text{ch}(K)$.

When $K^0$ is injective, then we have the following lemma.

Lemma 7.3.3. ([13], 1.4.16.1) Let $K^\cdot \in C(X)$ and assume that $K^0$ is injective. Then the prestack $\mathcal{H}_K$ is already a stack.

Let $PI\mathcal{C}(X)$ denote the 2-category of Picard stacks on $X$, and let $C'(X) \subset C(X)$ denote the full sub-2-category of 2-term complexes $K^\cdot$ with $K^0$ injective.

Lemma 7.3.4. ([13], 1.4.17) The construction $\text{ch}$ described above induces a 2-equivalence of the 2-categories $PI\mathcal{C}(X)$ and $C'(X)$.

We will follow the notation of [11] and (continue to) refer to Deligne’s 2-functor $PI\mathcal{C}(X) \rightarrow C(X)$ by $\text{ch}$. Deligne also gives a characterization of the Picard stack $\text{Hom}(\text{ch}(K), \text{ch}(L))$, where $K^\cdot, L^\cdot \in C(X)$. Let $Sh_{Ab}(X)$ denote the category of abelian sheaves on $X$.

Lemma 7.3.5. ([13], 1.4.18.1) Assume that $L^0$ is injective. Then we have an equivalence

$$
\text{ch}(\tau \leq 1 \text{Hom}_{Sh_{Ab}(X)}(K^\cdot, L^\cdot) \xrightarrow{\sim} \text{Hom}_{PI\mathcal{C}(X)}(\text{ch}(K), \text{ch}(L)))
$$

(7–18)

The $\tau \leq 1$ above refers to a certain truncation operation; see [13] for details. For the rest of this section we will use this point of view while studying $\mathcal{G}$-2-gerbes, where $\mathcal{G}$ is strict Picard. Let $\mathcal{P}$ be a connected $\mathcal{G}$-2-gerbe where $\mathcal{G}$ is strict Picard. Let $K^\cdot$ be the 2-term complex associated to $\mathcal{G}$. We form the double Čech complex:

$$
\begin{align*}
C^i(K^0) & \xrightarrow{d} C^i(K^1) \rightarrow 0 \\
\downarrow \delta & \downarrow \delta \\
C^{ij}(K^0) & \xrightarrow{d} C^{ij}(K^1) \rightarrow 0 \\
\downarrow \delta & \downarrow \delta \\
C^{ijk}(K^0) & \xrightarrow{d} C^{ijk}(K^1) \rightarrow 0 \\
\downarrow \delta & \downarrow \delta \\
C^{ijkl}(K^0) & \xrightarrow{d} C^{ijkl}(K^1) \rightarrow 0
\end{align*}
$$

(7–19)
where $\delta$ denotes the Čech differential, and $d$ denotes the complex differential. This double complex is denoted $\mathcal{C}(K^\cdot)$. One can obtain an ordinary complex $C^\cdot$ from this double complex as follows. Put $C^m = \bigoplus_{p+q=n} \mathcal{C}^p(K^q)$, with the differential $D = \delta + (-1)^p d$ in bidegree $(p, q)$. The sign $(-1)^p$ is placed here to ensure that $D \circ D = 0$. This ordinary complex is called the total complex associated to the double complex $\mathcal{C}(K^\cdot)$. Then the Čech hypercohomology $\check{H}^p(K^\cdot)$ of the complex $K^\cdot$ is defined to be the cohomology of the total complex $\mathcal{C}^\cdot[7]$. A degree $n$ cocycle consists of a finite family $c_p \in \mathcal{C}^p(K^{n-p})$ such that

$$\delta(c_p) = (-1)^p d(c_{p+1}) \in \mathcal{C}^{p+1}(K^{n-p}). \quad (7\text{-}20)$$

By the definition of a Picard stack, the 3-cocycle terms $\lambda_{ij}$, $\tilde{m}_{ijk}$, and $\{\tilde{m}_{ijk}, g_{klm}\}$ are trivial. The remaining data $(\nu_{ijkl}, g_{ijk})$ satisfy the relation

$$\nu_{jklm}\nu_{ijlm}\nu_{ijkl} = \nu_{ijkl}\nu_{iklm}. \quad (7\text{-}21)$$

Also, the diagram

$$\begin{diagram}
  x_i & \xrightarrow{g_{ijl}} & x_i \\
  \downarrow{g_{ikl}} & & \downarrow{g_{ikl}} \\
  x_i & \xrightarrow{g_{ijk}} & x_i \\
  \downarrow{g_{ijl}} & & \downarrow{g_{ijl}} \\
  x_i & \xrightarrow{\nu_{ijkl}} & x_i
\end{diagram} \quad (7\text{-}22)$$

becomes

$$\begin{diagram}
  x_i & \xrightarrow{g_{ijl}} & x_i \\
  \downarrow{g_{ikl}} & & \downarrow{g_{ikl}} \\
  x_i & \xrightarrow{g_{ijk}} & x_i \\
  \downarrow{g_{ijl}} & & \downarrow{g_{ijl}} \\
  x_i & \xrightarrow{\nu_{ijkl}} & x_i
\end{diagram} \quad (7\text{-}23)$$

By Deligne’s construction, when we view this data as taking values in $K^\cdot$ instead of $\mathcal{G}$, we get that $\nu_{ijkl}$ are sections of $K^0$ and the $g_{ijk}$ are sections of $K^1$. Further, by the construction of $K^\cdot$, the diagram above says that:

$$d(\nu_{ijkl}) = g_{ijk} + g_{ikl} - g_{jkl} - g_{ijl}. \quad (7\text{-}24)$$
So $d(\nu) = \delta(g)$. Also, since the $\nu$ are sections of $K^0$, the relation

$$\nu_{jklm} \nu_{ijlm} \nu_{ijkl} = \nu_{ijkl} \nu_{iklm}, \quad (7-25)$$

says that

$$\nu_{jklm} + \nu_{ijlm} + \nu_{ijkl} = \nu_{ijkm} + \nu_{iklm}, \quad (7-26)$$

so

$$\nu_{jklm} + \nu_{ijlm} + \nu_{ijkl} - \nu_{ijkm} - \nu_{iklm} = 0, \quad (7-27)$$

that is, $\delta(\nu) = 0$. Thus the pair $(\nu_{ijkl}, g_{ijk})$ give us a hypercocycle in $K$.

We now discuss the coboundary relations. When $G$ is Picard, these relations simplify to the statement that the cocycles $(\nu_{ijkl}, g_{ijk})$ and $(\nu'_{ijkl}, \gamma_{ijk})$ are cohomologous if there exist:

1. A family of objects $h_{ij}$ in $G_{U_{ij}}$.
2. A family of morphisms $a_{ijk}$ that satisfy:

$$a_{ijk} : \gamma_{ijk} \circ h_{ik} \Rightarrow h_{ij} \circ h_{jk} \circ g_{ijk}, \quad (7-28)$$

in $G_{U_{ijk}}$, for which the following identity is satisfied:

$$a_{ijk} a_{ikl} \nu'_{ijkl} = \nu_{ijkl} a_{jkl} a_{ijl}, \quad (7-29)$$

As before, switching to additive notation after viewing the $h_{ij}$ and $a_{ijk}$ as sections in $K$, the first identity above says that

$$d(a_{ijk}) = \gamma_{ijk} + h_{ij} - h_{jk} - g_{ijk}, \quad (7-30)$$

So

$$d(a_{ijk}) = \gamma_{ijk} - g_{ijk} + \delta(h_{ij}), \quad (7-31)$$

So $d(a_{ijk}) - \delta(h_{ij}) = \gamma_{ijk} - g_{ijk}$. Similarly, the second identity says that

$$a_{ijk} + a_{ikl} + \nu'_{ijkl} = \nu_{ijkl} + a_{jkl} + a_{ijl} \quad (7-32)$$
so

\[ \nu'_{ijkl} - \nu_{ijkl} = a_{jkl} + a_{ijl} - a_{ijk} - a_{ikl} = \delta(a_{ijk}). \]  

(7–33)

Thus the coboundary conditions for the two cocycle pairs \((\nu_{ijkl}, g_{ijk})\) and \((\nu'_{ijkl}, \gamma_{ijk})\) in \(\mathcal{G}\) translate exactly into hyper-coboundary conditions in \(K\).

Thus we get a well-defined map:

\[ \tilde{H}^3(X, \mathcal{G}) \to \tilde{H}^3(X, K) \]  

(7–34)

that sends the cohomology class of each pair \((\nu_{ijkl}, g_{ijk})\) to its corresponding class \((\nu_{ijkl}, g_{ijk})\).

To see that this map gives an equivalence, observe that any hypercocycle \((\nu_{ijkl}, g_{ijk})\) in \(\tilde{H}^3(X, K)\) satisfies

\[ \nu_{jklm} + \nu_{ijlm} + \nu_{ijkl} - \nu_{ijkm} - \nu_{iklm} = 0, \]  

(7–35)

and

\[ d(\nu_{ijkl}) = g_{ijk} + g_{ikl} - g_{jkl} - g_{ijl}. \]  

(7–36)

Since this cocycle takes values in \(K\), by Deligne ([13]), we can view it as a cocycle pair \((\nu_{ijkl}, g_{ijk})\) that takes values in its associated Picard stack \(\mathcal{G}\) and satisfies the relations

\[ \nu_{jklm}\nu_{ijlm}\nu_{ijkl} = \nu_{ijkm}\nu_{iklm}. \]  

(7–38)

But by Breen [4], Theorem 5.6, such a cocycle with values in \(\mathcal{G}\) has an associated \(\mathcal{G}\)-2-gerbe \(\mathcal{P}\), where \(\mathcal{P}\) is constructed by precisely reversing the construction which associates a nonabelian 3-cocycle \((\nu_{ijkl}, \tilde{m}_{ijk}, g_{ijk}, \lambda_{ij})\) to a \(\mathcal{G}\)-2-gerbe \(\mathcal{P}\), of which the
\( \mathcal{G} \)-valued cocycle pair \((\nu_{ijkl}, g_{ijk})\) is a special case. Thus the map \( \hat{H}^3(X, \mathcal{G}) \to \hat{H}^3(X, K) \) is an isomorphism.
CHAPTER 8
CONCLUSION

We summarize the results of this thesis. The goal of this project was to define the 2-lien of a 2-gerbe. To this end, we needed to define the notion of a quotient by an action of a group stack on a stack. In chapter 3 we solved this problem locally: that is, we showed that coequalizers were representable in the 2-category $\text{CAT}$ of (small) categories by explicitly constructing a coequalizer for (small) pair of arrows. In chapter 4 we defined what it means to have an inner equivalence of a group category. We then gave a definition for the action of a group category on a category, and defined the quotient by such an action using the coequalizer constructed in chapter 3. In chapter 5 we carried out the stackification of these concepts: that is, we showed the representability of coequalizers in the 2-category $\text{STACKS}$, and gave a definition for the action of a group stack on a stack, and the quotient by such an action. In chapter 6 we finally defined the 2-lien of a space $X$. We showed how every 2-gerbe gives us such a 2-lien and did the work to establish that this 2-lien is in fact given up to canonical 2-equivalence. We then gave a more concrete description of a 2-lien using cocycles. Finally in chapter 7 we proved some results about $\mathcal{G}$-2-gerbes whose 2-lien arises from a strict Picard stack. We recalled Deligne’s correspondence between strict Picard stacks $\mathcal{G}$ and 2-term complexes of abelian sheaves $K$. We then proved that the set of equivalence classes $\check{H}^3(X, \mathcal{G})$ of connected gerbes with 2-lien $\mathcal{G}$ is isomorphic to the hypercohomology group $\check{H}^3(X, K)$.

There are some natural directions that one could take as a continuance of this project. The above cohomological result of chapter 7 has only been proven for connected 2-gerbes, and it would be nice to prove it after removing the connectedness hypothesis. The proof would involve working with hypercovers (open covers of each open set) but the result would still seem to be true for $\mathcal{G}$ Picard. Then as a next step, one could relax the strict Picard assumption and only assume that the 2-lien $\mathcal{G}$ is braided, and try to find a nice cohomological description for 2-gerbes with 2-lien $\mathcal{G}$.
REFERENCES


BIOGRAPHICAL SKETCH

I was born in Mumbai, India. I graduated from Truman State University in May 2001 and started graduate school at UF in the fall. I started working with Dr. Crew in 2004.