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LIST OF SYMBOLS, NOMENCLATURE, OR ABBREVIATIONS

Let $G$ be a finite group, $\ell$ be a prime number, and $\mathbb{F}$ be an algebraically closed field of characteristic $\ell$. We write

$Z(G)$ : for the center of $G$

$O_\ell(G)$ : for the maximal normal $\ell$-subgroup of $G$

$Irr(G)$ : for the set of irreducible complex characters of $G$, or the set of irreducible $\mathbb{C}G$-representation

$IBr_\ell(G)$ : for the set of irreducible $\ell$-Brauer characters of $G$, or the set of absolutely irreducible $G$-representation in characteristic $\ell$

$d_{\ell}(G)$ : for the smallest degree of irreducible $\mathbb{F}G$-modules of dimension $> 1$

$m_{\ell}(G)$ : for the largest degree of irreducible $\mathbb{F}G$-modules

$d_{2,\ell}(G)$ : for the second smallest degree of irreducible $\mathbb{F}G$-modules of dimension $> 1$

$d_{C}(G), m_{C}(G), d_{2,C}(G)$ : for $d_{0}(G), m_{0}(G), d_{2,0}(G)$, respectively

$\widehat{\chi}$ : for the restriction of a character $\chi \in Irr(G)$ to $\ell$-regular elements of $G$

Furthermore, we write

$GL_n^+(q)$ : for $GL_n(q)$

$GL_n^-(q)$ : for $GU_n(q)$

$CSp_{2n}(q)$ : see the discussion before Lemma 5.1.4

$PCSp_{2n}(q)$ : for the quotient of $CSp_{2n}(q)$ by its center

$CO_{2n}^\pm(q), CO_{2n}^\pm(q)^0$ : see the discussion before Lemma 5.1.5

$P(CO_{2n}^\pm(q)^0)$ : for the quotient of $CO_{2n}^\pm(q)^0$ by its center

$Spin_{n_1}^\pm(q)$ : for the Spin group. Modulo some exceptions, it is the universal cover of the simple orthogonal group $PO_{n}^\pm(q)$

**semi-simple element**: for an element which is diagonalizable

**unipotent character**: see the discussion at the beginning of §5.1
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REPRESENTATIONS OF FINITE GROUPS OF LIE TYPE

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Major: Mathematics

Let $G$ be a finite quasi-simple group of Lie type. One of the important problems in modular representation theory is to determine when the restriction of an absolutely irreducible representation of $G$ to its proper subgroups is still irreducible. The solution of this problem is a key step towards classifying all maximal subgroups of finite classical groups. We solve this problem for the cases where $G$ is a Lie group of the following types: $G_2(q)$, $^2B_2(q)$, $^2G_2(q)$, and $^3D_4(q)$.

One of the main tools to approach the above problem, and many others, is the classification of low-dimensional representations of finite groups of Lie type. Low-dimensional complex representations were first studied in [70] for finite classical groups and then in [50] for exceptional groups. We extend the results in [70] for symplectic and orthogonal groups to a larger bound. More explicitly, we classify irreducible complex characters of the symplectic groups $Sp_{2n}(q)$ and orthogonal groups $Spin_n^\pm(q)$ of degrees up to the bound $D$, where $D = (q^n - 1)q^{4n-10}/2$ for symplectic groups, $D = q^{4n-8}$ for orthogonal groups in odd dimension, and $D = q^{4n-10}$ for orthogonal groups in even dimension.
1.1 Overview and Motivation

Finite primitive permutation groups have been studied since the pioneering work of Galois and Jordan on group theory; they have had important applications in many different areas of mathematics.

If $G$ is a primitive permutation group with a point stabilizer $M$ then $M$ is a maximal subgroup of $G$. Thanks to works of Aschbacher, O’Nan, Scott [3], and of Liebeck, Praeger, Saxl, and Seitz [47], [48], most problems involving such a $G$ can be reduced to the case where $G$ is a finite classical group. If $G$ is a finite classical group, a fundamental theorem of Aschbacher [2] states that any maximal subgroup $M$ of $G$ is a member of either one of eight families $C_i$, $1 \leq i \leq 8$, of naturally defined subgroups of $G$ or the collection $S$ of certain quasi-simple subgroups of $G$. Conversely, if $M \in \bigcup_{i=1}^{8} C_i$, then the maximality of $M$ has been determined by Kleidman and Liebeck in [40]. It remains to determine which $M \in S$ are indeed maximal subgroups of $G$. This question leads to a number of important problems concerning modular representations of finite quasi-simple groups. One of them is the irreducible restriction problem.

**Problem A.** Let $\mathbb{F}$ be an algebraically closed field of characteristic $\ell$. Classify all triples $(G, V, M)$ where $G$ is a finite quasi-simple group, $V$ is an irreducible $\mathbb{F}G$-module of dimension greater than one, and $M$ is a proper subgroup of $G$ such that the restriction $V|_M$ is irreducible.

We recall that a finite group $G$ is called quasi-simple if $G = [G, G]$ and $G/Z(G)$ is simple. Moreover, any non-abelian finite simple groups is one of alternating groups, 26 sporadic groups, or finite simple groups of Lie type.

The solution for Problem A when $G/Z(G)$ is a sporadic group is largely computational. When $G$ is a cover of a symmetric or alternating group, Problem A is quite complicated and almost done in [6], [41], [42], and [57]. Our main focus is on the case where $G$ is a
finite group of Lie type in characteristic different from $\ell$. This has been solved recently in [43] when $G$ is of type $A$.

One of the important ingredients needed to solve Problem A is the classification of low-dimensional irreducible representations of finite groups of Lie type. This is the reason why we have been working simultaneously on the irreducible restriction problem and the problem of determining low-dimensional representations.

Suppose that $G = G(q)$ is a finite group of Lie type defined over a field of $q$ elements, where $q = p^n$ is a prime power. Lower bounds for the degrees of nontrivial irreducible representations of $G$ in cross characteristic $\ell$ (i.e. $\ell \neq p$) were found by Landazuri, Seitz and Zalesskii in [45], [59] and improved later by many people. These bounds have proved to be very useful in various applications. We are interested in not only the smallest representation, but more importantly, the low-dimensional representations.

Low-dimensional complex representations were first studied by Tiep and Zalesskii [70] for finite classical groups and then by Lübeck [50] for exceptional groups. For representations over fields of cross characteristics, this problem has been studied recently in [5], [23] for $SL_n(q)$; [25], [32] for $SU_n(q)$; [25] for $Sp_{2n}(q)$ with $q$ odd; and [24] for $Sp_n(q)$ with $q$ even. Let $H$ be one of these groups, and let the smallest degree of nontrivial irreducible representations of $H$ in cross characteristic be denoted by $d(H)$. The main purpose of these papers is to classify the irreducible representations of $H$ of degrees close to $d(H)$, and to prove that there is a relatively “big” gap between the degrees of these representations and the next degree.

A common application of low-dimensional representations is as follows. Suppose we want to prove some statement $\mathcal{P}$ involving representations $\varphi$ of finite groups $G$. First, one tries to reduce to the case when $G$ is quasi-simple. Then, with $G$ being quasi-simple, one shows that $\mathcal{P}$ holds if $\varphi$ has degree greater than a certain bound $d$. At this stage, results on low-dimensional representations should be applied to identify the representation $\varphi$ of degree $\leq d$ and to establish $\mathcal{P}$ directly for these representations. We refer to
[68] for a survey of recent progress and more detailed applications of low-dimensional representations of finite groups of Lie type.

1.2 Results

In this dissertation, we have completed Problem A for the cases where $G$ is a Lie type group of the following types: $G_2(q)$, $^2B_2(q)$, $^2G_2(q)$ and $^3D_4(q)$.

The smallest degree of non-trivial irreducible characters of $G_2(q)$ was determined by Hiss. Based on results of Hiss and Shamash about character tables, decomposition numbers, and Brauer trees of $G_2(q)$, we found the second smallest degree of irreducible characters of $G_2(q)$ in cross characteristic. This second degree plays a crucial role in solving Problem A when $G = G_2(q)$. Another ingredient is the modular representation theory of various large subgroups of $G_2(q)$, such as $SL_3(q)$ [64] and $SU_3(q)$ [20]. The following theorem is the focus of chapter 2.

**Theorem B.** Let $G = G_2(q)$, $q = p^n$, $q \geq 5$, $p$ a prime number. Let $\varphi$ be an irreducible character of $G$ in cross characteristic $\ell$ and $M$ a maximal subgroup of $G$. Assume that $\varphi(1) > 1$. Then $\varphi|_M$ is irreducible if and only if one of the following holds:

(i) $q \equiv -1 (\text{mod } 3)$, $M = SL_3(q) : 2$ and $\varphi$ is the unique character of the smallest degree $q^3 - 1$.

(ii) $q \equiv 1 (\text{mod } 3)$, $M = SU_3(q) : 2$ and $\varphi$ is the unique character of the smallest degree.

In this case, $\varphi(1) = q^3$ when $\ell = 3$ and $\varphi(1) = q^3 + 1$ when $\ell \neq 3$.

The solution of Problem A for Suzuki and Ree groups is less complicated than the case of $G_2(q)$ and straightforward. The next two theorems are proved in chapter 3.

**Theorem C.** Let $G = Sz(q) = ^2B_2(q)$ be the Suzuki group where $q = 2^n$ and $n \geq 3$ is odd. Let $\varphi$ be an irreducible character of $G$ in characteristic $\ell \neq 2$ and $M$ a maximal subgroup of $G$. Assume that $\varphi(1) > 1$. Then $\varphi|_M$ is irreducible if and only if $M$ is $G$-conjugate to the maximal parabolic subgroup of $G$ and $\varphi$ is the reduction modulo $\ell$ of any of the two irreducible complex characters of degree $(q - 1)\sqrt{q/2}$.
**Theorem D.** Let $G = {^2}G_2(q)$ be the Ree group where $q = 3^n$ and $n \geq 3$ is odd. Let $\varphi$ be an irreducible character of $G$ in characteristic $\ell \neq 3$ and $M$ a maximal subgroup of $G$. Assume that $\varphi(1) > 1$. Then $\varphi|_M$ is irreducible if and only if $M$ is $G$-conjugate to the maximal parabolic subgroup of $G$ and $\varphi$ is the nontrivial constituent (of degree $q^2 - q$) of the reduction modulo $\ell = 2$ of the unique irreducible complex character of degree $q^2 - q + 1$.

Unlike the cases of $G_2(q)$, $^2B_2(q)$, and $^2G_2(q)$, the cross-characteristic representation theory of $^3D_4(q)$ is still not very well understood. To solve Problem A for the case $^3D_4(q)$, we need to apply fundamental results of Broué and Michel [4] on unions of $\ell$-blocks, as well as results of Geck [21] on basic sets of Brauer characters in $\ell$-blocks. Recent results of Himstedt [26] on character tables of parabolic subgroups of $^3D_4(q)$ have also proved useful. It turns out that there are no irreducible restrictions for $^3D_4(q)$ and this will be shown in chapter 4.

**Theorem E.** Let $G = ^3D_4(q)$ and let $\varphi$ be any irreducible representation of $G$ in characteristic $\ell$ coprime to $q$. If $M$ is any proper subgroup of $G$ and $\deg(\varphi) > 1$, then $\varphi|_M$ is reducible.

Now we move on to the problem of determining low-dimensional complex representations, or equivalently, low-dimensional complex characters. Let us temporarily denote by $G$ either the symplectic groups $Sp_{2n}(q)$ or the orthogonal groups $Spin_n^{\pm}(q)$. The smallest nontrivial complex character of $G$ was determined by Tiep and Zalesskii in [70]. Furthermore, when $G$ is the odd characteristic symplectic group, they classified all irreducible complex characters of degrees less than $(q^{2n} - 1)/2(q + 1)$. It turns out that, up to this bound, $G$ has four irreducible characters of degrees $(q^n \pm 1)/2$, which are the so-called Weil characters, and the smallest unipotent character of degree $(q^n - 1)(q^n - q)/2(q + 1)$.

We want to extend these results to a larger bound. More precisely, we classify the irreducible complex characters of $G$ of degrees up to the bound $D$, where $D = (q^n - 1)q^{4n-10}/2$ for symplectic groups, $D = q^{4n-8}$ for orthogonal groups in odd dimension, and
$D = q^{4n-10}$ for orthogonal groups in even dimension. When $q$ is even, the low-dimensional complex characters of $Sp_{2n}(q) \cong Spin_{2n+1}(q)$ were classified up to an already good enough bound $(q^{2n} - 1)(q^{n-1} - 1)(q^{n-1} - q^2)/2(q^4 - 1)$ in [24] and therefore it is not in our consideration.

Our main results on this topic are the following theorems, which will be proved in chapters 5 and 6. We note that $\pm$ could be understood as $\pm 1$ and vice versa, depending on the context.

**Theorem F.** Let $\chi$ be an irreducible complex character of $G = Sp_{2n}(q)$, where $n \geq 6$ and $q$ is an odd prime power. Then either $\chi(1) > (q^n - 1)q^{4n-10}/2$ or $\chi$ belongs to an explicit list of $q^2 + 12q + 36$ characters displayed at the Tables 5-1, 5-2, 5-3 (at the end of chapter 5).

**Theorem G.** Let $\chi$ be an irreducible complex character of $G = Spin_{2n+1}(q)$, where $n \geq 5$ and $q$ is an odd prime power. Then $\chi(1) = 1$ (1 character), $\chi(1) = (q^{2n} - 1)/(q^2 - 1)$ (1 character), $\chi(1) = q(q^{2n} - 1)/(q^2 - 1)$ (1 character), $\chi(1) = (q^n + \alpha_1)(q^n + \alpha_2)/2(q + \alpha_1\alpha_2)$ (4 characters, $\alpha_{1,2} = \pm 1$), $\chi(1) = (q^{2n} - 1)/(q + \alpha)$ ($(q + \alpha - 2)/2$ characters for each $\alpha = \pm 1$), or $\chi(1) > q^{4n-8}$.

**Theorem H.** Let $\chi$ be an irreducible complex character of $G = \Omega_{2n}^\pm(q)$, where $n \geq 5, \alpha = \pm, (n, q, \alpha) \neq (5, 2, +)$, and $q$ is a power of 2. Let

$$D(n, q, \alpha) = \begin{cases} (q - 1)(q^2 + 1)(q^3 - 1)(q^4 + 1), & n = 5, \alpha = -, \\ q^{4n-10} + 1, & \text{otherwise.} \end{cases}$$

Then $\chi(1) = 1$ (1 character), $\chi(1) = (q^n - \alpha)(q^{n-1} + \alpha q)/(q^2 - 1)$ (1 character), $\chi(1) = (q^{2n} - q^2)/(q^2 - 1)$ (1 character), $\chi(1) = (q^n - \alpha)(q^{n-1} + \alpha \beta)/(q - \beta)$ ($(q - \beta - 1)/2$ characters for each $\beta = \pm 1$), or $\chi(1) \geq D(n, q, \alpha)$.

Furthermore, when $(n, \alpha) = (5, -)$, $G$ has exactly $q$ characters of degree $D(5, q, -)$, 1 character of degree $q^2(q^4 + 1)(q^5 + 1)/(q + 1)$, and no more characters of degrees up to $q^{10}$. 

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Theorem I. Let \( \chi \) be an irreducible complex character of \( G = \text{Spin}_{2n}^\alpha(q) \), where \( \alpha = \pm \), \( n \geq 5 \), and \( q \) is an odd prime power. Let

\[
D(n, q, \alpha) = \begin{cases} 
(q - 1)(q^2 + 1)(q^3 - 1)(q^4 + 1), & n = 5, \alpha = -, \\
q^{4n-10} + 1, & \text{otherwise.}
\end{cases}
\]

Then \( \chi(1) = 1 \) (1 character), \( \chi(1) = (q^n - \alpha)(q^{n-1} + \alpha q)/(q^2 - 1) \) (1 character), \( \chi(1) = (q^n - q^2)/(q^2 - 1) \) (1 character), \( \chi(1) = (q^n - \alpha)(q^{n-1} - \alpha)/2(q + 1) \) (2 characters), \( \chi(1) = (q^n - \alpha)(q^{n-1} + \alpha)/2(q - 1) \) (2 characters), \( \chi(1) = (q^n - \alpha)(q^{n-1} + \alpha \beta)/(q - \beta) \) ((\( q - \beta - 2 \))/2 characters for each \( \beta = \pm 1 \)), or \( \chi(1) \geq D(n, q, \alpha) \). Furthermore, when \((n, \alpha) = (5, -)\), \( G \) has exactly \( q \) characters of degree \( D(5, q, -) \), 1 character of degree \( q^2(q^4 + 1)(q^5 + 1)/(q + 1) \), and no more characters of degrees up to \( q^{10} \).
CHAPTER 2
IRREDUCIBLE RESTRICTIONS FOR $G_2(q)$

Let $G$ be a finite group of Lie type $G_2(q)$ defined over a finite field with $q = p^n$ elements, where $p$ is a prime number and $n$ is a positive integer. Let $M$ be a maximal subgroup of $G$ and let $\varphi$ be an absolutely irreducible representation of $G$ in cross characteristic $\ell$. The purpose of this chapter is to find all possibilities of $\varphi$ and $M$ such that $\varphi|_M$ is also irreducible.

In order to do that, we use the results about maximal subgroups, character tables, blocks, and Brauer trees of $G$ obtained by many authors. The list of maximal subgroups of $G$ is determined by Cooperstein in [13] for $p = 2$ and Kleidman in [39] for $p$ odd.

The complex character table of $G$ is determined by Chang and Ree ([10]) for $p \geq 5$, by Enomoto ([16]) for $p = 3$, and by Enomoto and Yamada ([17]) for $p = 2$. In a series of papers [29], [33], [34], [61], [62], and [63], Hiss and Shamash have determined the blocks, Brauer trees and (almost completely) the decomposition numbers for $G$.

In order to solve our problem, it turns out to be useful to know the degrees of low-dimensional irreducible Brauer characters of $G$. The smallest (or the first) degree of non-trivial characters of $G$ was determined by Hiss and will be used here. For the second smallest degree, we compare directly the degrees of irreducible Brauer characters and get the exact formula for it which is given in Theorem 2.2.1. Using the first degree of $G$, we can exclude many maximal subgroups of $G$ by Reduction Theorem 2.3.4. The remaining maximal subgroups are treated individually by various tools, including the second degree and modular representation theory of large subgroups of $G$ such as $SL_3(q)$ [64] and $SU_3(q)$ [20].

This chapter is organized as follows. In §1, we state some lemmas which will be used later. In §2, we collect the degrees of irreducible Brauer characters and give formulas for the first and second degrees of $G_2(q)$. §3 is devoted to prove Theorem B. Small groups $G_2(3)$ and $G_2(4)$ and their covers will be treated in §4.
2.1 Preliminaries

We record a few statements which will be used throughout until chapter 4. Some of these statements are well-known but we include their proofs for completeness. Hereafter, $\mathbb{F}$ is an algebraically closed field of characteristic $\ell$.

**Lemma 2.1.1.** Let $G$ be a finite group. Suppose $V$ is an irreducible $\mathbb{F}G$-module of dimension greater than one and $H$ is a proper subgroup of $G$ such that the restriction $V|_H$ is irreducible. Then

$$\sqrt{|H/Z(H)|} \geq m_c(H) \geq m_\ell(H) \geq \dim(V) \geq d_\ell(G).$$

**Proof.** Suppose $\varphi \in \text{IBr}_\ell(H)$ of degree $m_\ell(H)$. Then there exists $\chi \in \text{Irr}(H)$ such that $d_{\chi\varphi} \neq 0$, where $d_{\chi\varphi}$ is the decomposition number associated with $\chi$ and $\varphi$. Since $\hat{\chi} = \sum_{\varphi' \in \text{IBr}_\ell(H)} d_{\chi\varphi}\varphi'$, it follows that $m_c(H) \geq \chi(1) = \hat{\chi}(1) \geq \varphi(1) = m_\ell(H)$. Other inequalities are obvious. \qed

**Lemma 2.1.2.** Let $G$ be a simple group and $V$ an irreducible $\mathbb{F}G$-module of dimension greater than one. Then $Z_G(V) := \{g \in G \mid g|_V = \lambda \cdot Id_V \text{ for some } \lambda \in \mathbb{F}\} = 1$.

**Proof.** Let $\mathfrak{X}$ be the $\mathbb{F}G$-representation associated with $V$. Then $\text{Ker}(\mathfrak{X}) = \{g \in G \mid \mathfrak{X}(g) = \mathfrak{X}(1)\}$ is a normal subgroup of $G$. Since $G$ is simple, $\text{Ker}(\mathfrak{X})$ is either trivial or the whole group $G$. The case $\text{Ker}(\mathfrak{X}) = G$ cannot happen since $\mathfrak{X}$ is non-trivial and irreducible. So, $\text{Ker}(\mathfrak{X}) = 1$.

Assume $g_0$ is any element in $Z_G(V)$. Then $\mathfrak{X}(g_0)$ is a scalar matrix and therefore it commutes with $\mathfrak{X}(g)$ for every $g \in G$. Hence $g_0$ commutes with $g$ for every $g \in G$ since $\text{Ker}(\mathfrak{X}) = 1$. In other words, $Z_G(V) \subseteq Z(G)$, which implies the lemma. \qed

**Lemma 2.1.3.** Let $G$ be a simple group and $V$ an irreducible $\mathbb{F}G$-module of dimension greater than one. Suppose $H$ is a subgroup of $G$ such that $V|_H$ is irreducible. Then $Z(H) = C_G(H) = 1$.

**Proof.** This is a corollary of Lemma 2.1.2. \qed
Lemma 2.1.4. Let $G$ be a finite group and $1 \neq A \trianglelefteq H \leq G$. Let $V$ be a faithful irreducible $\mathbb{F}G$-module.

(i) Suppose that $C_V(A) := \{v \in V \mid a(v) = v \text{ for every } a \in A\} \neq 0$. Then $V|_H$ is reducible.

(ii) Suppose that $O_\ell(H) \neq 1$. Then $V|_H$ is reducible.

Proof. (i) We will argue by contradiction. Suppose that $V|_H$ is irreducible. By Clifford’s theorem, $V|_A = e \bigoplus_{i=1}^t V_i$ where $e$ is the multiplicity of $V_i$ in $V$ and $\{V_1, ..., V_t\}$ is the $H$-orbit of $V_1$ under the action of $H$ on the set of all irreducible $\mathbb{F}A$-modules. Since $A$ acts trivially on $C_V(A) \neq 0$, at least one of the $V_i$ is the trivial $A$-module. It follows that all $V_1, ..., V_t$ are trivial $A$-modules and therefore $A$ acts trivially on $V$. This contradicts the faithfulness of $V$.

(ii) Assume the contrary: $V|_H$ is irreducible. Then it is well-known that $O_\ell(H)$ acts trivially on $V$. It follows that $C_V(O_\ell(H)) = V$. This and the hypothesis $O_\ell(H) \neq 1$ lead to a contradiction by (i). \qed

Lemma 2.1.5. Let $G$ be a finite group and $\chi \in \text{Irr}(G)$. Let $H$ be a normal $\ell'$-subgroup of $G$ and suppose that $\chi|_H = \sum_{i=1}^t \theta_i$, where $\theta_1$ is irreducible and $\theta_1, \theta_2, ..., \theta_t$ are the distinct $G$-conjugates of $\theta_1$. Then $\hat{\chi}$ is also irreducible.

Proof. We have $\hat{\chi}|_H = \sum_{i=1}^t \theta_i$ as $H$ is an $\ell'$-group. Let $\psi$ be an irreducible constituent of $\hat{\chi}$. Then there is an $i \in \{1, 2, ..., t\}$ such that $\theta_i$ is an irreducible constituent of $\psi|_H$. Therefore, by Clifford’s theory, all $\theta_1, \theta_2, ..., \theta_t$ are contained in $\psi|_H$. This implies that $\hat{\chi}|_H = \psi|_H$. So $\hat{\chi} = \psi$ as desired. \qed

Lemma 2.1.6. Let $\tau$ be an automorphism of a finite group $G$ which fuses two conjugacy classes of subgroups of $G$ with representatives $M_1, M_2$. Let $A$ and $B$ be subsets of $\text{IBr}_\ell(G)$ such that $\tau(A) = B$, $\tau(B) = A$.

(i) Assume $\varphi|_{M_1}$ is irreducible (resp. reducible) for all $\varphi \in A \cup B$. Then $\varphi|_{M_2}$ is irreducible (resp. reducible) for all $\varphi \in A \cup B$. 

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(ii) Assume \( \varphi|_{M_1} \) is irreducible for all \( \varphi \in A \) and \( \varphi|_{M_1} \) is reducible for all \( \varphi \in B \). Then \( \varphi|_{M_2} \) is reducible for all \( \varphi \in A \) and \( \varphi|_{M_2} \) is irreducible for all \( \varphi \in B \).

**Proof.** It is enough to show that \( \varphi|_{M_1} \) is irreducible (resp. reducible) for all \( \varphi \in A \) if and only if \( \varphi|_{M_2} \) is irreducible (resp. reducible) for all \( \varphi \in B \). This is true because for every \( \varphi \in A \), we have \( \varphi|_{M_1} = \psi \circ \tau|_{\tau(M_2)} = (\psi|_{M_2}) \circ \tau \) for some \( \psi \in B \). \( \square \)

**Lemma 2.1.7.** Let \( G \) be a finite group. Suppose that the universal cover of \( G \) is \( M.G \) where \( M \) is the Schur multiplier of \( G \) of prime order. Then every maximal subgroup of \( M.G \) is the pre-image of a maximal subgroup of \( G \) under the natural projection \( \pi : M.G \to G \).

**Proof.** Let \( H \) be a maximal subgroup of \( M.G \). Then \( \pi(H) \) is a subgroup of \( G \). We will show that \( \pi(H) \) is a proper subgroup of \( G \). Assume the contrary \( \pi(H) = G \). First suppose that there exists an element \( g \in G \) such that \( \pi^{-1}(g) \cap H \) consists of at least two different elements, say \( g_1 \) and \( g_2 \). Then \( g_2 = z.g_1 \) where \( 1 \neq z \in M \). Since \( |M| \) is prime, \( < z > = M \) and therefore \( M \leq H \). It follows that \( H = M.G \) and we get a contradiction.

Next we suppose that for each \( g \in G \), \( \pi^{-1}(g) \cap H \) consists of exactly one element. Then \( \pi|_H \) is an isomorphism. In other words, \( H \simeq G \) and \( M.G = M : H \) becomes a split extension. Without loss we may identify \( H \) with \( G \). Now every projective complex irreducible representation \( \Phi \) of \( G \) lifts to a linear representation \( \Psi \) of \( M : G \). Then \( \Phi \) also lifts to the linear representation \( \Psi|_G \) of \( G \). Thus the Schur multiplier \( M \) of \( G \) is trivial, a contradiction again. We have shown that \( \pi(H) \) is a proper subgroup of \( G \). Then \( \pi(H) \) is contained in a maximal subgroup of \( G \), say \( K \). We have \( H \leq \pi^{-1}(K) \). Since \( H \) is maximal, \( H = \pi^{-1}(K) \) and we are done. \( \square \)

**Lemma 2.1.8.** [18] Let \( G \) be a finite group and \( H \triangleleft G \). Suppose \( |G : H| = p \) is prime and \( \chi \in \text{IBr}_\ell(G) \). Then either

(i) \( \chi|_H \) is irreducible or

(ii) \( \chi|_H = \sum_{i=1}^{p} \theta_i \), where the \( \theta_i \) are distinct and irreducible.
Lemma 2.1.9. [35, p. 190] Let $G$ be a finite group and $H$ be a normal subgroup of $G$. Let $\chi \in \text{Irr}(G)$ and $\theta \in \text{Irr}(H)$ be a constituent of $\chi|_H$. Then $\chi(1)/\theta(1)$ divides $|G/H|$. 

Lemma 2.1.10. [56] Let $B$ be an $\ell$-block of group $G$. Assume that all $\chi \in B \cap \text{Irr}(G)$ are of the same degree. Then $B \cap \text{IBr}_\ell(G) = \{ \phi \}$ and $\hat{\chi} = \phi$ for every $\chi \in B \cap \text{Irr}(G)$.

Lemma 2.1.11. [69, Theorem 1.6] Let $G$ be a finite group of Lie type, of simply connected type. Assume that $G$ is not of type $A_1, A_2, B_2, G_2, B_2$. If $Z$ is a long-root subgroup and $V$ is a nontrivial irreducible representation of $G$, then $Z$ must have nonzero fixed points on $V$.

### 2.2 Degrees of Irreducible Brauer Characters of $G_2(q)$

In this section, we collect the degrees of irreducible characters of $G_2(q)$ (both complex and Brauer characters) obtained by many people. Then we will recall the value of $d_\ell(G)$ and determine the value of $d_{2,\ell}(G)$ when $\ell$ is coprime to $q$.

The degrees of irreducible complex characters of $G_2(q)$ can be read off from [10], [16], [17] and are listed in Table 2-1. From this table, we get

$$d_C(G) = \begin{cases} \frac{q^3 + 1}{6}, & q \equiv 1(\text{mod } 3), \\ q^3 - 1, & q \equiv 2(\text{mod } 3), \\ q^4 + q^2 + 1, & q \equiv 0(\text{mod } 3), \end{cases} \quad (2.1)$$

and

$$d_{2,C}(G) = \begin{cases} \frac{1}{6}q(q - 1)^2(q^2 - q + 1), & p = 2, 3 \text{ or } q = 5, 7, \\ q^4 + q^2 + 1, & p \geq 5, q > 7, \end{cases} \quad (2.2)$$

for every $q \geq 5$. Moreover, $\text{Irr}(G)$ contains a unique character of degree larger than 1 but less than $d_{2,C}(G)$ and this character has degree $d_C(G)$.

The degrees of irreducible $\ell$-Brauer characters of $G_2(q)$ when $\ell \mid |G|$ and $\ell \not| q$ can be read off from [29], [33], [34], [61], [62], and [63]. They are listed in Tables 2-2, 2-3, 2-4, and 2-5. Comparing these degrees directly, we easily get the value of $d_\ell(G)$ when $\ell \mid |G|$, $\ell \not| q$. Combining this value with formula 2.1, we get that if $q \geq 5$ and $\ell \not| q$ then
\[
\mathfrak{d}_\ell(G) = \begin{cases}
q^3 + 1, & q \equiv 1(\text{mod } 3), \ell \neq 3, \\
q^3, & q \equiv 1(\text{mod } 3), \ell = 3, \\
q^3 - 1, & q \equiv 2(\text{mod } 3), \forall \ell, \\
q^4 + q^2, & q \equiv 0(\text{mod } 3), \ell = 2, \\
q^4 + q^2 + 1, & q \equiv 0(\text{mod } 3), \ell \neq 2.
\end{cases}
\] (2.3)

When \(\ell \nmid |G|\), we know that \(\text{IBr}_\ell(G) = \text{Irr}(G)\) and the value of \(\mathfrak{d}_{2,C}(G)\) is given in formula (2.2). Formula (2.2) and direct comparison of the degrees of irreducible \(\ell\)-Brauer characters of \(G\) when \(\ell \nmid |G|\) yield the following theorem. We omit the details of this direct computation.

**Theorem 2.2.1.** Let \(q \geq 5\). We have

\[
\mathfrak{d}_{2,2}(G) = \begin{cases}
\frac{1}{6}q(q-1)^2(q^2 - q + 1), & p = 3 \text{ or } q = 5, 7, \\
q^4 + q^2, & p \geq 5, q \geq 11,
\end{cases}
\]

\[
\mathfrak{d}_{2,3}(G) = \begin{cases}
= \frac{1}{6}q(q-1)^2(q^2 - q + 1), & q = 5, 7, \text{ or } p = 2, q \equiv -1(\text{mod } 3), \\
= q^4 + q^2 + 1, & p \geq 5, q \equiv -1(\text{mod } 3) \text{ and } q \geq 11, \\
\geq q^4 - q^3 + q^2, & q \geq 13 \text{ and } q \equiv 1(\text{mod } 3),
\end{cases}
\]

and if \(\ell = 0\) or \(\ell \geq 5\), \(\ell \nmid q\) then

\[
\mathfrak{d}_{2,\ell}(G) = \mathfrak{d}_{2,C}(G) = \begin{cases}
\frac{1}{6}q(q-1)^2(q^2 - q + 1), & p = 2, 3 \text{ or } q = 5, 7, \\
q^4 + q^2 + 1, & p \geq 5, q \geq 11.
\end{cases}
\]

Moreover, \(\text{IBr}_\ell(G)\) contains a unique character denoted by \(\psi\) of degree larger than 1 but less than \(\mathfrak{d}_{2,\ell}(G)\) and this character has degree \(\mathfrak{d}_\ell(G)\).

Also from the results about blocks and Brauer trees of \(G_2(q)\), we see that if \(3 \nmid q\) then \(\psi = \overline{X}_{32}\) in all cases except when \(\ell = 3\) and \(q \equiv 1(\text{mod } 3)\) where \(\psi = \overline{X}_{32} - \overline{1}_G\). If \(3 \mid q\) then \(\psi = \overline{X}_{22}\) except when \(\ell = 2\) where \(\psi = \overline{X}_{22} - \overline{1}_G\). We also notice that \(\varphi_{18} = \overline{X}_{18}\) is irreducible in all cases.
2.3 Proofs

In the next Lemmas, we use the notation in [17], [20], [30], and [64].

Lemma 2.3.1. (i) When $q \equiv -1(\text{mod } 3)$, the character $X_{32}|_{SL_3(q)}$ is irreducible and equal to $\chi_{\frac{q^2-1}{3}}$.

(ii) When $q \equiv 1(\text{mod } 3)$, the character $X_{32}|_{SU_3(q)}$ is irreducible and equal to $\chi_{\frac{q^2-1}{3+1}}$.

Proof. (i) First we assume $q$ is odd. We need to find the fusion of conjugacy classes of $SL_3(q)$ in $G$. By [64, p. 487], $SL_3(q)$ has the following conjugacy classes: $C_1^{(0)}$, $C_2^{(0)}$, $C_3^{(0,0)}$, $C_4^{(k)}$, $C_5^{(k)}$, $C_6^{(k,l,m)}$, $C_7^{(k)}$ and $C_8^{(k)}$. For any $x \in G$, we denote by $K(x)$ the conjugacy class containing $x$. Then we have $C_1^{(0)} \subseteq K(1)$, $C_2^{(0)} \subseteq K(u_1)$, $C_3^{(0,0)} \subseteq K(u_6)$, $C_4^{(k)} \subseteq K(k_2) \cup K(h_{1b})$, $C_5^{(k)} \subseteq K(k_{2,1}) \cup K(h_{1b,1})$, $C_6^{(k,l,m)} \subseteq K(h_1)$, $C_7^{(k)} \subseteq K(h_b) \cup K(h_{2b})$ and $C_8^{(k)} \subseteq K(h_3)$. Taking the values of $X_{32}$ and $\chi_{\frac{q^2-1}{3}}$ on these classes, we get

$$
\chi_{\frac{q^2-1}{3}}(C_1^{(0)}) = X_{32}(1) = q^3 - 1,
$$
$$
\chi_{\frac{q^2-1}{3}}(C_2^{(0)}) = X_{32}(u_1) = -1,
$$
$$
\chi_{\frac{q^2-1}{3}}(C_3^{(0,0)}) = X_{32}(u_6) = -1,
$$
$$
\chi_{\frac{q^2-1}{3}}(C_4^{(k)}) = X_{32}(k_2) = X_{32}(h_{1b}) = q - 1,
$$
$$
\chi_{\frac{q^2-1}{3}}(C_5^{(k)}) = X_{32}(k_{2,1}) = X_{32}(h_{1b,1}) = -1,
$$
$$
\chi_{\frac{q^2-1}{3}}(C_6^{(k,l,m)}) = X_{32}(h_1) = 0,
$$
$$
\chi_{\frac{q^2-1}{3}}(C_7^{(k)}) = X_{32}(h_b) = X_{32}(h_{2b}) = -(\theta^{k \frac{q^2-1}{3}} + \theta^{k \frac{q^2+1}{3+1}}) = (\omega^k + w^{-k}),
$$
$$
\chi_{\frac{q^2-1}{3}}(C_8^{(k)}) = X_{32}(h_3) = 0,
$$

where $\theta$ is a primitive $(q^2 - 1)$-root of unity and $\omega = \theta^{\frac{q^2-1}{3}}$. We have shown that

$$
X_{32}|_{SL_3(q)} = \chi_{\frac{q^2-1}{3}} \text{ if } q \equiv -1(\text{mod } 3) \text{ and } q \text{ odd. The case } q \text{ is even is proved similarly.}
$$

(ii) Now we show that $X_{32}|_{SU_3(q)}$ is irreducible when $q \equiv 1(\text{mod } 3)$. More precisely,

$$
X_{32}|_{SU_3(q)} = \chi_{\frac{q^2-1}{3+1}}.
$$

Let us consider the case $q$ is odd. We need to find the fusion of conjugacy classes of $SU_3(q)$ in $G$. From [20, p. 565], $SU_3(q)$ has the following conjugacy classes: $C_1^{(0)}$, $C_2^{(0)}$, $C_3^{(0,0)}$, $C_4^{(k)}$, $C_5^{(k)}$, $C_6^{(k,l,m)}$, $C_7^{(k)}$ and $C_8^{(k)}$. We have $C_1^{(0)} \subseteq K(1)$, $C_2^{(0)} \subseteq K(u_1)$, $C_3^{(0,0)} \subseteq K(u_4)$, $C_4^{(k)} \subseteq K(k_2) \cup K(h_{2b})$, $C_5^{(k)} \subseteq K(k_{2,1}) \cup K(h_{2a,1})$, and
$C_{6}^{(k,l,m)} \subseteq K(h_{2})$, $C_{7}^{(k)} \subseteq K(h_{a}) \cup K(h_{1a})$ and $C_{8}^{(k)} \subseteq K(h_{6})$. Taking the values of $X_{32}$ and $\chi_{q^{3}+1}^{(\frac{q^{2}-1}{3})}$ on these classes, we get

$$
\chi_{q^{3}+1}^{(\frac{q^{2}-1}{3})}(C_{1}^{(0)}) = X_{32}(1) = q^{3} + 1,
\chi_{q^{3}+1}^{(\frac{q^{2}-1}{3})}(C_{2}^{(0)}) = X_{32}(u_{1}) = 1,
\chi_{q^{3}+1}^{(\frac{q^{2}-1}{3})}(C_{3}^{(0,0)}) = X_{32}(u_{4}) = 1,
\chi_{q^{3}+1}^{(\frac{q^{2}-1}{3})}(C_{4}^{(k)}) = X_{32}(h_{2}) = X_{32}(h_{2a}) = q + 1,
\chi_{q^{3}+1}^{(\frac{q^{2}-1}{3})}(C_{5}^{(k)}) = X_{32}(h_{2a}) = X_{32}(h_{2a,1}) = 1,
\chi_{q^{3}+1}^{(\frac{q^{2}-1}{3})}(C_{6}^{(k,l,m)}) = X_{32}(h_{2}) = 0,
\chi_{q^{3}+1}^{(\frac{q^{2}-1}{3})}(C_{7}^{(k)}) = X_{32}(h_{a}) = X_{32}(h_{1a}) = (\theta^{k\frac{q^{2}-1}{3}} + \theta^{-qk\frac{q^{2}-1}{3}}) = (\omega^{k} + w^{-k}),
\chi_{q^{3}+1}^{(\frac{q^{2}-1}{3})}(C_{8}^{(k)}) = X_{32}(h_{6}) = 0,
$$

where $\theta$ is a primitive $(q^{2} - 1)$-root of unity and $\omega = \theta^{\frac{q^{2}-1}{3}}$. We have shown that, when $q \equiv 1(\text{mod} \ 3)$ and $q$ is odd, $X_{32}|_{SU_{3}(q)} \in \text{Irr}(SU_{3}(q))$ and $X_{32}|_{SU_{3}(q)} = \chi_{q^{3}+1}^{(\frac{q^{2}-1}{3})}$. The case $q$ even is proved similarly. We omit the details.

**Lemma 2.3.2.** Suppose that $q \equiv -1(\text{mod} \ 3)$. Then $\chi_{q^{3}-1}^{(\frac{q^{2}-1}{3})} \in \text{IBr}_{\ell}(SL_{3}(q))$ for $\ell \nmid q$.

**Proof.** Since $q \equiv -1(\text{mod} \ 3)$, $GL_{3}(q) = SL_{3}(q) \times Z(GL_{3}(q))$ with $Z(GL_{3}(q)) \simeq \mathbb{Z}_{q-1}$. Let $V$ be a $\mathbb{C}GL_{3}(q)$-module affording the irreducible character $\chi_{q^{3}-1}^{(\frac{q^{2}-1}{3})} \cdot 1_{\mathbb{Z}_{q-1}}$. We have $\dim V = q^{3} - 1$. According to [23, §4], $V$ has the form $(S_{C}(s, 1) \circ S_{C}(t, 1)) \uparrow G$ where $s \in \mathbb{F}_{q}^{\times}$ and $t$ has degree 2 over $\mathbb{F}_{q}$. Using Corollary 2.7 of [23], since 1 and 2 are coprime, we see that $V$ is irreducible in any cross characteristic. This implies that $V|_{SL_{3}(q)}$ is also irreducible in any cross characteristic and therefore $\chi_{q^{3}-1}^{(\frac{q^{2}-1}{3})} \in \text{IBr}_{\ell}(SL_{3}(q))$ for every $\ell \nmid q$.

**Note:** 2-Brauer characters of $SU_{3}(q)$ were not considered in [20].

**Lemma 2.3.3.** Suppose that $q \equiv 1(\text{mod} \ 3)$. Then $\chi_{q^{3}+1}^{(\frac{q^{2}-1}{3})} \in \text{IBr}_{2}(SU_{3}(q))$ when $\ell = 2$.

**Proof.** We denote the character $\chi_{q^{3}+1}^{(\frac{q^{2}-1}{3})}$ by $\rho$ for short. Since $q \equiv 1(\text{mod} \ 3)$, $GU_{3}(q) = SU_{3}(q) \times Z(GU_{3}(q))$ with $Z(GU_{3}(q)) \simeq \mathbb{Z}_{q+1}$. Hence, $SU_{3}(q)$ has the same degrees
of irreducible Brauer characters as $GU_3(q)$ does. It is shown in [72] that every $\varphi \in \text{IBr}_2(GU_3(q))$ either lifts to characteristic 0 or $\varphi(1) = q(q^2 - q + 1) - 1$. This and the character table of $SU_3(q)$ gives the possible values for degrees of irreducible 2-Brauer characters of $SU_3(q)$: $1, q^2 - q, q^2 - q + 1, q(q^2 - q + 1) - 1, q(q^2 - q + 1), (q - 1)(q^2 - q + 1), q^3, q^3 + 1, \text{and } (q + 1)^2(q - 1)$.

Assume $\hat{\rho}$ is reducible when $\ell = 2$. Then $\hat{\rho}$ is the sum of more than one irreducible 2-Brauer characters of $SU_3(q)$. In other words, $\rho(1) = q^3 + 1$ is the sum of more than one values listed above. This implies that $\hat{\rho}$ must include a character of degree either 1, or $q^2 - q$, or $q^2 - q + 1$. Once again, by [72], this character lifts to a complex character that we denote by $\alpha$. Clearly, $\rho$ and $\alpha$ belong to the same 2-block of $SU_3(q)$. We will use central characters to show that this can not happen.

Let $R$ be the full ring of algebraic integers in $\mathbb{C}$ and $\pi$ a maximal ideal of $R$ containing $2R$. It is known that $\alpha$ and $\rho$ are in the same 2-block if and only if

$$\omega_{\rho}(K) - \omega_{\alpha}(K) \in \pi,$$

(3.4)

where $K$ is any class sum and $\omega_\chi$ is the central character associated with $\chi$. The value of $\omega_\chi$ on a class sum is given below:

$$\omega_\chi(K) = \frac{\chi(g)|K|}{\chi(1)},$$

where $K$ is the conjugacy class with class sum $K$ and $g$ is an element in $K$. Therefore, (3.4) implies that

$$\frac{\rho(g)}{\rho(1)}|g^G| - \frac{\alpha(g)}{\alpha(1)}|g^G| \in \pi,$$

(3.5)

where $g \in G$ and $|g^G|$ denotes the length of the conjugacy class containing $g$.

Consider the first case when $\alpha(1) = 1$. It means that $\alpha$ is the trivial character. In (3.5), take $g$ to be any element in the conjugacy class $C_\gamma^{(1)}$ (notation in [20]), we have

$$\frac{\rho(g)}{\rho(1)}|g^G| - \frac{\alpha(g)}{\alpha(1)}|g^G| = -\frac{1}{q + 1}q^3(q^3 + 1) - \frac{1}{q^2 + 1}q^3(q^3 + 1) = -q^3 - q^3(q^3 + 1) \in \pi. \text{ Note that } \pi \cap \mathbb{Z} = 2\mathbb{Z}. \text{ Since } -q^3 - q^3(q^3 + 1) \text{ is an odd number, we get a contradiction.}$$

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Secondly, if $\alpha(1) = q^2 - q$ then $\alpha = \chi_{q^2-q}$. In (3.5), take $g$ to be any element in the conjugacy class $C^{(1)}_{7}$, we have $\frac{\varphi(g)}{\varphi(1)}|gG| - \frac{\alpha(g)}{\alpha(1)}|gG| = \frac{1}{q^{3+1}}q^3(q^3 + 1) - \frac{0}{q^2-q}q^3(q^3 + 1) = -q^3 \in \pi$. Since $-q^3$ is an odd number, we again get a contradiction.

Finally, if $\alpha(1) = q^2 - q + 1$ then $\alpha = \chi^{(u)}_{q^2-q+1}$ for some $1 \leq u \leq q$. In (3.5), take $g$ to be any element in the conjugacy class $C^{(q+1)}_{7}$. Note that since $q \equiv 1(\mod 3)$, we have $\rho(G) - \alpha(G) = -1q^3 + 1q^3(q^3 + 1) - \frac{1}{q^2-q+1}q^3(q^3 + 1) = -q^3 - q^3(q+1) \in \pi$, which leads to a contradiction since $-q^3 - q^3(q+1)$ is odd. Proof is completed.

**Theorem 2.3.4** (Reduction Theorem). Let $G = G_2(q)$, $q = p^n$, $q \geq 5$, and $p$ a prime number. Let $\varphi$ be an irreducible character of $G$ in cross characteristic and $M$ a maximal subgroup of $G$. Assume that $\varphi(1) > 1$ and $\varphi|_M$ is also irreducible. Then $M$ is $G$-conjugate to one of the following groups:

(i) maximal parabolic subgroups $P_a, P_b$,
(ii) $SL_3(q) : 2$, $SU_3(q) : 2$,
(iii) $G_2(q_0)$ with $q = q_0^2$, $p \neq 3$.

**Proof.** By Lemma 2.1.1, we have $d_\ell(G) \leq \sqrt{\left|\mathbb{C}(M)\right|}$. Moreover, from formulas (2.1) and (2.3), we have $d_\ell(G) \geq q^3 - 1$ if $3 \nmid q$ and $d_\ell(G) \geq q^4 + q^2$ if $3 \mid q$ for every $\ell \mid q$. Therefore,

$$\sqrt{\left|\mathbb{C}(M)\right|} \geq \begin{cases} q^3 - 1 & \text{if } 3 \nmid q, \\ q^4 + q^2 & \text{if } 3 \mid q. \end{cases} \quad (3.6)$$

Here, we will only give the proof for the case $p \geq 5$. The proofs for $p = 2$ and $p = 3$ are similar. According to [39], if $M$ is a maximal subgroup of $G = G_2(q)$, $q = p^n$, $p \geq 5$, then $M$ is $G$-conjugate to one of the following groups:

1. $P_a, P_b$, maximal parabolic subgroups,
2. $(SL_2(q) \circ SL_2(q)) : 2$, involution centralizer,
3. $2^3 \cdot L_3(2)$, only when $p = q$,
4. $SL_3(q) : 2$, $SU_3(q) : 2$,
5. $G_2(q_0)$, $q = q_0^2$, $\alpha$ prime.
6. $PGL_2(q), p \geq 7, q \geq 11,$
7. $L_2(8), p \geq 5,$
8. $L_2(13), p \neq 13,$
9. $G_2(2), q = p \geq 5,$
10. $J_1, q = 11.$

Consider for instance the case 5) with $\alpha \geq 3$. Then $\sqrt{|M|} = \sqrt{q^6(q^8 - 1)(q^7 - 1)} < q^7 < q^{3^3} < q^3 - 1$ for every $q \geq 5$. This contradicts (3.6). The cases 2), 3), 6) - 10) are excluded similarly.

**Proof of Theorem B.** By Reduction Theorem, $M$ must be $G$-conjugate to one of the following subgroups:

(i) maximal parabolic subgroups $P_a, P_b,$
(ii) $SL_3(q) : 2, SU_3(q) : 2,$
(iii) $G_2(q_0)$ with $q = q_0^2, p \neq 3.$

**Case 1:** $M = P_a.$ Let $Z := Z(P'_a)$, the center of the derived subgroup of $P_a$. We know that $P_a$ is the normalizer of $Z$ in $G$ and therefore $Z$ is nontrivial. In fact, $Z$ is a long-root subgroup of $G$. Let $V$ be an irreducible $\mathbb{F}_G$-module affording the character $\varphi$. By Lemma 2.1.11, $Z$ must have nonzero fixed points on $V$. In other words, $C_V(Z) = \{v \in V \mid a(v) = v \text{ for every } a \in Z\} \neq 0$. Therefore $V|_{P_a}$ is reducible by Lemma 2.1.4. Equivalently, $\varphi|_{P_a}$ is reducible.

**Case 2:** $M = P_b.$ Using the results about character tables of $P_b$ in [1], [16], and [17], we have $m_C(P_b) = q(q - 1)(q^2 - 1)$ for $q \geq 5$. Suppose that $\varphi|_{P_b}$ is irreducible, then $\varphi(1) \leq q(q - 1)(q^2 - 1)$ by Lemma 2.1.1. If $3 \mid q$ then $d_G(G) \geq q^4 + q^2$ because of formula (2.3). Then we have $d_G(G) \geq q^4 + q^2 > q(q - 1)(q^2 - 1) \geq \varphi(1)$ and this cannot happen. So $q$ must be coprime to 3.

It is easy to check that $m_C(P_b) < d_{2,\ell}(G)$ for every $q \geq 8$. Therefore, when $q \geq 8$, the inequality $\varphi(1) \leq q(q - 1)(q^2 - 1)$ can hold only if $\varphi$ is the nontrivial character of smallest degree. When $q = 5$ or 7, checking directly, we see that besides the nontrivial
character of smallest degree, \( \varphi \) can be \( \varphi_{18} = \overline{X}_{18} \) of degree \( \frac{1}{6}q(q - 1)^2(q^2 - q + 1) \). Recall that \( |P_b| = q^6(q^2 - 1)(q - 1) \), which is not divisible by \( X_{18}(1) \) for \( q = 5 \) or 7. It follows that \( X_{18}|_{P_b} \) is reducible and so is \( \varphi_{18}|_{P_b} \). We have shown that the unique possibility for \( \varphi \) is the nontrivial character \( \psi \) of smallest degree when \( 3 \nmid q \). Recall that \( X_{32}(1) = q^3 + \epsilon \), which does not divide \( |P_b| \) and therefore \( X_{32}|_{P_b} \) is reducible.

If \( \ell = 3 \) and \( q \equiv 1(\text{mod } 3) \) then \( \psi = \overline{X}_{32} - \hat{1}_G \). Assume that \( \psi|_{P_b} = \overline{X}_{32}|_{P_b} - \hat{1}_{P_b} \) is irreducible. The reducibility of \( X_{32}|_{P_b} \) and the irreducibility of \( \overline{X}_{32}|_{P_b} - \hat{1}_{P_b} \) implies that \( X_{32}|_{P_b} = \lambda + \mu \) where \( \hat{\lambda} = \hat{1}_{P_b} \), \( \mu \in \text{Irr}(P_b) \) and \( \hat{\mu} \in \text{IBr}_3(P_b) \). We then have \( \mu(1) = X_{32}(1) - 1 = q^3 \), which is a contradiction since \( P_b \) has no irreducible complex character of degree \( q^3 \).

It remains to consider the case when \( \ell \neq 3 \) or \( q \) is not congruent to 1 modulo 3. Then \( \psi|_{P_b} = \overline{X}_{32}|_{P_b} \), which is reducible as noted above.

**Case 3:** \( M = \text{SL}_3(q) : 2 \). From [64], we know that \( m_\mathbb{C}(\text{SL}_3(q)) = (q+1)(q^2 + q + 1) \) for every \( q \geq 5 \). Therefore, \( m_\mathbb{C}(\text{SL}_3(q) : 2) \leq 2(q+1)(q^2 + q + 1) \). Since \( \varphi|_{\text{SL}_3(q) : 2} \) is irreducible, \( \varphi(1) \leq 2(q+1)(q^2 + q + 1) \). Similar arguments as above show that \( q \) is not divisible by 3.

By Theorem 2.2.1, the inequality \( \varphi(1) \leq 2(q+1)(q^2 + q + 1) \) can hold only if \( \varphi \) is the nontrivial character \( \psi \) of smallest degree or \( \varphi_{18} \) when \( q = 5 \). Note that \( X_{18}(1) = 280 \) and \( |\text{SL}_3(q) : 2| = 744,000 \) when \( q = 5 \) and therefore \( X_{18}(1) \nmid |\text{SL}_3(q) : 2| \). Hence, \( X_{18}|_{\text{SL}_3(q) : 2} \) is reducible and so is \( \varphi_{18}|_{\text{SL}_3(q) : 2} \) when \( q = 5 \). Again, the unique possibility for \( \varphi \) is \( \psi \) when \( 3 \nmid q \).

If \( \ell = 3 \) and \( q \equiv 1(\text{mod } 3) \) then \( \psi = \overline{X}_{32} - \hat{1}_G \). Assume that \( \psi|_{\text{SL}_3(q) : 2} \) is irreducible. Let \( V \) be an irreducible \( \overline{FG} \)-module, \( \text{char}(\mathbb{F}) = 3 \), affording the character \( \psi \). Then \( V|_{\text{SL}_3(q) : 2} \) is an irreducible \( \mathbb{F}(\text{SL}_3(q) : 2) \)-module. Let \( \sigma \) be a generator for the multiplicative group \( \mathbb{F}_q^\times \) and \( I \) be the identity matrix in \( \text{SL}(3, \mathbb{F}_q) \). Consider the matrix \( T = \sigma^{\frac{q-1}{2}} \cdot I \). We have \( \langle T \rangle = Z(\text{SL}_3(q)) \) and hence \( \langle T \rangle \leq \text{SL}_3(q) : 2 \). Since \( \text{ord}(T) = 3 \), \( \langle T \rangle \leq O_3(\text{SL}_3(q) : 2) \) and therefore \( O_3(\text{SL}_3(q) : 2) \) is nontrivial. It follows that \( V|_{\text{SL}_3(q) : 2} \) is reducible by Lemma 2.1.4.
If \( \ell \neq 3 \) and \( q \equiv 1(\text{mod } 3) \) then \( \psi = \widehat{X}_{32} \). So \( \psi|_{SL_3(q); 2} = \widehat{X}_{32}|_{SL_3(q); 2} \). Note that \( X_{32}(1) = q^3 + 1 \) and \( |SL_3(q) : 2| = 2q^3(q^3 - 1)(q^2 - 1) \). Therefore \( X_{32}(1) \nmid |SL_3(q) : 2| \) for every \( q \geq 5 \). Hence \( X_{32}|_{SL_3(q); 2} \) as well as \( \psi|_{SL_3(q); 2} \) are reducible in this case.

When \( q \equiv -1(\text{mod } 3) \), we have \( \psi = \widehat{X}_{32} \). By Lemmas 2.3.1 and 2.3.2, we get that \( \psi|_{SL_3(q)} = \widehat{X}_{32}|_{SL_3(q)} = \chi_{q^3 + 1} \in \text{IBr}_t(SL_3(q)) \) for \( \ell \nmid q \), as we claim in the item (i) of Theorem B.

**Case 4:** \( M = SU_3(q) : 2 \). According to [64], we have \( m_C(SU_3(q)) = (q + 1)^2(q - 1) \) for every \( q \geq 5 \) and therefore \( m_C(SU_3(q)) : 2 \leq 2(q + 1)^2(q - 1) \) for \( q \geq 5 \). Hence \( \varphi(1) \leq 2(q + 1)^2(q - 1) \) by the irreducibility of \( \varphi|_{SU_3(q); 2} \). Again, \( q \) must be coprime to 3.

By Theorem 2.2.1, the inequality \( \varphi(1) \leq 2(q + 1)^2(q - 1) \) can hold only if \( \varphi \) is the nontrivial character \( \psi \) of smallest degree or \( \varphi_{18} \) of degree \( \frac{1}{6}q(q - 1)^2(q^2 - q + 1) \) when \( q = 5 \). Note that \( \varphi_{18} = \widehat{X}_{18} \). By [20], the degrees of irreducible complex characters of \( SU_3(5) \) are: 1, 20, 125, 21, 105, 84, 126, 144, 28 and 48. When \( q = 5 \), \( X_{18}(1) = 280 \).

Therefore, \( X_{18}|_{SU_3(5)} \) is the sum of at least 3 irreducible characters. Since \( SU_3(5) \) is a normal subgroup of index 2 of \( SU_3(5) : 2 \), by Clifford’s theorem, \( X_{18}|_{SU_3(5); 2} \) is reducible when \( q = 5 \). This implies that \( \varphi_{18}|_{SU_3(5); 2} \) is also reducible when \( q = 5 \). In summary, the unique possibility for \( \varphi \) is \( \psi \) when \( 3 \nmid q \).

If \( q \equiv -1(\text{mod } 3) \) then \( \psi = \widehat{X}_{32} \) of degree \( q^3 - 1 \). Recall that \( |SU_3(q) : 2| = 2q^3(q^3 + 1)(q^2 - 1) \), which is not divisible by \( q^3 - 1 \) for every \( q \geq 5 \). Hence \( X_{32}|_{SU_3(q); 2} \) is reducible and so is \( \psi|_{SU_3(q); 2} \). Now it remains to consider \( q \equiv 1(\text{mod } 3) \).

First we assume that \( \ell = 2 \). Then \( \psi = \widehat{X}_{32} \). By Lemmas 2.3.1 and 2.3.3, we have \( \psi|_{SU_3(q)} = \widehat{X}_{32}|_{SU_3(q)} = \chi_{q^3 + 1} \in \text{IBr}_2(SU_3(q)) \). Therefore \( \psi|_{SU_3(q); 2} \) is irreducible when \( \ell = 2 \). Next, if \( \ell = 3 \) then \( \psi = \widehat{X}_{32} - \widehat{1}_G \). By Lemma 2.3.1, we have \( \psi|_{SU_3(q)} = \widehat{X}_{32}|_{SU_3(q)} - \widehat{1}|_{SU_3(q)} = \chi_{q^3 + 1} - \widehat{1}|_{SU_3(q)} = \chi_q^3 \), which is an irreducible 3-Brauer character of \( SU_3(q) \) by [20, p. 573]. Finally, if \( \ell \neq 2, 3 \) then \( \psi = \widehat{X}_{32} \). By Lemma 2.3.1, we have \( \widehat{X}_{32}|_{SU_3(q)} = \chi_{q^3 + 1} \) which is irreducible again by [20]. Therefore, \( \psi|_{SU_3(q); 2} \) is also irreducible, as we claim in the item (ii) of Theorem B.
Case 5: \( M = G_2(q_0) \) with \( q = q_0^2, 3 \nmid q \). Since \( \varphi|_M \) is irreducible, \( \varphi(1) \leq \sqrt{|G_2(q_0)|} = \sqrt{q_0^6(q_0^6 - 1)(q_0^2 - 1)} < \sqrt{q^7} < \vartheta_{2, \ell}(G) \) for every \( q \geq 5 \). Therefore, by Theorem 2.2.1, the unique possibility for \( \varphi \) is \( \psi \). Since \( q = q_0^2, q \equiv 1(\text{mod } 3) \).

If \( \ell = 3 \) then \( \psi = \widehat{X}_{32} - \widehat{1}_G \). Hence, \( \psi|_{G_2(q_0)} = \widehat{X}_{32}|_{G_2(q_0)} - \widehat{1}_{G_2(q_0)} \). Recall that \( X_{32}(1) = q^3 + 1 \) and \( |G_2(q_0)| = q_0^6(q_0^6 - 1)(q_0^2 - 1) = q^6(q^3 - 1)(q - 1) \). It is easy to see that \( (q^3 + 1) \nmid q^3(q^3 - 1)(q - 1) \) for every \( q \geq 5 \). So \( X_{32}|_{G_2(q_0)} \) is reducible. Assume that \( \overline{X}_{32}|_{G_2(q_0)} - \widehat{1}_{G_2(q_0)} \) is irreducible. Then \( X_{32}|_{G_2(q_0)} = \lambda + \mu \) where \( \hat{\lambda} = \widehat{1}_{G_2(q_0)}, \mu \in \text{Irr}(G_2(q_0)) \) and \( \mu \in \text{IBr}_3(G_2(q_0)) \). We then have \( \mu(1) = X_{32}(1) - 1 = q^3 - q_0^6 \). So \( \mu \) is the Steinberg character. From [33], we know that the reduction modulo 3 of the Steinberg character is reducible, which contradicts \( \hat{\mu} \in \text{IBr}_3(G_2(q_0)) \).

If \( \ell \neq 3 \) then \( \psi = \widehat{X}_{32} \). Therefore, \( \psi|_{G_2(q_0)} = \widehat{X}_{32}|_{G_2(q_0)} \). From the reducibility of \( X_{32}|_{G_2(q_0)} \) as noted above, \( \psi|_{G_2(q_0)} \) is also reducible. \( \square \)

2.4 Small Groups \( G_2(3) \) and \( G_2(4) \)

In this section, we mainly use results and notation of [12] and [36]. We notice that the universal cover of \( G_2(3) \), which is \( 3 \cdot G_2(3) \), has two pairs of complex conjugate irreducible characters of degree 27. The purpose of this section is to prove the following theorem.

**Theorem 2.4.1.** Let \( G \in \{ G_2(3), 3 \cdot G_2(3), G_2(4), 2 \cdot G_2(4) \} \). Let \( \varphi \) be a faithful irreducible character of \( G \) in cross characteristic \( \ell \) of degree greater than 1 and \( M \) a maximal subgroup of \( G \). Then we have

(i) When \( G = G_2(3) \), \( \varphi|_M \) is irreducible if and only if one of the following holds:

(a) \( M = U_3(3) : 2, \varphi \) is the unique irreducible character of degree 14 when \( \ell = 7 \) or \( \varphi \) is any of the irreducible characters of degree 14, 64 when \( \ell \neq 3, 7 \);

(b) \( M = 2^2 \cdot L_3(2), \varphi \) is the unique irreducible character of degree 14 when \( \ell \neq 2, 3 \).

(ii) When \( G = 3 \cdot G_2(3) \), the universal cover of \( G_2(3) \), \( \varphi|_M \) is irreducible if and only if one of the following holds:

(a) \( M = 3.P \) or \( 3.Q \) where \( P, Q \) are maximal parabolic subgroups of \( G_2(3) \), \( \varphi \) is any of the four irreducible characters of degree 27;
(b) $M = 3.(U_3(3) : 2)$, $\varphi$ is any of the two complex conjugate irreducible characters of degree 27 when $\ell \neq 2, 3, 7$;
(c) $M = 3.(L_3(3) : 2)$, $\varphi$ is any of the two complex conjugate irreducible characters of degree 27 when $\ell \neq 2, 3, 13$;
(d) $M = 3.(L_2(8) : 3)$, $\varphi$ is any of the four irreducible characters of degree 27 when $\ell \neq 2, 3, 7$.

(iii) When $G = G_2(4)$, $\varphi|_M$ is irreducible if and only if one of the following holds:
(a) $M = U_3(4) : 2$, $\varphi$ is the unique irreducible character of degree 64 when $\ell = 3$ or $\varphi$ is the unique irreducible character of degree 65 when $\ell \neq 2, 3$;
(b) $M = J_2$, $\varphi$ is any of the two irreducible characters of degree 300 when $\ell \neq 2, 3, 7$.

(iv) When $G = 2 \cdot G_2(4)$, the universal cover of $G_2(4)$, $\varphi|_M$ is irreducible if and only if one of the following holds:
(a) $M = 2.P$ or $2.Q$ where $P, Q$ are maximal parabolic subgroups of $G_2(4)$, $\varphi$ is the unique irreducible character of degree 12;
(b) $M = 2.(U_3(4) : 2)$, $\varphi$ is the unique irreducible character of degree 12 or $\varphi$ is any of the two irreducible characters of degree 104 when $\ell \neq 2, 5$;
(c) $M = 2.(SL_3(4) : 2)$, $\varphi$ is the unique irreducible character of degree 12 when $\ell \neq 2, 3$.

**Lemma 2.4.2.** Theorem 2.4.1 holds in the case $G = G_2(3)$, $M = U_3(3) : 2$.

**Proof.** According to [12, p. 14], we have $m_G(U_3(3)) = 32$ and $m_G(U_3(3) : 2) = 64$. Thus, if $\varphi|_M$ is irreducible then $\varphi(1) \leq 64$. Inspecting the character tables of $G_2(3)$ in [12, p. 60] and [36, p. 140, 142, 143], we see that $\varphi(1) = 14$ or 64.

Note that $G_2(3)$ has a unique irreducible complex character of degree 14 which is denoted by $\chi_2$ and every reduction modulo $\ell \neq 3$ of $\chi_2$ is still irreducible. Now we will show that $\chi_2|_{U_3(3)} = \chi_6$, which is the unique irreducible character of degree 14 of $U_3(3)$. Suppose that $\chi_2|_{U_3(3)} \neq \chi_6$, then $\chi_2|_{U_3(3)}$ is reducible and it is the sum of more than one
irreducible characters of degree less than 14. Note that $U_3(3)$ has exactly one conjugacy
class of elements of order 6, which is denoted by $6A$. If $\chi_2|U_3(3)$ is sum of two irreducible
characters, then the degree of these characters is 7. So $\chi_2|U_3(3)(6A)$ is 0, 2 or 4. This
cannot happen since the value of $\chi_2$ on any class of elements of order 6 of $G_2(3)$ is 1 or
$-2$. If $\chi_2|U_3(3)$ is sum of more than two irreducible characters, then $\chi_2|U_3(3)(6A) \geq 2$
which cannot happen, either. In summary, we have $\chi_2|U_3(3) = \chi_6$. We also see that every
reduction modulo $\ell \neq 3$ of $\chi_6$ is still irreducible. Hence, if $\ell \neq 3$ and $\varphi$ is the $\ell$-Brauer
character of $G_2(3)$ of degree 14, then $\varphi|U_3(3)$ is irreducible and so is $\varphi|U_3(3):2$.

By [12, p. 14], $U_3(3) : 2$ has one complex characters of degree 64 which we denote by
$\chi$. Also, $G_2(3)$ has two irreducible complex characters of degree 64 that are $\chi_3$ and $\chi_4$ as
denoted in [12, p. 60]. We will show that $\chi_3,4|U_3(3):2 = \chi$. Note that the two conjugacy
classes of $(U_3(3) : 2)$-subgroups of $G_2(3)$ are fused under an outer automorphism $\tau$
of $G_2(3)$, which stabilizes each of $\chi_3$ and $\chi_4$. Hence without loss we may assume that
$U_3(3) : 2$ is the one considered in [16, p. 237]. Checking directly, it is easy to see that
the values of $\chi_3$ and $\chi_4$ coincide with those of $\chi$ at every conjugacy classes except the
class of elements of order 3 at which we need to check more. $U_3(3) : 2$ has two classes of
elements of order 3, $3A$ and $3B$. Using [16, p. 237] to find the fusion of conjugacy classes
of $U_3(3)$ in $G_2(3)$ and the values of $\chi_3$ and $\chi_4$ (which are $\theta_{12}(k)$ in [16]), we see that the
classes $3A, 3B$ of $U_3(3)$ are contained in the classes $3A, 3E$ of $G_2(3)$, respectively. We also
have $\chi_3(3A) = \chi_4(3A) = \chi(3A) = -8$ and $\chi_3(3E) = \chi_4(3E) = \chi(3B) = -2$. Thus,
$\chi_3|U_3(3):2 = \chi_4|U_3(3):2 = \chi$. By [36, p. 140, 142, 143], any reduction modulo $\ell \neq 3$ of $\chi_3$ as
well as $\chi_4$ is irreducible. Also, the reduction modulo $\ell$ of $\chi$ is irreducible for every $\ell \neq 3,
7$. Therefore, $\overline{\chi_3}|U_3(3):2$ and $\overline{\chi_4}|U_3(3):2$ are the same and irreducible for every $\ell \neq 3, 7$. When
$\ell = 7$, $m_7(U_3(3)) = 28$ and $m_7(U_3(3) : 2) = 56$. Since $\overline{\chi_3}(1) = \overline{\chi_4}(1) = 64$,
$\overline{\chi_3}|U_3(3):2$ and $\overline{\chi_4}|U_3(3):2$ are reducible when $\ell = 7$.

\begin{flushright}
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**Lemma 2.4.3.** *Theorem 2.4.1 holds in the case $G = G_2(3), M = 2^3 \cdot L_3(2)$.*
Proof. We have \(|2^3 \cdot L_3(2)| = 1344\). So \(m_C(2^3 \cdot L_3(2)) \leq \sqrt{1344} < 37\) and therefore the unique possibility for \(\varphi\) is the reduction modulo \(\ell \neq 3\) of the character \(\chi_2\) of degree 14.

When \(\ell \nmid |G_2(3)|\) (\(\ell \neq 2, 3, 7, \text{ and } 13\)), from \([44]\), we know that \(\widehat{\chi}_2|_{2^3 \cdot L_3(2)}\) is irreducible. When \(\ell = 2\), we have \(m_2(2^3 \cdot L_3(2)) = m_2(L_3(2)) \leq \sqrt{168} < 13\). So \(\widehat{\chi}_2|_{2^3 \cdot L_3(2)}\) is reducible when \(\ell = 2\). When \(\ell = 13\), since 13 \(\nmid \) \(|2^3 \cdot L_3(2)|\), \(\widehat{\chi}_2|_{2^3 \cdot L_3(2)}\) is irreducible. The last case we need to consider is \(\ell = 7\). Let \(E = 2^3 \leq 2^3 \cdot L_3(2)\) which is an elementary abelian group of order \(2^3\). Since \(\chi_2|_{E \cdot L_3(2)}\) is irreducible and \(L_3(2)\) acts transitively on \(E\setminus\{1\}\) and \(\text{Irr}(E)\setminus\{1_E\}\), \(\chi_2|_E = 2 \cdot \sum_{\alpha \in \text{Irr}(E)\setminus\{1_E\}} \alpha\). Let \(I\) be the inertia group of \(\alpha\) in \(E \cdot L_3(2)\). By Clifford’s theory, we have \(\chi_2|_{E \cdot L_3(2)} = \text{Ind}_I^{E \cdot L_3(2)}(\rho)\) for some \(\rho \in \text{Irr}(I)\) and \(\rho|_E = 2\alpha\). We also have \(|I| = \frac{|E \cdot L_3(2)|}{|\text{Irr}(E)\setminus\{1_E\}|} = 2^6 \cdot 3\). Thus, the reduction modulo 7 of \(\rho\) is itself and therefore it is irreducible. Hence \(\chi_2|_{E \cdot L_3(2)}\) is also irreducible when \(\ell = 7\).

\[\square\]

Lemma 2.4.4. Theorem 2.4.1 holds in the case \(G = 3 \cdot G_2(3), M = 3.P\) or \(3.Q\) where \(P, Q\) are maximal parabolic subgroups of \(G_2(3)\).

Proof. First, we consider \(M = 3.P\) where \(P\) is one of two maximal parabolic subgroups which is specified in \([16, \text{ p. } 217]\). If \(\varphi|_{3.P}\) is irreducible then \(\varphi(1) \leq m_C(3.P) \leq \sqrt{|(3.P)/Z(3.P)|} \leq \sqrt{|P|} = \sqrt{11664} = 108\). Inspecting the character tables (both complex and Brauer) of \(3 \cdot G_2(3)\) in \([12]\) and \([36]\) (note that we only consider faithful characters), we have \(\varphi(1) = 27\) and \(\varphi\) is actually the reduction modulo \(\ell \neq 3\) of one of four irreducible complex characters of degree 27 of \(3 \cdot G_2(3)\). From now on, we denote these characters by \(\chi_{24}, \overline{\chi_{24}}\) (corresponding to the line \(\chi_{24}\) in \([12, \text{ p. } 60]\)) and \(\chi_{25}, \overline{\chi_{25}}\) (corresponding to the line \(\chi_{25}\) in \([12, \text{ p. } 60]\)).

Now we will show that \(\chi_{24}|_{3.P}\) is irreducible. Note that if \(g_1\) and \(g_2\) are the pre-images of an element \(g \in G_2(3)\) under the natural projection \(\pi: 3 \cdot G_2(3) \rightarrow G_2(3)\), then \(\chi_{24}(g_1) = \omega \chi_{24}(g_2)\) where \(\omega\) is a cubic root of unity. Therefore we have \(\chi_{24}|_{3.P}, \chi_{24}|_{3.P}|_{3.P} = \frac{1}{|3.P|} \sum_{x \in 3.P} \chi_{24}(x) \overline{\chi_{24}(x)} = \frac{1}{|P|} \sum_{g \in P} \chi_{24}(\overline{g}) \overline{\chi_{24}(\overline{g})}\), where \(\overline{g}\) is a fixed pre-image of \(g\) under \(\pi\). We choose \(\overline{g}\) so that the value of \(\chi_{24}\) at \(\overline{g}\) is printed in \([12, \text{ p. } 60]\). The fusion
of conjugacy classes of $P$ in $G_2(3)$ is given in [16, p. 217]. By comparing the orders of centralizers of conjugacy classes of $G_2(3)$ in [16, p. 239] with those in [12, p. 60], we can find a correspondence between conjugacy classes of $G_2(3)$ in these two papers. The length of each conjugacy class of $P$ can be computed from [16, p. 217, 218]. All the above information is collected in Table 2-6.

From this table, we see that the value of $\chi_{24}$ is zero at any element $\bar{g}$ for which the order of $g$ is 3, 6 or 9. We denote by $\chi_{24}(X)$ the value $\chi_{24}(g)$ for some $g \in X$. Then we have

$$\sum_{g \in P} \chi_{24}(g)\overline{\chi_{24}(g)} = |A_1|(\chi_{24}(A_1))^2 + |B_{11}|(\chi_{24}(B_{11}))^2 + |B_{21}|(\chi_{24}(B_{21}))^2 + |D_{1}|(\chi_{24}(D_{1}))^2 + |D_{2}|(\chi_{24}(D_{2}))^2 + 2|E_2(1)|(\chi_{24}(E_2(1)))^2.$$ (4.7)

By [16, p. 217], the class $D_1$ of $P$ is contained in the class $D_{11}$ of $G_2(3)$. This class is the class $4A$ or $4B$ according to the notation in [12, p. 60]. First we assume $D_{11}$ is $4A$. Then by looking at the values of one character of degree 273 of $G_2(3)$ both in [16] and [12], it is easy to see that $D_2 \subset D_{12} = 12A$ and $E_2(i) \subset E_2 = 8A$. Therefore, Equation (4.7) becomes

$$\sum_{g \in P} \chi_{24}(g)\overline{\chi_{24}(g)} = 27^2 + 324 \cdot 3^2 + 81 \cdot 3^2 + 486 \cdot (-1)^2 + 2 \cdot 1458 \cdot 1^2 = 11,664.$$ 

Next, we assume $D_{11}$ is $4B$. Then $D_{12} = 12B$ and $E_2 = 8B$ and Equation (4.7) becomes

$$\sum_{g \in P} \chi_{24}(g)\overline{\chi_{24}(g)} = 27^2 + 324 \cdot 3^2 + 81 \cdot 3^2 + 486 \cdot 3^2 + 972 \cdot 0^2 + 2 \cdot 1458 \cdot 1^2 = 11,664.$$ 

So in any case, we have $[\chi_{24}|_{3.P}, \chi_{24}|_{3.P}]_{3.P} = \frac{1}{|P|} \sum_{g \in P} \chi_{24}(g)\overline{\chi_{24}(g)} = 1$. That means $\chi_{24}|_{3.P}$ is irreducible.

Note that $\hat{\chi}_{24}$ is irreducible for every $\ell \neq 3$ by [36, p. 140, 142, 143]. We will show that $\hat{\chi}_{24}|_{3.P}$ is also irreducible for every $\ell \neq 3$. The structure of $P$ is $[3^5] : 2S_4$. We denote by $O_3$ the maximal normal 3-subgroup (of order $3^5$) of $P$. Then the order of any element
in $O_3$ is 1, 3 or 9, and $3.O_3$ is the maximal normal 3-subgroup of $3.P$. Since the values of $\chi_{24}$ is zero at any element $\bar{g}$ in which the order of $g$ is 3 or 9, we have

$$[\chi_{24}|_{3.O_3}, \chi_{24}|_{3.O_3}]_{3.O_3} = \frac{1}{|O_3|} \sum_{g \in O_3} \chi_{24}^g \chi_{24}(\bar{g}) = \frac{1}{3^2} 27^2 = 3.$$ 

By Clifford’s theory, $\chi_{24}|_{3.O_3} = e \cdot \sum_{i=1}^t \theta_i$ where $e = [\chi_{24}|_{3.O_3}, \theta_1]_{3.O_3}$ and $\theta_1, \theta_2, \ldots, \theta_t$ are the distinct conjugates of $\theta_1$ in $3.P$. So $e^2 \cdot t = 3$ and therefore $e = 1, t = 3$. Thus, $\chi_{24}|_{3.O_3} = \theta_1 + \theta_2 + \theta_3$. When $\ell \neq 3$, $\hat{\theta}_i$ is clearly irreducible. By Lemma 2.1.5, $\hat{\chi}_{24}|_{3.P}$ is irreducible when $\ell \neq 3$.

Using similar arguments, we also have $\hat{\chi}_{25}|_{3.P}$ is irreducible for every $\ell \neq 3$, and therefore $\hat{\chi}_{24}|_{3.P}$ as well as $\hat{\chi}_{25}|_{3.P}$ are irreducible for every $\ell \neq 3$. Note that an outer automorphism $\tau$ of $G_2(3)$ sends $P$ to $Q$ and fixes $\{\chi_{24}, \chi_{25}, \chi_{24}, \chi_{25}\}$. Hence the Lemma also holds when $M = 3.Q$ by Lemma 2.1.6. 

**Lemma 2.4.5.** Theorem 2.4.1 holds in the case $G = 3 \cdot G_2(3), M = 3.(U_3(3) : 2)$.

**Proof.** Since the Schur multiplier of $U_3(3)$ is trivial and $\mathbb{Z}_3 = Z(3 \cdot G_2(3)) \leq Z(M)$, $3.(U_3(3) : 2) = 3 \times (U_3(3) : 2)$. So $m_C(M) = m_C(U_3(3) : 2) = 64$. Therefore if $\varphi|_M$ is irreducible then $\varphi(1) = 27$ and $\varphi$ is the reduction modulo $\ell \neq 3$ of one of four complex characters of degree 27 which we denote by $\chi_{24}, \chi_{25}, \chi_{24}$ and $\chi_{25}$ as before.

When $\ell = 2$ or 7, $U_3(3) : 2$ does not have any irreducible representation in characteristic $\ell$ of degree 27. Therefore, $\ell \neq 2, 3$ and 7. That means $\ell \nmid |M|$ and $\text{IBr}_\ell(M) = \text{Irr}(M)$. So we only need to consider the complex case $\ell = 0$.

It is obvious that $\varphi|_{3 \times (U_3(3); 2)}$ is irreducible if and only if $\varphi|_{(U_3(3); 2)}$ is irreducible. Moreover, since $\varphi(1) = 27, \varphi|_{(U_3(3); 2)}$ is irreducible if and only if $\varphi|_{U_3(3)}$ is irreducible. Now we will show that either $\chi_{24}|_{U_3(3)}$ or $\chi_{25}|_{U_3(3)}$ is irreducible and the other is reducible.

From [12, p. 14], we know that $U_3(3)$ has the unique irreducible character of degree 27, which is denoted by $\chi_{10}$. This character is extended to two characters of degree 27 of $U_3(3) : 2$. It is easy to see that the classes $4A, 4B$ of $U_3(3)$ is contained in the same class of $G_2(3)$, which is $4A$ or $4B$. For definiteness, we suppose that this class is $4A$. Then the
class $4C$ of $U_3(3)$ is contained in the class $4B$ of $G_2(3)$. In $U_3(3)$ we have $(8A)^2 \subset 4A$, $(8B)^2 \subset 4B$ and in $G_2(3)$, $(8A)^2 \subset 4A$, $(8B)^2 \subset 4B$. So the classes $8A$, $8B$ of $U_3(3)$ are contained in the class $8A$ of $G_2(3)$. Similarly, the classes $12A$, $12B$ of $U_3(3)$ are contained in the class $12A$ of $G_2(3)$. Comparing the values of $\chi_{24}|_{U_3(3)}$ as well as $\chi_{25}|_{U_3(3)}$ with those of $\chi_{10}$ on every conjugacy class of $U_3(3)$, we see that $\chi_{25}|_{U_3(3)} = \chi_{10}$ and $\chi_{24}|_{U_3(3)}$ is reducible.

Note that $G_2(3)$ has two non-conjugate maximal subgroups which are isomorphic to $U_3(3) : 2$. We denote these groups by $M_1$ and $M_2$. Suppose that $\chi_{25}|_{M_1}$ is irreducible and $\chi_{24}|_{M_1}$ is reducible. Let $\tau$ be an automorphism of $G_2(3)$ such that $\tau(M_1) = M_2$. Then $\tau(M_2) = M_1$, $\chi_{24} \circ \tau = \chi_{25}$ and $\chi_{25} \circ \tau = \chi_{24}$. By Lemma 2.1.6, $\chi_{25}|_{M_2}$ is reducible and $\chi_{24}|_{M_2}$ is irreducible. \qed

**Lemma 2.4.6.** Theorem 2.4.1 holds in the case $G = 3 \cdot G_2(3)$, $M = 3.(L_3(3) : 2)$.

**Proof.** Since the Schur multiplier of $L_3(3)$ is trivial and $Z_3 = Z(3 \cdot G_2(3)) \leq Z(M)$, $3.(L_3(3) : 2) = 3 \cdot (L_3(3) : 2)$. So we have $m_C(M) = m_C((L_3(3) : 2)) = 52$ by [12, p. 13]. Therefore, if $\varphi|_M$ is irreducible then $\varphi(1) = 27$ and $\varphi$ is restriction modulo $\ell \neq 3$ of $\chi_{24}$, $\chi_{25}$, $\chi_{24}$ or $\chi_{25}$.

When $\ell = 2$ or $13$, $L_3(3) : 2$ does not have any irreducible $\ell$-Brauer character of degree 27. Therefore $\ell \neq 2, 3, 13$ and we have $\text{IBr}_\ell(M) = \text{Irr}(M)$. Thus we only need to consider the complex case $\ell = 0$.

Since $\varphi(1) = 27$, it is obvious that $\varphi|_{3 \cdot (L_3(3) : 2)}$ is irreducible if and only if $\varphi|_{L_3(3)}$ is irreducible. Now we will show that either $\chi_{24}|_{L_3(3)}$ or $\chi_{25}|_{L_3(3)}$ is irreducible and the other is reducible.

From [12, p. 13], we know that $L_3(3)$ has a unique irreducible character of degree 27, which is denoted by $\chi_{11}$. This character is extended to two characters of degree 27 of $L_3(3) : 2$. The class $4A$ of $L_3(3)$ is contained in a class of elements of order 4 of $G_2(3)$, which is $4A$ or $4B$. For definiteness, we suppose that this class is $4A$. In $L_3(3)$ we have $(8A)^2 \subset 4A$, $(8B)^2 \subset 4A$ and in $G_2(3)$, $(8A)^2 \subset 4A$, $(8B)^2 \subset 4B$. So the classes $8A$, $8B$ of $U_3(3)$ is contained in the class $4A$ of $G_2(3)$. In $U_3(3)$ we have $(8A)^2 \subset 4A$, $(8B)^2 \subset 4A$. So the classes $8A$, $8B$ of $U_3(3)$ are contained in the class $4A$ of $G_2(3)$. Similarly, the classes $12A$, $12B$ of $U_3(3)$ are contained in the class $12A$ of $G_2(3)$. Comparing the values of $\chi_{24}|_{U_3(3)}$ as well as $\chi_{25}|_{U_3(3)}$ with those of $\chi_{10}$ on every conjugacy class of $U_3(3)$, we see that $\chi_{25}|_{U_3(3)} = \chi_{10}$ and $\chi_{24}|_{U_3(3)}$ is reducible.

Note that $G_2(3)$ has two non-conjugate maximal subgroups which are isomorphic to $U_3(3) : 2$. We denote these groups by $M_1$ and $M_2$. Suppose that $\chi_{25}|_{M_1}$ is irreducible and $\chi_{24}|_{M_1}$ is reducible. Let $\tau$ be an automorphism of $G_2(3)$ such that $\tau(M_1) = M_2$. Then $\tau(M_2) = M_1$, $\chi_{24} \circ \tau = \chi_{25}$ and $\chi_{25} \circ \tau = \chi_{24}$. By Lemma 2.1.6, $\chi_{25}|_{M_2}$ is reducible and $\chi_{24}|_{M_2}$ is irreducible. \qed

**Lemma 2.4.6.** Theorem 2.4.1 holds in the case $G = 3 \cdot G_2(3)$, $M = 3.(L_3(3) : 2)$.

**Proof.** Since the Schur multiplier of $L_3(3)$ is trivial and $Z_3 = Z(3 \cdot G_2(3)) \leq Z(M)$, $3.(L_3(3) : 2) = 3 \times (L_3(3) : 2)$. So we have $m_C(M) = m_C((L_3(3) : 2)) = 52$ by [12, p. 13]. Therefore, if $\varphi|_M$ is irreducible then $\varphi(1) = 27$ and $\varphi$ is restriction modulo $\ell \neq 3$ of $\chi_{24}$, $\chi_{25}$, $\chi_{24}$ or $\chi_{25}$.

When $\ell = 2$ or $13$, $L_3(3) : 2$ does not have any irreducible $\ell$-Brauer character of degree 27. Therefore $\ell \neq 2, 3, 13$ and we have $\text{IBr}_\ell(M) = \text{Irr}(M)$. Thus we only need to consider the complex case $\ell = 0$.

Since $\varphi(1) = 27$, it is obvious that $\varphi|_{3 \times (L_3(3) : 2)}$ is irreducible if and only if $\varphi|_{L_3(3)}$ is irreducible. Now we will show that either $\chi_{24}|_{L_3(3)}$ or $\chi_{25}|_{L_3(3)}$ is irreducible and the other is reducible.

From [12, p. 13], we know that $L_3(3)$ has a unique irreducible character of degree 27, which is denoted by $\chi_{11}$. This character is extended to two characters of degree 27 of $L_3(3) : 2$. The class $4A$ of $L_3(3)$ is contained in a class of elements of order 4 of $G_2(3)$, which is $4A$ or $4B$. For definiteness, we suppose that this class is $4A$. In $L_3(3)$ we have $(8A)^2 \subset 4A$, $(8B)^2 \subset 4A$ and in $G_2(3)$, $(8A)^2 \subset 4A$, $(8B)^2 \subset 4B$. So the classes $8A$, $8B$ of $U_3(3)$ is contained in the class $4A$ of $G_2(3)$. In $U_3(3)$ we have $(8A)^2 \subset 4A$, $(8B)^2 \subset 4A$. So the classes $8A$,
$8B$ of $L_3(3)$ is contained in the class $8A$ of $G_2(3)$. Comparing the values of $\chi_{24}$ as well as $\chi_{25}$ with those of $\chi_{11}$ on every conjugacy classes of $L_3(3)$, we see that $\chi_{24}|_{L_3(3)} = \chi_{11}$ and $\chi_{25}|_{L_3(3)}$ is reducible.

Note that $G_2(3)$ has two non-conjugate maximal subgroups which are isomorphic to $L_3(3) : 2$. We denote these groups by $M_1$ and $M_2$. Suppose that $\chi_{24}|_{M_1}$ is irreducible and $\chi_{25}|_{M_1}$ is reducible. Let $\tau$ be an automorphism of $G_2(3)$ such that $\tau(M_1) = M_2$. Then $\tau(M_2) = M_1$, $\chi_{24} \circ \tau = \chi_{25}$ and $\chi_{25} \circ \tau = \chi_{24}$. By Lemma 2.1.6, we have $\chi_{24}|_{M_2}$ is reducible and $\chi_{25}|_{M_2}$ is irreducible.

Lemma 2.4.7. Theorem 2.4.1 holds in the case $G = 3 \cdot G_2(3)$, $M = 3.(L_2(8) : 3)$.

Proof. By [12, p. 6], the Schur multiplier of $L_2(8)$ is trivial. So $3.(L_2(8)) = 3 \times L_2(8)$. Therefore, $m_C(M) \leq 3m_C(3 \times L_2(8)) = 27$. Hence if $\varphi|M$ is irreducible then $\varphi(1) = 27$ and $\varphi$ must be the restriction modulo $\ell \neq 3$ of one of characters $\chi_{24}$, $\chi_{25}$, $\overline{\chi_{24}}$ and $\overline{\chi_{25}}$.

When $\ell = 2, 7$, $M$ does not have any irreducible $\ell$-Brauer character of degree 27. So $\ell \neq 2, 3, 7$ and therefore it is enough to consider the complex case $\ell = 0$.

Inspecting character values at elements of order 7, it is easy to see that $\chi_{24}|_{L_2(8)} = \chi_{25}|_{L_2(8)}$ is the sum of three irreducible characters of degree 9 which are fused in $M$. Therefore, $\chi_{24}|_{M}$ as well as $\chi_{25}|_{M}$ are irreducible. \qed

Lemma 2.4.8. Theorem 2.4.1 holds in the case $G = G_2(4)$, $M = U_3(4) : 2$.

Proof. We have $m_C(U_3(4)) = 75$ and $m_C(U_3(4) : 2) = 150$. So if $\varphi|M$ is irreducible then $\varphi(1) \leq 150$. Inspecting character table of $G_2(4)$, we see that $\varphi(1) = 64, 78$ when $\ell = 3$ or $\varphi(1) = 65, 78$ when $\ell \neq 2, 3$. Arguing as in Case 4 of the proof of Theorem B, we see that the restriction of the character of smallest degree of $G_2(4)$ to $U_3(4)$ is irreducible (of course when $\ell \neq 2$).

Now it remains to consider the case when $\varphi$ is a reduction modulo $\ell \neq 2$ of the character $\chi_3$ (as denoted in [12, p. 98]) of degree 78. Since $U_3(4) : 2$ has no complex
irreducible character of degree 78, it follows that $\chi_3|_{U_3(4):2}$ is reducible and so is $\hat{\chi}_3|_{U_3(4):2}$ for every $\ell \neq 2$.

Lemma 2.4.9. Theorem 2.4.1 holds in the case $G = G_2(4)$, $M = J_2$.

Proof. Comparing the degrees of irreducible $\ell$-Brauer characters of $G_2(4)$ with those of $J_2$, we see that if $\varphi|_{J_2}$ is irreducible then $\varphi(1)$ can only be one of two irreducible characters of degree 300 with $\ell \neq 2, 3, 7$. When $\ell = 0$, these two complex characters are denoted by $\chi_4$ and $\chi_5$ in [12, p. 98]. First, we show that $\chi_4|_{J_2}$ is actually irreducible. More precisely, $\chi_4|_{J_2} = \chi_{20}$, where $\chi_{20}$ is the unique irreducible complex character of $J_2$ of degree 300.

It is easy to see that the values of $\chi_4$ and $\chi_{20}$ are the same at conjugacy classes of elements of order 5, 7, 8, 10, and 15. The unique class $12A$ of elements of order 12 in $J_2$ is real and therefore it is contained in a real class of $G_2(4)$. Hence it is contained in class $12A$ of $G_2(4)$ and we have $\chi_4(12A) = \chi_{20}(12A) = 1$. We see that $(12A)^3 = 4A$ in both $G_2(4)$ and $J_2$. Therefore the class $4A$ of $J_2$ is contained in the class $4A$ of $G_2(4)$ and we also have $\chi_4(4A) = \chi_{20}(4A) = 4$. Now we move to classes of elements of order 2, 3 and 6. Since $J_2$ is a subgroup of $G_2(4)$, either $2 \cdot J_2$ (the universal cover of $J_2$) or $2 \times J_2$ is a subgroup of $2 \cdot G_2(4)$ (the universal cover of $G_2(4)$). Note that $d_c(2 \cdot G_2(4)) = 12$ and $d_c(J_2) = d_c(2 \times J_2) = 14$. So $2 \times J_2$ cannot be a subgroup of $2 \cdot G_2(4)$ and therefore $2 \cdot J_2$ is a subgroup of $G_2(4)$. From [12], the class $2A$ of $G_2(4)$ lifts to two involution classes of $2 \cdot G_2(4)$ and the class $2B$ of $G_2(4)$ lifts to a class of elements of order 4 of $2 \cdot G_2(4)$. In the same way, the class $2A$ of $J_2$ lifts to two involution classes of $2 \cdot J_2$ and the class $2B$ of $J_2$ lifts to a class of elements of order 4 of $2 \cdot J_2$. These imply that the classes $2A$ and $2B$ of $J_2$ are contained in the classes $2A$ and $2B$ of $G_2(4)$, respectively. Again, we have $\chi_4(2A) = \chi_{20}(2A) = -20$ and $\chi_4(2B) = \chi_{20}(2B) = 0$. Using similar arguments, we also can show that the classes $6A$ and $6B$ of $J_2$ are contained in the classes $6A$ and $6B$ of $G_2(4)$ respectively and we also have $\chi_4(6A) = \chi_{20}(6A) = 1$, $\chi_4(6B) = \chi_{20}(6B) = 0$. In both $G_2(4)$ and $J_2$, we have $(6A)^2 = 3A$ and $(6B)^2 = 3B$. That means the classes $3A$ and $3B$ of $J_2$ are contained in the classes $3A$ and $3B$ of $G_2(4)$, respectively. One more time,
\(\chi_4(3A) = \chi_20(3A) = -15\) and \(\chi_4(3B) = \chi_20(3B) = 0\). We have shown that the values of \(\chi_4\) and \(\chi_20\) agree at all conjugacy classes of \(J_2\). Therefore \(\chi_4|_{J_2} = \chi_20\).

Note that \(\chi_5 = \overline{\chi_4}\) and all irreducible characters of \(J_2\) are real. Therefore we also have \(\chi_5|_{J_2} = \chi_20\). When \(\ell \neq 2, 3,\) and \(7\), the reductions modulo \(\ell\) of \(\chi_4, \chi_5\) and \(\chi_20\) are still irreducible. Thus, \(\widehat{\chi_4}|_{J_2}\) and \(\widehat{\chi_5}|_{J_2}\) are irreducible for every \(\ell \neq 2, 3,\) and \(7\).

\[\square\]

**Lemma 2.4.10.** Theorem 2.4.1 holds in the case \(G = 2 \cdot G_2(4), M = 2.P\).

**Proof.** We have \(m_{C(2.P)} \leq \sqrt{|2.P/Z(2.P)|} \leq \sqrt{|P|} = \sqrt{184,320} < 430\). Therefore if \(\varphi|_M\) is irreducible then \(\varphi(1) < 430\). There are five cases as follows:

- \(\varphi(1) = 12\) when \(\ell \neq 2\). Then \(\varphi\) must be a reduction modulo \(\ell \neq 2\) of the unique irreducible complex character of degree 12 of \(2 \cdot G_2(4)\). Throughout the proof of this Lemma we denote this character by \(\chi\). Now we will show that \(\chi|_{2.P}\) is irreducible.

Note that if \(g_1\) and \(g_2\) are pre-images of an element \(g \in G_2(4)\) under the natural projection \(\pi : 2 \cdot G_2(4) \rightarrow G_2(4)\), then \(\chi(g_1) = \pm \chi(g_2)\). Therefore \([\chi|_{2.P}, \chi|_{2.P}]_{2.P} = \frac{1}{|Z|} \sum_{x \in 2.P} \chi(x)\overline{\chi(x)} = \frac{1}{|P|} \sum_{\overline{g} \in P} \chi(\overline{g})\overline{\chi(\overline{g})}\), where \(\overline{g}\) is a pre-image of \(g\) under \(\pi\). We choose \(\overline{g}\) so that the value of \(\chi\) at \(\overline{g}\) is printed in [12, p. 98]. In [17, p. 357], we have the fusion of conjugacy classes of \(P\) in \(G_2(4)\). By comparing the orders of centralizers of conjugacy classes of \(G_2(4)\) in [17, p. 364] with those in [12, p. 98] and looking at the values of irreducible characters of degrees 65, 78, we can find a correspondence between conjugacy classes of \(G_2(4)\) in these two papers. The length of each conjugacy class of \(P\) can be computed from [17, p. 357]. All the above information is collected in Table 2-7. From this table, we get

\[
\sum_{g \in P} \chi(\overline{g})\overline{\chi(\overline{g})} = 12^2 + 3 \cdot (-4)^2 + 60 \cdot (-4)^2 + 120 \cdot (-4)^2 + 1440 \cdot (-4)^2 + 5720 \cdot 2^2 + 320 \cdot (-6)^2 + 960 \cdot 2^2 + 3840 \cdot 2^2 + 3840 \cdot 2^2 + 3840 \cdot 2^2 + 3840 \cdot 2^2 + 3840 \cdot 2^2 + 3840 \cdot 2^2 + 3840 \cdot 2^2 + 3840 \cdot 2^2 + 9216 \cdot 1^2 = 184,320.
\]

Therefore, \([\chi|_{2.P}, \chi|_{2.P}]_{2.P} = \frac{1}{|P|} \sum_{g \in P} \chi(\overline{g})\overline{\chi(\overline{g})} = 1\). It follows that \(\chi|_{2.P}\) is irreducible.

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Next, we show that $\hat{\chi}|_{2,P}$ is also irreducible for every $\ell \neq 2$. Set $O_2 = 2^{2+8}$ to be the maximal normal 2-subgroup of $P$. Then $O_2$ is a union of conjugacy classes of $P$. Note that $O_2$ is a 2-group of exponent 4 and so the orders of elements in $O_2$ are 1, 2 or 4. Hence $O_2$ is either $A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$ or $A_0 \cup A_1 \cup A_2 \cup A_5$. If the latter case happens then $[\chi|_{2,O_2}, \chi|_{2,O_2}] = \frac{1}{|O_2|} \sum g \in O_2 \chi(g)\hat{\chi}(g) = \frac{12^2+3(-4)^2+60(-4)^2}{1024} = \frac{1452}{1024}$ which is not an integer. Therefore the first case must happen and we have $[\chi|_{2,O_2}, \chi|_{2,O_2}]_{2,O_2} = e \cdot \sum t_i \theta_i$, where $e = [\chi|_{2,O_2}, \theta_1|_{2,O_2}$ and $\theta_1, \theta_2, ..., \theta_t$ are the distinct conjugates of $\theta_1$ in $2.P$. So $e^2 t = 3$ and $e = 1, t = 3$. Thus $\chi|_{2,O_2} = \theta_1 + \theta_2 + \theta_3$. By Lemma 2.1.5, we have $\hat{\chi}|_{2,P}$ is irreducible for every $\ell \neq 2$.

- $\varphi(1) = 104$ when $\ell \neq 2, 5$. In this case, $\varphi$ is actually a reduction modulo $\ell$ of an irreducible character complex of degree 104. Note that $104 \nmid 368,640 = |2.P|$, so the restriction of any complex irreducible character of degree 104 to 2.P is reducible. It follows that $\hat{\varphi}|_{2,P}$ is also reducible.

- $\varphi(1) = 364$ when $\ell \neq 2, 3$. In this case, $\varphi$ is actually a reduction modulo $\ell$ of the unique faithful irreducible complex character of degree 364. Note that $364 \nmid 368,640 = |2.P|$, so the restriction of any complex irreducible character of degree 364 to 2.P is reducible. Therefore there is no examples in this case.

- $\varphi(1) = 352$ when $\ell = 3$. Denote by $\chi_{364}$ the unique faithful irreducible complex character of degree 364 of $2 \cdot G_2(4)$. Then $\varphi = \hat{\chi_{364}} - \hat{\chi}$. Suppose that $\varphi|_{2,P}$ is irreducible. Since $\chi_{364}|_{2,P}$ is reducible and $\hat{\chi}|_{2,P}$ is irreducible, $\chi_{364}|_{2,P} = \lambda + \mu$ where $\lambda$ and $\mu$ are complex characters of 2.P such that $\hat{\lambda} = \hat{\chi}|_{2,P}$ and $\hat{\mu} \in \text{IBr}_3(2.P)$. So $\mu \in \text{Irr}(2.P)$ and $\mu(1) = 352$. But $352 \nmid |2.P|$ and we get a contradiction.

- $\varphi(1) = 92$ when $\ell = 5$. Denote by $\chi_{104}$ one of the two irreducible complex characters of degree 104 of $2 \cdot G_2(4)$. Then $\varphi = \hat{\chi_{104}} - \hat{\chi}$. Suppose that $\varphi|_{2,P}$ is irreducible. Because $\chi_{104}|_{2,P}$ is reducible and $\hat{\chi}|_{2,P}$ is irreducible, $\chi_{104}|_{2,P} = \lambda + \mu$ where $\lambda$ and $\mu$ are complex characters of 2.P such that $\hat{\lambda} = \hat{\chi}|_{2,P}$ and $\hat{\mu} \in \text{IBr}_3(2.P)$. So $\mu \in \text{Irr}(2.P)$ and $\mu(1) = 92$. But $92 \nmid |2.P|$ and we get a contradiction.
character of $2.P$ such that $\hat{\lambda} = \chi|_{2.P}$ and $\hat{\mu} \in \text{IBr}_5(2.P)$. So $\mu \in \text{Irr}(2.P)$ and $\mu(1) = 92$. But $92 \nmid |2.P|$ so we get a contradiction. \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill

**Lemma 2.4.11.** Theorem 2.4.1 holds in the case $G = 2 \cdot G_2(4)$, $M = 2.Q$.

**Proof.** By similar arguments as in Lemma 2.4.10, we only need to show that $\chi|_{2.Q}$ is irreducible for every $\ell \neq 2$, where $\chi$ is the unique irreducible character of degree 12 of $2 \cdot G_2(4)$. First, let us prove that $\chi|_{2.Q}$ is irreducible.

We have $[\chi|_{2.Q}, \chi|_{2.Q}]_{2.Q} = \frac{1}{|Q|} \sum_{x \in 2.Q} \chi(x)\overline{\chi(x)} = \frac{1}{|Q|} \sum_{g \in Q} \chi(g)\overline{\chi(g)}$, where $g$ is a pre-image of $g$ under $\pi$. We choose $g$ so that the value of $\chi$ at $g$ is printed in [12, p. 98]. In [17, p. 361], we have the fusion of conjugacy classes of $Q$ in $G_2(4)$ and the length of each conjugacy class of $Q$. This will be collected in Table 2-8. From this table, we get

$$\sum_{g \in Q} \chi(g)\overline{\chi(g)} = 12^2 + 15 \cdot (-4)^2 + 360 \cdot (-4)^2 + 240 \cdot (-4)^2 + 240 \cdot (-4)^2$$
$$+ 5760 \cdot 2^2 + 2 \cdot 64 \cdot (-6)^2 + 2 \cdot 960 \cdot 2^2 + 2 \cdot 3840 \cdot 2^2$$
$$+ 2 \cdot 3840 \cdot 2^2 + 2 \cdot 3072 \cdot 2^2 + 4 \cdot 12288 \cdot (-1)^2$$
$$= 184,320.$$

Therefore, $[\chi|_{2.Q}, \chi|_{2.Q}]_{2.Q} = \frac{1}{|Q|} \sum_{g \in Q} \chi(g)\overline{\chi(g)} = 1$. In other words $\chi|_{2.Q}$ is irreducible.

Next, we show that $\hat{\chi}|_{2.Q}$ is also irreducible. Set $O_2 = 2^{4+6}$ to be the maximal normal 2-subgroup of $Q$. Since $\chi|_{2.Q}$ is irreducible, by Clifford’s theorem, $\chi|_{2.O_2} = e \cdot \sum_{i=1}^t \theta_i$, where $e = [\chi|_{2.O_2}, \theta_1|_{2.O_2}$ and $\theta_1, \theta_2, \ldots, \theta_t$ are the distinct conjugates of $\theta_1$ in $2.Q$. We have $12 = \chi(1) = et\theta_1(1)$. Note that $O_2$ is a 2-group of exponent 4 and so $\theta_1(1) \in \{1, 2, 4\}$. Therefore $et \in \{3, 6, 12\}$. Set $m = e^2t = [\chi|_{2.O_2}, \chi|_{2.O_2}]_{2.O_2} = \frac{1}{|O_2|} \sum_{g \in O_2} \chi(g)\overline{\chi(g)}$. From Table 2-8, we see that $\chi(g) = 0$ or $-4$ for every element $g$ of order 2 or 4. Therefore, $m = \frac{1}{|O_2|} \sum_{g \in O_2} \chi(g)\overline{\chi(g)} = \frac{1}{1024} (12^2 + n \cdot (-4)^2)$ where $n$ is the sum of the lengths of conjugacy classes in $O_2$ at which the values of $\chi$ is $-4$. Hence $n = (1024 \cdot m - 144)/16$. We also have $n \leq 15 + 360 + 240 + 240 = 855$ and $5 \mid n$. This implies that $m \leq 13$. Since $et \in \{3, 6, 12\}$, it follows that $e^2t \in \{3, 6, 9, 12\}$. If $m = 3$, resp. 9, 12, then $n = 183$, resp.
556, 759, which is coprime to 5. So \( m = 6 \) and then \( n = 375 \) which is the sum of lengths of the classes \( A_1 \) and \( A_{32} \).

We have shown that \([\chi|_{2.O_2}, \chi|_{2.O_2}|_{2.O_2}] = 6\). So \( e^2.t = 6 \) and therefore \( e = 1, t = 6 \). Thus \( \chi|_{2.O_2} = \sum_{i=1}^{6} \theta_i \). By Lemma 2.1.5, \( \widehat{\chi}|_{2.Q} \) is irreducible for every \( \ell \neq 2 \).

**Lemma 2.4.12.** Theorem 2.4.1 holds in the case \( G = 2 \cdot G_2(4), M = 2.(U_3(4) : 2) \).

**Proof.** We note that the Schur multiplier of \( U_3(4) \) is trivial. So \( M = 2.(U_3(4) : 2) = (2 \times U_3(4)).2 \). Therefore \( m_c(M) \leq 2m_c(2 \times U_3(4)) = 150 \) by [12, p. 30]. Hence, if \( \varphi|M \) is irreducible then \( \varphi(1) \leq 150 \). From the character table of \( 2 \cdot G_2(4) \), we have \( \varphi(1) = 12 \) when \( \ell \neq 2 \), \( \varphi(1) = 104 \) when \( \ell \neq 2, 5 \) or \( \varphi(1) = 92 \) when \( \ell = 5 \).

- \( \varphi(1) = 92 \) when \( \ell = 5 \). Inspecting the 5-Brauer character table of \( U_3(4) \) in [36, p. 72], we see that \( M \) does not have any irreducible 5-Brauer character of degree 92.

- \( \varphi(1) = 12 \) when \( \ell \neq 2 \). In this case, \( \varphi \) is the reduction modulo \( \ell \neq 2 \) of the unique irreducible complex character \( \chi \) of degree 12 of \( 2 \cdot G_2(4) \). Inspecting the character tables of \( U_3(4) \), we have \( d(\chi|_{U_3(4)}) = 12 \) for every \( \ell \neq 2 \). This implies \( \widehat{\chi}|_{U_3(4)} \) is irreducible and so is \( \widehat{\chi}|_{M} \).

- \( \varphi(1) = 104 \) when \( \ell \neq 2, 5 \). Then \( \varphi \) is a reduction modulo \( \ell \) of one of two faithful irreducible complex characters of degree 104 of \( 2 \cdot G_2(4) \), which are denoted by \( \chi_{34} \) and \( \chi_{35} \) as in [12, p. 98]. First, we show that \( \chi_{34}|_{M} \) is irreducible.

From Lemma 2.4.8, we know that the restriction of the unique irreducible character \( \chi_2 \) of degree 65 of \( G_2(4) \) to \( U_3(4) \) is irreducible and equal to the unique rational irreducible character of degree 65 of \( U_3(4) \). By looking at the values of these characters, we see that the involution class \( 2A \) of \( U_3(4) \) is contained in the class \( 2A \) of \( G_2(4) \) and the class \( 5E \) of \( U_3(4) \) is contained in the class \( 5A \) or \( 5B \) of \( G_2(4) \). For definiteness, we suppose that this class is \( 5A \). Now assume that \( \chi_{34}|_{U_3(4)} \) contains the unique irreducible character of degree 64 of \( U_3(4) \). Since \( \chi_{34}(2A) = 8 \), by looking at the values of irreducible characters of \( U_3(4) \) at the involution class \( 2A \), we see that the other irreducible constituents of \( \chi_{34}|_{U_3(4)} \) are
of degrees 1 and 39. But then we get a contradiction by looking at the values of these characters at the class 5E of $U_3(4)$.

We have shown that $\chi_{34}|_{U_3(4)}$ does not contain the irreducible character of degree 64 of $U_3(4)$. Inspecting the character table of $U_3(4)$ in [12, p. 30], we see that $U_3(4)$ has four irreducible characters of degree 52, which are denoted by $\chi_9$, $\chi_{10}$, $\chi_{11}$ and $\chi_{12}$. Note that $\chi_{10} = \overline{\chi_9}$ and $\chi_{12} = \overline{\chi_{11}}$. We also see that the values of the irreducible characters (of degree different from 64) of $U_3(4)$ at the class 5E are: 1, 2, $-b_5$, $b_5+1$, $-b_5+1$, $b_5+2$, $b_5$, $-b_5-1$, 0 where $b_5 = \frac{1}{2}(-1 + \sqrt{5})$. Moreover, this value is $b_5$ if and only if the character is $\chi_9$ or $\chi_{10}$. Note that $\chi_{34}(5A) = 2b_5$. Now we suppose that $2b_5 = x_1 \cdot 1 + x_2 \cdot 2 + x_3 \cdot (-b_5) + x_4 \cdot (b_5 + 1) + x_5 \cdot (-b_5 + 1) + x_6 \cdot (b_5 + 2) + x_7 \cdot b_5 + x_8 \cdot (-b_5 - 1)$ where $x_i$s $(i = 1, 2, ..., 8)$ are nonnegative integer numbers. Then $x_1 + 2x_2 + x_4 + x_5 + 2x_6 - x_8 = 0$ and $-x_3 + x_4 - x_5 + x_6 + x_7 - x_8 = 2$. These equations imply that $2 = -x_3 + x_4 - x_5 + x_6 + x_7 - (x_1 + 2x_2 + x_4 + x_5 + 2x_6) = -x_1 - 2x_2 - 2x_5 + x_7$. Therefore $x_7 \geq 2$. In other words, there are at least two irreducible constituents of degree 52 in $\chi_{34}|_{U_3(4)}$. Since $\chi_{34}(1) = 104$, $\chi_{34}|_{U_3(4)}$ must be a sum of two irreducible characters of degree 52. Since $\chi_{34}$ is real, it is easy to see that $\chi_{34}|_{U_3(4)} = \chi_9 + \chi_{10}$. Recall that $M = (2 \times U_3(4)) \cdot 2$ and the “2” is an outer automorphism of $U_3(4)$ that fuses $\chi_9$ and $\chi_{10}$. We have $\chi_{34}|M$ is irreducible.

It is easy to see that $\chi_{35} = \ast \circ \chi_{34}$ and $\chi_{11} + \chi_{12} = \ast \circ (\chi_9 + \chi_{10})$ where the operator $\ast$ is the algebraic conjugation: $r + s\sqrt{5} \mapsto r - s\sqrt{5}$ for $r, s \in \mathbb{Q}$. Since $\chi_{34}|_{U_3(4)} = \chi_9 + \chi_{10}$, $\chi_{35}|_{U_3(4)} = \chi_{11} + \chi_{12}$. Now arguing similarly as above, we also have that $\chi_{35}|M$ is irreducible.

When $\ell \neq 2, 5$, the reductions modulo $\ell$ of $\chi_{34}$ and $\chi_{35}$ are still irreducible. Similar arguments show that $\hat{\chi}_{34}|M$ as well as $\hat{\chi}_{35}|M$ are irreducible. \hfill \square

**Lemma 2.4.13.** Theorem 2.4.1 holds in the case $G = 2 \cdot G_2(4)$, $M = 2.(SL_3(4) : 2)$.

**Proof.** We have $SL_3(4) = 3 \cdot L_3(4)$. So $M = (2 \times (3 \cdot L_3(4))) \cdot 2$ or $M = (6 \cdot L_3(4)) \cdot 2$. Inspecting the character table of $3 \cdot L_3(4)$ in [12, p. 24, 25], we see that $d_C(3 \cdot L_3(4)) = 15$. 

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But \( \varnothing_C(2 \cdot G_2(4)) = 12 \). So \( 3 \cdot L_3(4) \) is not a subgroup of \( 2 \cdot G_2(4) \) and therefore 
\[ M = (6 \cdot L_3(4)).2. \]

Inspecting the character table of \( 6 \cdot L_3(4) \) in [12, p. 25], we have 
\( m_C(M) \leq 2m_C(6 \cdot L_3(4)) = 120 \). Therefore if \( \varphi|_M \) is irreducible then 
\( \varphi(1) \leq 120 \). Hence \( \varphi(1) = 12 \) when \( \ell \neq 2 \), \( \varphi(1) = 104 \) when \( \ell \neq 2, 5 \) or \( \varphi(1) = 92 \) when \( \ell = 5 \). Also from character tables of 
\( 6 \cdot L_3(4) \), we see that \( M \) does not have any faithful irreducible character of degree 92 or 104. So the unique possibility for \( \varphi \) is a reduction modulo \( \ell \neq 2 \) of the unique irreducible character \( \chi \) of degree 12 of \( 2 \cdot G_2(4) \).

We have \( 1 \neq O_3(\mathbb{Z}_6) \triangleleft M \). So \( O_3(M) \neq 1 \) and therefore by Lemma 2.1.4, \( \hat{\chi}|_M \) 
is reducible when \( \ell = 3 \). Now we suppose \( \ell \neq 2, 3 \). From [12, p. 23], \( M \) can be either 
\( 6.L_3(4).2_1 \) or \( 6.L_3(4).2_2 \) or \( 6.L_3(4).2_3 \). Note that \( 6.L_3(4).2_1 \) has elements of order 24 and 
\( 2 \cdot G_2(4) \) does not have any element of order 24. Therefore, \( M \) is \( 6.L_3(4).2_2 \) or \( 6.L_3(4).2_3 \). 
In either case, by [12, p. 25] and [36, p. 56, 58], we have \( \varnothing_\ell(M) = 12 \) for every \( \ell \neq 2, 3 \). 
Hence \( \hat{\chi}|_M \) is irreducible for every \( \ell \neq 2, 3 \). \( \square \)

**Proof of Theorem 2.4.1.**

(i) According to [12, p. 61], if \( M \) is a maximal subgroup of \( G_2(3) \) then \( M \) 
is \( G_2(3) \)-conjugate to one of the following groups:

1. \( P = [3^5] : 2S_4 \), \( Q = [3^5] : 2S_4 \), maximal parabolic subgroups,
2. \( U_3(3) : 2 \), two non-conjugate subgroups,
3. \( L_3(3) : 2 \), two non-conjugate subgroups,
4. \( L_2(8) : 3 \),
5. \( 2^3 \cdot L_3(2) \),
6. \( L_2(13) \),
7. \( [2^5] : 3^2.2. \)

By Lemmas 2.4.2 – 2.4.3, we need to consider the following cases:

1) \( M = P \) or \( Q \). The structure of \( P \) as well as \( Q \) is \( 3^5 : 2S_4 \). So they are solvable. 
It is well-known that every Brauer character of them is liftable to complex characters.
Therefore the degree of any irreducible Brauer character divides $|P| = |Q| = 11,664$ and less than $\sqrt{11,664} = 108$. Checking both the complex and Brauer character tables of $G_2(3)$, we see that there is no characters satisfying these conditions.

3) $M = L_3(3) : 2$. From [12, p. 13], we have $m_{\mathbb{C}}(L_3(3) : 2) = 52$. So if $\varphi|M$ is irreducible then $\varphi(1) \leq 52$ and therefore $\varphi(1) = 14$. So $\varphi$ is a reduction modulo $\ell \neq 3$ of the unique nontrivial irreducible character of degree 14, which is denoted by $\chi_2$ in [12, p. 60]. Since $14 \nmid |L_3(3) : 2| = 11,232$, $\chi_2|L_3(3):2$ is reducible and so is $\widehat{\chi_2}|L_3(3):2$.

4) $M = L_2(8) : 3$. From [12, p. 6], we have $m_{\mathbb{C}}(L_2(8) : 3) = 27$. Again, if $\varphi|M$ is irreducible then $\varphi(1) = 14$. Inspecting [12, p. 6] and [36, p. 6], we see that $L_2(8) : 3$ does not have any irreducible Brauer character of degree 14. Therefore, $\varphi|L_2(8):3$ is reducible for every $\varphi \in \text{IBr}_\ell(G_2(3))$ with $\ell \neq 3$.

6) $M = L_2(13)$. From [12, p. 8], we have $m_{\mathbb{C}}(L_2(13)) = 14$. So if $\varphi|M$ is irreducible then $\varphi(1) = 14$. That means the unique possibility for $\varphi$ is the reduction modulo $\ell \neq 3$ of $\chi_2$. Now we will show that $\chi_2|L_2(13)$ is reducible and therefore $\widehat{\chi_2}|L_2(13)$ is also reducible for every $\ell \neq 3$. Suppose $\chi_2|L_2(13)$ is irreducible. Then it is one of the two characters of $L_2(13)$ of degree 14. The values of these characters on the unique class of elements of order 3 of $L_2(13)$ is $-1$. On the other hand, by [12, p. 60, 61], this class is contained in the class $3D$ of $G_2(3)$ and $\chi_2(3D) = 2$, a contradiction.

7) $M = [2^5] : 3^2.2$. We have $m_{\mathbb{C}}([2^5] : 3^2.2) \leq \sqrt{576} = 24$. Hence if $\varphi|M$ is irreducible then $\varphi(1) \leq 24$. Therefore $\varphi(1) = 14$ and $\varphi = \widehat{\chi_2}$ for $\ell \neq 3$. Note that $14 \nmid 576 = |[2^5] : 3^2.2|$. So $\chi_2|M$ is reducible and so is $\widehat{\chi_2}|M$ for every $\ell \neq 3$.

(ii) In this part, we only consider faithful irreducible characters of $3 \cdot G_2(3)$. They are characters which are not inflated from irreducible characters of $G_2(3)$. By Lemma 2.1.7, a maximal subgroup of $3 \cdot G_2(3)$ is the pre-image of a maximal subgroup of $G_2(3)$ under the natural projection $\pi : 3 \cdot G_2(3) \rightarrow G_2(3)$. We denote by $3.X$ the pre-image of $X$ under $\pi$. By Lemmas 2.4.4 – 2.4.7, we need to consider the following cases:
5) $M = 3.(2^3 \cdot L_3(2))$. If $\varphi|_M$ is irreducible then $\varphi(1) \leq m_C(M) \leq \sqrt{|3.(2^3 \cdot L_3(2))|} = \sqrt{4032} < 64$. So $\varphi(1) = 27$ and $\varphi$ is a reduction modulo $\ell \neq 3$ of an irreducible character $\chi$ of degree 27 of $G_2(3)$. When $\ell = 0$, since $27 \nmid |M|$, $\chi|_M$ is reducible. It follows that $\hat{\chi}|_M$ is also reducible.

6) $M = 3.L_2(13)$. From [12, p. 8], the Schur multiplier of $L_2(13)$ has order 2. So $3.L_2(13) = 3 \times L_2(13)$. Therefore we have $m_C(M) = m_C(L_2(13)) = 14$. On the other hand, the degree of any faithful irreducible Brauer character of $3 \cdot G_2(3)$ is at least 27. So we do not have any example in this case.

7) $M = 3.(2^5 : 3^2.2)$. We have $m_C(M) \leq \sqrt{|M : Z(M)|} \leq \sqrt{|2^5 : 3^2.2|} = \sqrt{576} = 24$. Again, since the degree of any faithful irreducible Brauer character of $3 \cdot G_2(3)$ is at least 27, there is no example in this case.

(iii) According to [12, p. 97], if $M$ is a maximal subgroup of $G_2(4)$ then $M$ is $G_2(4)$-conjugate to one of the following groups:

1. $P = 2^{2+8} : (3 \times A_5)$, $Q = 2^{4+6} : (A_5 \times 3)$, maximal parabolic subgroups.
2. $U_3(4) : 2$,
3. $SL_3(4) : 2$,
4. $U_3(3) : 2$,
5. $A_5 \times A_5$,
6. $L_2(13)$,
7. $J_2$.

By Lemmas 2.4.8 – 2.4.9, we need to consider the following cases.

1) $M = P$ or $Q$. From [17], it is easy to see that $m_C(P)$ as well as $m_C(Q)$ are less than 256. So if $\varphi|_M$ is irreducible then $\varphi(1) < 256$. Inspecting the complex and Brauer character tables of $G_2(4)$, we have two cases:

- $\ell \neq 2, 3$ and $\varphi$ is a reduction modulo $\ell$ of one of the two characters of degrees 65 and 78, which are $\chi_2$ and $\chi_3$ as denoted in [12, p. 98]. Note that both 65 and 78 do not divide $|P| = |Q| = 184,320$. So both $\hat{\chi}_2|_{P,Q}$ and $\hat{\chi}_3|_{P,Q}$ are reducible.
\( \ell = 3 \) and \( \varphi(1) = 64 \) or 78. The case \( \varphi(1) = 78 \) cannot happen by a similar reason as above. If \( \varphi(1) = 64 \) then \( \varphi = \hat{\chi}_2 - \hat{1}_{G_2(4)} \). Assume that the restriction of this character to \( P \) is irreducible. Since \( \chi_2|_P \) is reducible, \( \chi_2|_P = \lambda + \mu \) where \( \lambda, \mu \in \text{Irr}(P) \), \( \hat{\lambda} = \hat{1}_P \) and \( \hat{\mu} \in \text{IBr}_3(P) \). Then \( \mu(1) = 64 \). Inspecting the character table of \( P \) in [17, p. 358], we see that there is no irreducible character of \( P \) of degree 64 and we get a contradiction. The arguments for \( Q \) is exactly the same.

3) \( M = SL_3(4) : 2 \). This case is treated similarly as Case 3 when \( q \equiv 1(\text{mod } 3) \) in the proof of Theorem B.

4) \( M = U_3(3) : 2 \). We have \( m_C(U_3(3) : 2) = 64 \). So the unique possibility for \( \varphi \) is the character of smallest degree 64 when \( \ell = 3 \). But \( m_3(U_3(3) : 2) = 30 \) and therefore there is no example in this case.

5) \( M = A_5 \times A_5 \). We have \( m_C(A_5) = 5 \) and therefore \( m_C(A_5 \times A_5) = 25 \). On the other hand \( d_\ell(G_2(4)) \geq 64 \) for every \( \ell \neq 2 \). So \( \varphi|_{A_5 \times A_5} \) is reducible for every \( \varphi \in \text{IBr}_\ell(G_2(4)) \) with \( \ell \neq 2 \).

6) \( M = L_2(13) \). We have \( m_C(L_2(13)) = 14 \) and \( d_\ell(G_2(4)) \geq 64 \) for every \( \ell \neq 2 \). So \( \varphi|_{L_2(13)} \) is reducible for every \( \varphi \in \text{IBr}_\ell(G_2(4)) \) with \( \ell \neq 2 \).

(iv) In this part, we only consider faithful irreducible characters of \( 2 \cdot G_2(4) \). They are characters which are not inflated from irreducible characters of \( G_2(4) \). By Lemma 2.1.7, a maximal subgroup of \( 2 \cdot G_2(4) \) is the pre-image of a maximal subgroup of \( G_2(4) \) under the natural projection \( \pi : 2 \cdot G_2(4) \to G_2(4) \). We denote by \( 2 \cdot X \) the pre-image of \( X \) under \( \pi \).

By Lemmas 2.4.10 – 2.4.13, we need to consider the following cases:

4) \( M = 2.(U_3(3) : 2) \). Since the Schur multiplier of \( U_3(3) \) is trivial, \( 2.U_3(3) = 2 \times U_3(3) \) and \( 2.(U_3(3) : 2) = (2 \times U_3(3)).2 \). So if \( \varphi|_M \) is irreducible then \( \varphi(1) \leq 2m_C(U_3(3)) = 64 \). Therefore \( \varphi \) is the reduction modulo \( \ell \neq 2 \) of \( \chi \), the unique irreducible complex character of degree 12 of \( 2 \cdot G_2(4) \). Assume \( \chi|_M \) is irreducible. Using the character table of \( U_3(3) \) in [12, p. 14], it is easy to see that \( \chi|_{U_3(3)} = 2\chi_2 \), where \( \chi_2 \) is the unique irreducible character of degree 6 of \( U_3(3) \). Therefore \( \chi|_{2 \times U_3(3)} = 2(\sigma \otimes \chi_2) \), where \( \sigma \) is the nontrivial irreducible
character of $Z_2 = Z(2 \cdot G_2(4))$. This and Lemma 2.1.8 imply that $\chi |_M$ is reducible, a contradiction.

5) $M = 2.(A_5 \times A_5)$. We denote by $A$ and $B$ the pre-images of the first and second terms $A_5$ (in $A_5 \times A_5$), respectively, under the projection $\pi$. For every $a \in A$, $b \in B$, we have $\pi([a, b]) = [\pi(a), \pi(b)] = 1$. Therefore, $[a, b] \in Z_2$ where $Z_2 = Z(2 \cdot G_2(4)) \leq Z(M)$.

This implies that $[[A, B], A] = [[B, A], A] = 1$. By 3-subgroup lemma, we have $[[A, A], B] = 1$. Since the Schur multiplier of $A_5$ is 2, $A$ is $2 \times A_5$ or $2.A_5$, the universal cover of $A_5$. If $A = 2.A_5$ then $[A, A] = A$. If $A = 2 \times A_5$ then $[A, A] = A_5$. So, in any case, $[A, B] = 1$ or in other words $A$ centralizes $B$. That means $M = (A \times B)/Z_2$.

We have $m_C(M) \leq \sqrt{|M/Z(M)|} \leq \sqrt{|A_5 \times A_5|} = 60$. Therefore if $\varphi |_M$ is irreducible then $\varphi$ is the reduction modulo $\ell \neq 2$ of $\chi$, the unique complex irreducible character of degree 12 of $2 \cdot G_2(4)$. Now we will show that $\chi |_M$ is reducible.

Suppose $\lambda$ is any irreducible character of $M$ of degree 12. Then we can regard $\lambda$ as an irreducible character of $A \times B$ with $Z_2 \subset \text{Ker} \lambda$. Assume that $\lambda = \lambda_A \otimes \lambda_B$ where $\lambda_A \in \text{Irr}(A)$ and $\lambda_B \in \text{Irr}(B)$. There are two possibilities:

• One of $\lambda_A(1)$ and $\lambda_B(1)$ is 2 and the other is 6. With no loss of generality, assume that $\lambda_A(1) = 2$ and $\lambda_B(1) = 6$. From the character tables of $A_5$ and $2.A_5$ in [12, p. 2], we see that $A_5$ does not have any irreducible character of degree 2 or 6. So $A = 2.A_5$. The value of any irreducible character of degree 2 of $2.A_5$ at any conjugacy class of elements of order 5 is $\frac{1}{2}(-1 \pm \sqrt{5})$. Therefore, $\lambda |_A = 6\lambda_A$ is not rational in this case.

• One of $\lambda_A(1)$ and $\lambda_B(1)$ is 4 and the other is 3. With no loss of generality, assume that $\lambda_A(1) = 4$ and $\lambda_B(1) = 3$. From the character tables of $A_5$ and $2.A_5$ in [12, p. 2], we see that the value of any irreducible character of degree 3 of $A_5$ or $2.A_5$ at any conjugacy class of elements of order 5 is $\frac{1}{2}(1 \pm \sqrt{5})$. Therefore, $\lambda |_B = 4\lambda_B$ is not rational in this case.

We have shown that any irreducible character of degree 12 of $M$ is not rational. On the other hand, $\chi$ is rational. So $\chi |_M$ is reducible and therefore $\hat{\chi} |_M$ is also reducible for every $\ell \neq 2$. 

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6) $M = 2.L_2(13)$. From [12, p. 8], we have $m_C(2.L_2(13)) = 14$. So if $\varphi|M$ is irreducible then $\varphi$ is the reduction modulo $\ell \neq 2$ of $\chi$. The value of $\chi$ at the unique conjugacy class of elements of order 7 of $G_2(4)$ is $-2$. Inspecting the character tables of $L_2(13)$ and its universal cover [12, p. 8], we see that $M$ must be the universal cover of $L_2(13)$ and $\chi|M$ is a sum of two irreducible characters of degree 6. That means $\chi|M$ is reducible and therefore there is no example in this case.

7) $M = 2.J_2$. We already know from the proof of Lemma 2.4.9 that $M$ is the universal cover of $J_2$. Inspecting the character table of $2.J_2$ in [12, p. 43] and [36, p. 103, 104, 105], we see that the degrees of faithful irreducible characters of $2 \cdot G_2(4)$ and those of $2.J_2$ are different from each other. Therefore $\varphi|_{2.J_2}$ is reducible for every faithful irreducible character $\varphi$ of $2 \cdot G_2(4)$. \qed
Table 2-1. Degrees of irreducible complex characters of $G_2(q)$

<table>
<thead>
<tr>
<th>Character</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{11}$</td>
<td>1</td>
</tr>
<tr>
<td>$X_{12}$</td>
<td>$q^6$</td>
</tr>
<tr>
<td>$X_{13}$</td>
<td>$\frac{1}{2}q(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X_{14}$</td>
<td>$\frac{1}{2}q(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X_{15}$</td>
<td>$\frac{3}{2}q(q + 1)^2(q^2 - q + 1)$</td>
</tr>
<tr>
<td>$X_{16}$</td>
<td>$\frac{1}{2}q(q + 1)^2(q^2 + q + 1)$</td>
</tr>
<tr>
<td>$X_{17}$</td>
<td>$\frac{1}{2}q(q - 1)^2(q^2 + q + 1)$</td>
</tr>
<tr>
<td>$X_{18}$</td>
<td>$\frac{1}{2}q(q - 1)^2(q^2 - q + 1)$</td>
</tr>
<tr>
<td>$X_{19}(k)$</td>
<td>$\frac{1}{2}q(q - 1)^2(q + 1)^2$</td>
</tr>
<tr>
<td>$X_{31}$</td>
<td>$q^3(q^2 + \epsilon)$</td>
</tr>
<tr>
<td>$X_{32}$</td>
<td>$q^3 + \epsilon$</td>
</tr>
<tr>
<td>$X_{33}$</td>
<td>$q(q + \epsilon)(q^3 + \epsilon)$</td>
</tr>
<tr>
<td>$X_{21}$</td>
<td>$q^2(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X_{22}$</td>
<td>$q^4 + q^2 + 1$</td>
</tr>
<tr>
<td>$X_{23}$</td>
<td>$q(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X_{24}$</td>
<td>$q(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X_{1a}$</td>
<td>$q(q + 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X'_{1a}$</td>
<td>$(q + 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X_{1b}$</td>
<td>$q(q + 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X'_{1b}$</td>
<td>$(q + 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X_{2a}$</td>
<td>$q(q - 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X'_{2a}$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X_{2b}$</td>
<td>$q(q - 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X'_{2b}$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X_1$</td>
<td>$(q + 1)^2(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X_2$</td>
<td>$(q - 1)^2(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$X_a$</td>
<td>$q^6 - 1$</td>
</tr>
<tr>
<td>$X_b$</td>
<td>$q^6 - 1$</td>
</tr>
<tr>
<td>$X_3$</td>
<td>$(q^2 - 1)^2(q^2 - q + 1)$</td>
</tr>
<tr>
<td>$X_6$</td>
<td>$(q^2 - 1)^2(q^2 + q + 1)$</td>
</tr>
</tbody>
</table>

where:

(i) $X_{21}, X_{22}, X_{23}, X_{24}$ appear only if $q$ is odd,
(ii) $X_{31}, X_{32}, X_{33}$ appear only if $q$ is not divisible by 3,
(iii) $q \equiv \epsilon (\text{mod } 3)$.

This table is collected from [10], [16], and [17].
<table>
<thead>
<tr>
<th>Character</th>
<th>$\varphi_1$</th>
<th>$\varphi_2$</th>
<th>$\varphi_3$</th>
<th>$\varphi_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_{11}$</td>
<td>$\frac{1}{6}(q-1)^2(6q^4 + (8-3\alpha - \beta)q^3)$</td>
<td>$\frac{1}{6}(q-1)^2(6q^4 + (8-3\alpha - \beta)q^3)$</td>
<td>$\frac{1}{6}(q-1)^2(6q^4 + (8-3\alpha - \beta)q^3)$</td>
<td>$\frac{1}{6}(q-1)^2(6q^4 + (8-3\alpha - \beta)q^3)$</td>
</tr>
<tr>
<td>$\varphi_{12}$</td>
<td>$(10 - 3\alpha + \beta)q^2$</td>
<td>$(10 - 3\alpha + \beta)q^2$</td>
<td>$(10 - 3\alpha + \beta)q^2$</td>
<td>$(10 - 3\alpha + \beta)q^2$</td>
</tr>
<tr>
<td>$\varphi_{13}$</td>
<td>$\frac{1}{3}(q-1)(q^4 + q^3 + 2q^2 + 2q + 3)$</td>
<td>$\frac{1}{3}(q-1)(q^4 + q^3 + 2q^2 + 2q + 3)$</td>
<td>$\frac{1}{3}(q-1)(q^4 + q^3 + 2q^2 + 2q + 3)$</td>
<td>$\frac{1}{3}(q-1)(q^4 + q^3 + 2q^2 + 2q + 3)$</td>
</tr>
<tr>
<td>$\varphi_{14}$</td>
<td>$\frac{1}{3}(q-1)(q^4 + q^3 + 2q^2 + 2q + 3)$</td>
<td>$\frac{1}{3}(q-1)(q^4 + q^3 + 2q^2 + 2q + 3)$</td>
<td>$\frac{1}{3}(q-1)(q^4 + q^3 + 2q^2 + 2q + 3)$</td>
<td>$\frac{1}{3}(q-1)(q^4 + q^3 + 2q^2 + 2q + 3)$</td>
</tr>
<tr>
<td>$\varphi_{15}$</td>
<td>$q^4 + q^2$</td>
<td>$q^4 + q^2$</td>
<td>$q^4 + q^2$</td>
<td>$q^4 + q^2$</td>
</tr>
<tr>
<td>$\varphi_{16}$</td>
<td>$\frac{1}{3}q(q-1)^2(q^2 + q + 1)$</td>
<td>$\frac{1}{3}q(q-1)^2(q^2 + q + 1)$</td>
<td>$\frac{1}{3}q(q-1)^2(q^2 + q + 1)$</td>
<td>$\frac{1}{3}q(q-1)^2(q^2 + q + 1)$</td>
</tr>
<tr>
<td>$\varphi_{18}$</td>
<td>$\frac{1}{3}q(q-1)^2(q^2 - q + 1)$</td>
<td>$\frac{1}{3}q(q-1)^2(q^2 - q + 1)$</td>
<td>$\frac{1}{3}q(q-1)^2(q^2 - q + 1)$</td>
<td>$\frac{1}{3}q(q-1)^2(q^2 - q + 1)$</td>
</tr>
<tr>
<td>$\varphi_{19}(k)$</td>
<td>$\frac{1}{3}q(q-1)^2(q+1)^2$</td>
<td>$\frac{1}{3}q(q-1)^2(q+1)^2$</td>
<td>$\frac{1}{3}q(q-1)^2(q+1)^2$</td>
<td>$\frac{1}{3}q(q-1)^2(q+1)^2$</td>
</tr>
<tr>
<td>$\varphi_{31}$</td>
<td>$q^6 - 1$ if $q \equiv 1(\text{mod } 3)$</td>
<td>$q^6 - 1$ if $q \equiv 1(\text{mod } 3)$</td>
<td>$(q-1)^2(q^2 + q + 1)$ if $q \equiv 1(\text{mod } 3)$</td>
<td>$(q-1)^2(q^2 + q + 1)$ if $q \equiv 1(\text{mod } 3)$</td>
</tr>
<tr>
<td>&amp; $(q-1)^2(q^2 + q + 1)(q^2 + q + 1)$ if $q \equiv -1(\text{mod } 3)$</td>
<td>$(q-1)^2(q^2 + q + 1)(q^2 + q + 1)$ if $q \equiv -1(\text{mod } 3)$</td>
<td>$(q-1)^2(q^2 + q + 1)$ if $q \equiv -1(\text{mod } 3)$</td>
<td>$(q-1)^2(q^2 + q + 1)$ if $q \equiv -1(\text{mod } 3)$</td>
<td></td>
</tr>
<tr>
<td>$\varphi_{32}$</td>
<td>$q^4 + \epsilon$</td>
<td>$q^4 + \epsilon$</td>
<td>$q^4 + \epsilon$</td>
<td>$q^4 + \epsilon$</td>
</tr>
<tr>
<td>$\varphi_{33}$</td>
<td>$q(q + \epsilon)(q^3 + \epsilon)$</td>
<td>$q(q + \epsilon)(q^3 + \epsilon)$</td>
<td>$q(q + \epsilon)(q^3 + \epsilon)$</td>
<td>$q(q + \epsilon)(q^3 + \epsilon)$</td>
</tr>
<tr>
<td>$\varphi_{1a}$</td>
<td>$(q^2 - 1)(q^2 + q^2 + 1)$</td>
<td>$(q^2 - 1)(q^2 + q^2 + 1)$</td>
<td>$(q^2 - 1)(q^2 + q^2 + 1)$</td>
<td>$(q^2 - 1)(q^2 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{1b}$</td>
<td>$(q + 1)(q^4 + q^2 + 1)$</td>
<td>$(q + 1)(q^4 + q^2 + 1)$</td>
<td>$(q + 1)(q^4 + q^2 + 1)$</td>
<td>$(q + 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{2a}$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{2b}$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{2c}$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{2d}$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{3a}$</td>
<td>$q^6 - 1$</td>
<td>$q^6 - 1$</td>
<td>$q^6 - 1$</td>
<td>$q^6 - 1$</td>
</tr>
<tr>
<td>$\varphi_{3b}$</td>
<td>$q^6 - 1$</td>
<td>$q^6 - 1$</td>
<td>$q^6 - 1$</td>
<td>$q^6 - 1$</td>
</tr>
<tr>
<td>$\varphi_{3c}$</td>
<td>$(q^2 - 1)(q^2 - q + 1)$</td>
<td>$(q^2 - 1)(q^2 - q + 1)$</td>
<td>$(q^2 - 1)(q^2 - q + 1)$</td>
<td>$(q^2 - 1)(q^2 - q + 1)$</td>
</tr>
<tr>
<td>$\varphi_{3d}$</td>
<td>$(q^2 - 1)^2(q^2 + q + 1)$</td>
<td>$(q^2 - 1)^2(q^2 + q + 1)$</td>
<td>$(q^2 - 1)^2(q^2 + q + 1)$</td>
<td>$(q^2 - 1)^2(q^2 + q + 1)$</td>
</tr>
</tbody>
</table>

where:

(i) $\varphi_{31}, \varphi_{32}, \varphi_{33}$ appear only if $q$ is not divisible by 3,
(ii) $0 \leq \alpha \leq q - 1$ if $p \neq 3$ and $0 \leq \alpha \leq 2q$ if $p = 3$,
(iii) $0 \leq \beta \leq \frac{1}{3}(q + 2),$
(iv) $1 \leq \gamma \leq \frac{1}{3}(q + 1).$

Therefore, if $p \neq 3$ then $\varphi_{12}(1) \geq \frac{1}{3}(q - 1)^2(q + 1)(q^3 + 2q^2 + 2q + 3)$ and if $p = 3$ then $\varphi_{12}(1) \geq \frac{1}{3}(q - 1)^2(q^3 + 2q^2 + 4q + 3).$ Moreover, when $4 \mid (q + 1)$ and $q \equiv -1(\text{mod } 3)$, we have $\varphi_{31}(1) \geq \frac{1}{3}(q - 1)^2(q^2 + q + 1)(2q^2 + 2q + 3).$

This table is collected from [34].
Table 2-3. Degrees of 3-Brauer characters of $G_2(q)$, $3 ∤ q$

| Character | $3 | (q - 1), 0 ≤ α ≤ 1$ | $3 | (q + 1), 1 ≤ α ≤ q + 1$ |
|-----------|--------------------------|-----------------------------|
| $\varphi_{11}$ | 1 | 1 |
| $\varphi_{12}$ | $\frac{1}{6}(q - 1)^2(6q^4 + (9 - \beta - 2\gamma)q^2 + (9 - \beta - 2\gamma)q + 6)$ | $\frac{1}{6}(q - 1)^2(6q^4 + (11 - \alpha - 2\beta - 3\gamma)q^2 + + (11 - \alpha - 2\beta - 3\gamma)q + 6)$ |
| $\varphi_{14}$ | $\frac{1}{6}(q(q^2 - q + 1)((1 - \alpha)q^2 q^2 + (1 - \alpha))$ | $\frac{1}{6}(q^2 - 1)(q^3 + 3q^2 - q + 6)$ |
| $\varphi_{15}$ | $\frac{1}{2}(q^4 + q^2 + q^2 - q - 2)$ | $\frac{1}{2}q(q + 1)^2(q^2 - q + 1)$ |
| $\varphi_{16}$ | $q^3$ | $q^3 - 1$ |
| $\varphi_{17}$ | $\frac{1}{2}q(q - 1)^2(q^2 + q + 1)$ | $\frac{1}{2}q(q - 1)^2(q^2 + q + 1)$ |
| $\varphi_{18}$ | $\frac{1}{2}q(q - 1)^2(q^2 - q + 1)$ | $\frac{1}{2}q(q - 1)^2(q^2 - q + 1)$ |
| $\varphi_{19}$ | $\frac{1}{2}q(q - 1)^2(q^2 + 1)^2$ | $\frac{1}{2}q(q - 1)^2(q^2 + 1)^2$ |
| $\varphi_{21}$ | $q^2(q^4 + q^2 + 1)$ | $(q - 1)^2(q^4 + q^2 + 1)$ |
| $\varphi_{22}$ | $q^4 + q^2 + 1$ | $q^4 + q^2 + 1$ |
| $\varphi_{23}$ | $q(q^4 + q^2 + 1)$ | $(q - 1)(q^4 + q^2 + 1)$ |
| $\varphi_{24}$ | $q(q^4 + q^2 + 1)$ | $(q - 1)(q^4 + q^2 + 1)$ |
| $\varphi_{1a}$ | $q(q + 1)(q^4 + q^2 + 1)$ | $(q - 1)(q^4 + q^2 + 1)$ |
| $\varphi_{1' a}$ | $(q + 1)(q^4 + q^2 + 1)$ | $(q + 1)(q^4 + q^2 + 1)$ |
| $\varphi_{1b}$ | $q(q + 1)(q^4 + q^2 + 1)$ | $(q - 1)(q^4 + q^2 + 1)$ |
| $\varphi_{1' b}$ | $(q + 1)(q^4 + q^2 + 1)$ | $(q + 1)(q^4 + q^2 + 1)$ |
| $\varphi_{2a}$ | $q(q - 1)(q^4 + q^2 + 1)$ | $(q - 1)^2(q^4 + q^2 + 1)$ |
| $\varphi_{2' a}$ | $(q - 1)(q^4 + q^2 + 1)$ | $(q - 1)^2(q^4 + q^2 + 1)$ |
| $\varphi_{2b}$ | $q(q - 1)(q^4 + q^2 + 1)$ | $(q - 1)^2(q^4 + q^2 + 1)$ |
| $\varphi_{2' b}$ | $(q - 1)(q^4 + q^2 + 1)$ | $(q - 1)^2(q^4 + q^2 + 1)$ |
| $\varphi_{1}$ | $(q + 1)^2(q^4 + q^2 + 1)$ | $(q + 1)^2(q^4 + q^2 + 1)$ |
| $\varphi_{2}$ | $(q - 1)^2(q^4 + q^2 + 1)$ | $(q - 1)^2(q^4 + q^2 + 1)$ |
| $\varphi_{2a}$ | $q^6 - 1$ | $q^6 - 1$ |
| $\varphi_{2a}$ | $q^6 - 1$ | $q^6 - 1$ |
| $\varphi_{3}$ | $(q^2 - 1)^2(q^2 - q + 1)$ | $(q^2 - 1)^2(q^2 - q + 1)$ |
| $\varphi_{6}$ | $(q^2 - 1)^2(q^2 + q + 1)$ | $(q^2 - 1)^2(q^2 + q + 1)$ |

where:

(i) $\varphi_{21}$, $\varphi_{22}$, $\varphi_{23}$, $\varphi_{24}$ appear only if $q$ is odd,

(ii) when $3 \mid (q - 1)$, $\varphi_{14}(1) \in \{\frac{1}{6}q(q^2 - q + 1)(q^2 + 4q + 1), q^2(q^2 - q + 1)\}$

and $\varphi_{12} \geq \frac{1}{4}(q - 1)^2(q^2 + 2q^3 + 3q + 2)$,

(iii) when $3 \mid (q + 1)$, $\varphi_{12} \geq \frac{1}{4}(q - 1)^2(q + 2)^2(q^2 + q + 1)$.

This table is collected from [33].
Table 2.4. Degrees of $\ell$-Brauer characters of $G_2(q)$, $\ell \geq 5$ and $\ell \nmid q$

<table>
<thead>
<tr>
<th>Character</th>
<th>$\ell \mid (q - 1)$</th>
<th>$\ell \mid (q + 1)$, $1 \leq \alpha \leq \frac{1}{2}q$</th>
<th>$1 \leq \beta \leq \frac{1}{3}(q + 2)$, $2 \leq \gamma \leq \frac{1}{3}(q + 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_{11}$</td>
<td>$q^0$</td>
<td>$\frac{1}{6}(q - 1)^2(6q^4 + (8 - 3\alpha - \beta)q^2$</td>
<td>$(10 - 3\alpha + \beta)q^2$</td>
</tr>
<tr>
<td>$\varphi_{12}$</td>
<td>$q^6$</td>
<td>$\frac{1}{5}(q - 1)(q^4 + q^2 + 2q + 3)$</td>
<td>$\frac{1}{5}(q - 1)(q^4 + q^3 + 2q^2 + 2q + 3)$</td>
</tr>
<tr>
<td>$\varphi_{13}$</td>
<td>$\frac{1}{5}q(q^4 + q^2 + 1)$</td>
<td>$\frac{1}{5}(q - 1)(q^4 + q^2 + 2q + 3)$</td>
<td>$\frac{1}{5}(q - 1)(q^4 + q^3 + 2q^2 + 2q + 3)$</td>
</tr>
<tr>
<td>$\varphi_{14}$</td>
<td>$\frac{1}{3}q(q + 1)^2(q^2 - q + 1)$</td>
<td>$\frac{1}{3}q(q + 1)^2(q^2 - q + 1)$</td>
<td>$\frac{1}{3}q(q + 1)^2(q^2 - q + 1)$</td>
</tr>
<tr>
<td>$\varphi_{15}$</td>
<td>$\frac{1}{3}q(q + 1)^2(q^2 - q + 1)$</td>
<td>$\frac{1}{3}q(q + 1)(q^2 + q + 1)$</td>
<td>$\frac{1}{3}q(q + 1)(q^2 + q + 1)$</td>
</tr>
<tr>
<td>$\varphi_{16}$</td>
<td>$\frac{1}{3}q(q - 1)^2(q^2 + q + 1)$</td>
<td>$\frac{1}{6}q(q - 1)^2(q^2 - q + 1)$</td>
<td>$\frac{1}{6}q(q - 1)^2(q^2 - q + 1)$</td>
</tr>
<tr>
<td>$\varphi_{17}$</td>
<td>$\frac{1}{3}q(q - 1)^2(q^2 - q + 1)$</td>
<td>$\frac{1}{3}q(q - 1)^2(q^2 + q + 1)$</td>
<td>$\frac{1}{3}q(q - 1)^2(q^2 + q + 1)$</td>
</tr>
<tr>
<td>$\varphi_{18}$</td>
<td>$\frac{1}{3}q(q - 1)^2(q^2 - q + 1)$</td>
<td>$\frac{1}{3}q(q - 1)^2(q^2 + q + 1)$</td>
<td>$\frac{1}{3}q(q - 1)^2(q^2 + q + 1)$</td>
</tr>
<tr>
<td>$\varphi_{19}(k)$</td>
<td>$\frac{1}{3}q(q - 1)^2(q + 1)^2$</td>
<td>$q^0 - 1$ if $q \equiv 1(\text{mod } 3)$</td>
<td>$(q^3 - 1)(q^3 - \gamma q^2 + \gamma q - 1)$</td>
</tr>
</tbody>
</table>

where:

(i) $\varphi_{21}, \varphi_{22}, \varphi_{23}, \varphi_{24}$ appear only if $q$ is odd,

(ii) $\varphi_{31}, \varphi_{32}, \varphi_{33}$ appear only if $q$ is not divisible by 3,

(iii) when $\ell \mid (q + 1)$, $\varphi_{12}(1) \geq \frac{1}{18}(q - 1)^2(13q^4 - 4q^3 + 26q^2 + 46q + 18)$,

(iv) when $\ell \mid (q + 1)$ and $q \equiv -1(\text{mod } 3)$, $\varphi_{31}(1) \geq \frac{1}{3}(q^3 - 1)(2q^3 + q - 3)$.

This table is collected from [29] and [62].
<table>
<thead>
<tr>
<th>Character</th>
<th>$\ell \mid (q^2 + q + 1)$</th>
<th>$\ell \mid (q^2 - q + 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_{11}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\varphi_{12}$</td>
<td>$\frac{1}{6}(6q^6 - 7q^5 - 3q^4 + 8q^3 - 3q^2 - 7q + 6)$</td>
<td>$q^6 - \frac{1}{5}q(q + 1)^2(q^2 + q + 1) + 1$</td>
</tr>
<tr>
<td>$\varphi_{13}$</td>
<td>$\frac{1}{2}q(q^4 + q^2 + 1)$</td>
<td>$\frac{1}{3}q(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{14}$</td>
<td>$\frac{1}{2}q(q^4 + q^2 + 1)$</td>
<td>$\frac{1}{2}q(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{15}$</td>
<td>$\frac{1}{2}q^2(q^5 + q^4 + q^2 + q - 2)$</td>
<td>$\frac{1}{2}q(q + 1)^2(q^2 + q + 1)$</td>
</tr>
<tr>
<td>$\varphi_{16}$</td>
<td>$\frac{1}{3}q(q + 1)^2(q^2 + q + 1)$</td>
<td>$\frac{1}{6}q(q + 1)^2(q^2 + q + 1) - 1$</td>
</tr>
<tr>
<td>$\varphi_{17}$</td>
<td>$\frac{1}{2}q(q - 1)^2(q^2 + q + 1)$</td>
<td>$\frac{1}{3}q(q - 1)^2(q^2 + q + 1)$</td>
</tr>
<tr>
<td>$\varphi_{18}$</td>
<td>$\frac{1}{2}q^2(q - 1)^2(q^2 - q + 1)$</td>
<td>$\frac{1}{3}q(q - 1)^2(q^2 - q + 1)$</td>
</tr>
<tr>
<td>$\varphi_{19}(k)$</td>
<td>$\frac{1}{2}q(q - 1)^2(q^2 + 1)$</td>
<td>$\frac{1}{3}q(q - 1)^2(q + 1)^2$</td>
</tr>
<tr>
<td>$\varphi_{31}$</td>
<td>$(q^3 - q^2 - q)(q^3 + 1)$</td>
<td>$q^3(q^3 + 1)$ if $q \equiv 1$(mod 3)</td>
</tr>
<tr>
<td>&amp; if $q \equiv 1$(mod 3)</td>
<td>$(q^3 - 1)^2(q - 1)$ if $q \equiv -1$(mod 3)</td>
<td></td>
</tr>
<tr>
<td>$\varphi_{32}$</td>
<td>$q^3 + \epsilon$</td>
<td>$q^3 + \epsilon$</td>
</tr>
<tr>
<td>$\varphi_{33}$</td>
<td>$(q^2 + q - 1)(q^3 + 1)$</td>
<td>$q(q + \epsilon)(q^3 + \epsilon)$</td>
</tr>
<tr>
<td>&amp; if $q \equiv 1$(mod 3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; $q(q - 1)(q^3 - 1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; if $q \equiv -1$(mod 3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\varphi_{21}$</td>
<td>$q^2(q^4 + q^2 + 1)$</td>
<td>$q^2(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{22}$</td>
<td>$q^4 + q^2 + 1$</td>
<td>$q^4 + q^2 + 1$</td>
</tr>
<tr>
<td>$\varphi_{23}$</td>
<td>$q(q^4 + q^2 + 1)$</td>
<td>$q(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{24}$</td>
<td>$q^2(q^4 + q^2 + 1)$</td>
<td>$q^2(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{1a}$</td>
<td>$q(q + 1)(q^4 + q^2 + 1)$</td>
<td>$q(q + 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{1b}$</td>
<td>$(q + 1)(q^4 + q^2 + 1)$</td>
<td>$(q + 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{1c}$</td>
<td>$q(q + 1)(q^4 + q^2 + 1)$</td>
<td>$q(q + 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{1d}$</td>
<td>$(q + 1)(q^4 + q^2 + 1)$</td>
<td>$(q + 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{2a}$</td>
<td>$q(q - 1)(q^4 + q^2 + 1)$</td>
<td>$q(q - 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{2b}$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{1}$</td>
<td>$(q^2 - 1)^2(q^4 + q^2 + 1)$</td>
<td>$(q^2 - 1)^2(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{2}$</td>
<td>$(q - 1)^2(q^4 + q^2 + 1)$</td>
<td>$(q - 1)^2(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{a}$</td>
<td>$q^8 - 1$</td>
<td>$q^8 - 1$</td>
</tr>
<tr>
<td>$\varphi_{b}$</td>
<td>$q^6 - 1$</td>
<td>$q^6 - 1$</td>
</tr>
<tr>
<td>$\varphi_{3}$</td>
<td>$(q^2 - 1)^2(q^2 - q + 1)$</td>
<td>$(q^2 - 1)^2(q^2 - q + 1)$</td>
</tr>
<tr>
<td>$\varphi_{6}$</td>
<td>$(q - 1)^2(q^2 + q + 1)$</td>
<td>$(q - 1)^2(q^2 + q + 1)$</td>
</tr>
</tbody>
</table>

where:

(i) $\varphi_{21}, \varphi_{22}, \varphi_{23}, \varphi_{24}$ appear only if $q$ is odd,

(ii) $\varphi_{31}, \varphi_{32}, \varphi_{33}$ appear only if $q$ is not divisible by 3.

This table is collected from [61].
Table 2-6. Fusion of conjugacy classes of $P$ in $G_2(3)$

<table>
<thead>
<tr>
<th>Fusion in $[16]$</th>
<th>Corresponding class in $[12]$</th>
<th>Length</th>
<th>Value of $\chi_{24}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1 \subset A_1$</td>
<td>$1A$</td>
<td>1</td>
<td>27</td>
</tr>
<tr>
<td>$A_2 \subset A_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_3 \subset A_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{41} \subset A_{31}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{42} \subset A_{32}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{43} \subset A_{32}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{51} \subset A_{41}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{52} \subset A_{42}$</td>
<td>3A,3B,3C,3D,3E</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>$A_{61} \subset A_{31}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{62} \subset A_{41}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{63} \subset A_{42}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{64} \subset A_{41}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{65}(t) \subset A_{32}, A_{41}, A_{42}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{66}(t) \subset A_{41}, A_{42}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{71} \subset A_{51}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{72} \subset A_{52}$</td>
<td>9A,9B,9C</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>$A_{73} \subset A_{53}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_{11} \subset B_1$</td>
<td>2A</td>
<td>324</td>
<td>3</td>
</tr>
<tr>
<td>$B_{12} \subset B_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_{13} \subset B_3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_{14} \subset B_4$</td>
<td>6A,6B,6C,6D</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>$B_{15} \subset B_5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_{21} \subset B_1$</td>
<td>2A</td>
<td>81</td>
<td>3</td>
</tr>
<tr>
<td>$B_{22} \subset B_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_{23} \subset B_3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_{24} \subset B_4$</td>
<td>6A,6B,6C,6D</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>$B_{25} \subset B_5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_1 \subset D_{11}$</td>
<td>4A,4B</td>
<td>486</td>
<td>-1, 3</td>
</tr>
<tr>
<td>$D_2 \subset D_{12}$</td>
<td>12A,12B</td>
<td>972</td>
<td>2, 0</td>
</tr>
<tr>
<td>$E_2(i) \subset E_2$ (two classes)</td>
<td>8A,8B</td>
<td>1458</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>
Table 2-7. Fusion of conjugacy classes of $P$ in $G_2(4)$

<table>
<thead>
<tr>
<th>Fusion in $[17]$</th>
<th>Corresponding class in $[12]$</th>
<th>Length</th>
<th>Value of $\chi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0 \subset A_0$</td>
<td>1A</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>$A_1 \subset A_1$</td>
<td>2A</td>
<td>3</td>
<td>-4</td>
</tr>
<tr>
<td>$A_2 \subset A_1$</td>
<td>2A</td>
<td>60</td>
<td>-4</td>
</tr>
<tr>
<td>$A_3 \subset A_2$</td>
<td>2B</td>
<td>240</td>
<td>0</td>
</tr>
<tr>
<td>$A_{41} \subset A_{31}$</td>
<td>4A</td>
<td>120</td>
<td>-4</td>
</tr>
<tr>
<td>$A_{42} \subset A_{32}$</td>
<td>4C</td>
<td>360</td>
<td>0</td>
</tr>
<tr>
<td>$A_5 \subset A_4$</td>
<td>4B</td>
<td>240</td>
<td>0</td>
</tr>
<tr>
<td>$A_{61} \subset A_2$</td>
<td>2B</td>
<td>960</td>
<td>0</td>
</tr>
<tr>
<td>$A_{62} \subset A_{31}$</td>
<td>4A</td>
<td>1440</td>
<td>-4</td>
</tr>
<tr>
<td>$A_{63} \subset A_{32}$</td>
<td>4C</td>
<td>1440</td>
<td>0</td>
</tr>
<tr>
<td>$A_{71} \subset A_{51}$</td>
<td>8A</td>
<td>5760</td>
<td>0</td>
</tr>
<tr>
<td>$A_{72} \subset A_{52}$</td>
<td>8B</td>
<td>5760</td>
<td>2</td>
</tr>
<tr>
<td>$B_0 \subset B_0$</td>
<td>3A</td>
<td>320</td>
<td>-6</td>
</tr>
<tr>
<td>$B_1 \subset B_1$</td>
<td>6A</td>
<td>960</td>
<td>2</td>
</tr>
<tr>
<td>$[B_2] \subset B_1$</td>
<td>6A</td>
<td>3840</td>
<td>2</td>
</tr>
<tr>
<td>$[B_3] \subset B_1$</td>
<td>6A</td>
<td>3840</td>
<td>2</td>
</tr>
<tr>
<td>$B_2(0) \subset B_2(0)$</td>
<td>12A</td>
<td>3840</td>
<td>2</td>
</tr>
<tr>
<td>$B_2(i) \subset B_2(i)$ (i = 1, 2)</td>
<td>12B, 12C</td>
<td>3840</td>
<td>0</td>
</tr>
<tr>
<td>$C_{31}(i) \subset C_{21}$ (two classes)</td>
<td>3B</td>
<td>5120</td>
<td>0</td>
</tr>
<tr>
<td>$C_{32}(i) \subset C_{22}$ (two classes)</td>
<td>6B</td>
<td>15360</td>
<td>0</td>
</tr>
<tr>
<td>$C_{41}(i) \subset C_{21}$ (two classes)</td>
<td>3B</td>
<td>1024</td>
<td>0</td>
</tr>
<tr>
<td>$C_{42}(i) \subset C_{22}$ (two classes)</td>
<td>6B</td>
<td>15360</td>
<td>0</td>
</tr>
<tr>
<td>$D_{11}(i) \subset D_{11}(i)$ (two classes)</td>
<td>5C, 5D</td>
<td>3072</td>
<td>-3</td>
</tr>
<tr>
<td>$D_{12}(i) \subset D_{12}(i)$ (two classes)</td>
<td>10A, 10B</td>
<td>9216</td>
<td>1</td>
</tr>
<tr>
<td>$E(i) \subset E_1(i)$ (four classes)</td>
<td>15C, 15D</td>
<td>12288</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 2-8. Fusion of conjugacy classes of $Q$ in $G_2(4)$

<table>
<thead>
<tr>
<th>Fusion in [17]</th>
<th>Corresponding class in [12]</th>
<th>Length</th>
<th>Value of $\chi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0 \subset A_0$</td>
<td>1A</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>$A_1 \subset A_1$</td>
<td>2A</td>
<td>15</td>
<td>-4</td>
</tr>
<tr>
<td>$A_2 \subset A_2$</td>
<td>2B</td>
<td>48</td>
<td>0</td>
</tr>
<tr>
<td>$A_{31} \subset A_2$</td>
<td>2B</td>
<td>240</td>
<td>0</td>
</tr>
<tr>
<td>$A_{32} \subset A_{31}$</td>
<td>4A</td>
<td>360</td>
<td>-4</td>
</tr>
<tr>
<td>$A_{33} \subset A_{32}$</td>
<td>4C</td>
<td>360</td>
<td>0</td>
</tr>
<tr>
<td>$A_{41} \subset A_1$</td>
<td>2A</td>
<td>240</td>
<td>-4</td>
</tr>
<tr>
<td>$A_{42}(0) \subset A_{31}$</td>
<td>4A</td>
<td>240</td>
<td>-4</td>
</tr>
<tr>
<td>$A_{42}(i) \subset A_4$</td>
<td>4B</td>
<td>240</td>
<td>0</td>
</tr>
<tr>
<td>$A_{5}(t) \subset A_2, A_{32}, A_4$ (four classes)</td>
<td>2B, 4C, 4B</td>
<td>720</td>
<td>0</td>
</tr>
<tr>
<td>$A_{61} \subset A_{51}$</td>
<td>8A</td>
<td>5760</td>
<td>0</td>
</tr>
<tr>
<td>$A_{62} \subset A_{52}$</td>
<td>8B</td>
<td>5760</td>
<td>2</td>
</tr>
<tr>
<td>$B_{0}(i) \subset B_0$ (i = 1, 2)</td>
<td>3A</td>
<td>64</td>
<td>-6</td>
</tr>
<tr>
<td>$B_{1}(i) \subset B_1$ (i = 1, 2)</td>
<td>6A</td>
<td>960</td>
<td>2</td>
</tr>
<tr>
<td>$B_{2}(i) \subset B_1$ (i = 1, 2)</td>
<td>6A</td>
<td>3840</td>
<td>2</td>
</tr>
<tr>
<td>$B_{3}(i, 0) \subset B_{2}(0)$ (i = 1, 2)</td>
<td>12A</td>
<td>3840</td>
<td>2</td>
</tr>
<tr>
<td>$B_{3}(i, j) \subset B_{2}(j)$ (i = 1, 2, j = 1, 2)</td>
<td>12B, 12C</td>
<td>3840</td>
<td>0</td>
</tr>
<tr>
<td>$C_{21} \subset C_{21}$</td>
<td>3B</td>
<td>5120</td>
<td>0</td>
</tr>
<tr>
<td>$C_{22} \subset C_{22}$</td>
<td>6B</td>
<td>15360</td>
<td>0</td>
</tr>
<tr>
<td>$C_{31}(i) \subset C_{21}$ (two classes)</td>
<td>3B</td>
<td>5120</td>
<td>0</td>
</tr>
<tr>
<td>$C_{32}(i) \subset C_{22}$ (two classes)</td>
<td>6B</td>
<td>15360</td>
<td>0</td>
</tr>
<tr>
<td>$D_{11}(i) \subset D_{21}(i)$ (two classes)</td>
<td>5A, 5B</td>
<td>3072</td>
<td>2</td>
</tr>
<tr>
<td>$D_{12}(i) \subset D_{22}(i)$ (two classes)</td>
<td>10C, 10D</td>
<td>9216</td>
<td>0</td>
</tr>
<tr>
<td>$E(i) \subset E_2(i)$ (four classes)</td>
<td>15A, 15B</td>
<td>12288</td>
<td>-1</td>
</tr>
</tbody>
</table>
Proof of Theorem C. According to [67], if \( M \) is a maximal subgroup of \( G \) then \( M \) is \( G \)-conjugate to one of the following groups:

1. \( P = [q^2], Z_{q-1} \), the maximal parabolic subgroup,
2. \( D_{2(q-1)} \),
3. \( Z_{q+\sqrt{q^2+1}}, Z_4 \),
4. \( Z_{q^2-\sqrt{q^2+1}}, Z_4 \),
5. \( Sz(q_0), q = q_0^\alpha, \alpha \) prime, \( q_0 \geq 8 \).

By Lemma 2.1.1 and the irreducibility of \( \varphi|_M \), we have \( \sqrt{|M|} \geq \vartheta_\ell(G) \), which is larger or equal to \( (q - 1)\sqrt{q/2} \) by [45]. It follows that \( |M| \geq q(q - 1)^2/2 \) and therefore \( M \) can only be the maximal parabolic subgroup of \( G \).

From the complex character table of \( P \) given in [52, p. 157], we have \( m_C(P) = (q - 1)\sqrt{q/2} \). Since \( \vartheta_\ell(G) \leq \varphi(1) \leq m_\ell(P) \leq m_C(P) \), we get \( \varphi(1) = (q - 1)\sqrt{q/2} \). Using the notation and results about Brauer trees of the Suzuki groups in [7], we obtain that \( \varphi = \hat{\Gamma}_1 \) or \( \hat{\Gamma}_2 \), where \( \Gamma_1 \) and \( \Gamma_2 \) are two irreducible complex characters of \( Sz(q) \) of degree \( (q - 1)\sqrt{q/2} \). Now we will show that the restrictions of these characters to \( P \) is indeed irreducible.

First we consider the case \( \ell = 0 \). Comparing directly the values of characters on conjugacy classes, we see that \( \Gamma_1|_P = \phi_2 \) and \( \Gamma_2|_P = \phi_3 \), where \( \phi_2 \) and \( \phi_3 \) are two complex irreducible characters of \( P \) of degree \( (q - 1)\sqrt{q/2} \) and their values are given in [52].

Next, we will show that \( \hat{\varphi}_i, i = 2, 3 \), are also irreducible when \( \ell \neq 0, 2 \). Assume the contrary that \( \hat{\varphi}_2 \) is reducible. Then it is the sum of more than one \( \ell \)-Brauer irreducible characters of \( P \). These characters have degrees less than \( (q - 1)\sqrt{\ell/2} \) and liftable because \( P \) is solvable. Consider the element \( f \) of order 4 in \( P \) which is given in [52, p. 157]. Since \( \ell \) is odd and \( \text{ord}(f) = 4 \), \( f \) is an \( \ell \)-regular element. Inspecting the character table of \( P \),
we see that the value at the element $f$ of any irreducible complex character of degree less than $(q - 1)\sqrt{q/2}$ is real. On the other hand, $\phi_2(f)$ is not real, a contradiction. In conclusion, $\hat{\phi}_2$ is irreducible and so is $\hat{\phi}_3$, as $\phi_3 = \overline{\phi}_2$.

**Note:** Most of Schur multipliers of the Suzuki groups and the Ree groups are trivial except the Schur multiplier of $Sz(8)$, which is an elementary abelian group of order $2^2$. Since $Z(2^2.Sz(8))$ is not cyclic, $2^2.Sz(8)$ does not have any faithful irreducible character. The multiplier $2^2$ has three cyclic quotients of order 2 which are corresponding to groups $2.Sz(8)$, $2'.Sz(8)$ and $2''.Sz(8)$. These groups are permuted by the automorphism group. Let us consider the irreducible restriction problem for one of them, say $2.Sz(8)$.

Inspecting the character table of $2.Sz(8)$, we see that if $\varphi$ is a faithful irreducible character of $2.Sz(8)$, then $\varphi(1) \geq 8$. Therefore if $\varphi|_M$ is irreducible then $\sqrt{|M/Z(M)|} \geq 8$. So the unique possibility for $M$ is $2.P$, where $P$ is the maximal parabolic subgroup of $Sz(8)$. Moreover, $\varphi(1) \leq mC(2.P) \leq \sqrt{|P|} = \sqrt{448} < 22$. Inspecting the character table of $2.Sz(8)$, we have that $\varphi(1) = 8$ when $\ell = 5$ or $\varphi(1) = 16$ when $\ell = 13$.

- If $\varphi(1) = 8$ with $\ell = 5$ then $\varphi = \varphi_{11}$ as denoted in [36, p. 64]. Note that $P = [2^6].7$ and $2.P = 2.([2^6].7) = [2^7].7$. Also from [36, p. 64], the value of $\varphi_{11}$ at any nontrivial 2-element is 0. It follows that $[\varphi_{11}|_{[2^7]}, \varphi_{11}|_{[2^7]}][_{[2^7]}] = 8^2/2^6 = 1$ and hence $\varphi_{11}|_{[2^7]}$ is irreducible. So $\varphi_{11}|_{2.P}$ is also irreducible.

- If $\varphi(1) = 16$ with $\ell = 13$ then $\varphi = \varphi_9$ as denoted in [36, p. 65]. Since $16 > \sqrt{2^7}$, $\varphi_9|_{[2^7]}$ is reducible. It follows that $\varphi_9|_{2.P}$ is also reducible by Lemma 2.1.8.

So when $G = 2.Sz(8)$, suppose that $\varphi$ is faithful, then $\varphi|_M$ is irreducible if and only if $M = 2.P$ and $\varphi$ is the unique irreducible 5-Brauer character of degree 8.

### 3.2 Ree Groups

**Proof of Theorem D.** According to [37, p. 181], if $M$ is a maximal subgroup of $G$, $M$ is $G$-conjugate to one of the following groups:

1. $P = [q^3] : Z_{q-1}$, the maximal parabolic subgroup,
2. $2 \times L_2(q)$, involution centralizer,
3. \((2^2 \times D_{(q+1)/2}) : 3\),

4. \(\mathbb{Z}_{q+\sqrt{q+1}} : \mathbb{Z}_6\),

5. \(\mathbb{Z}_{q-\sqrt{q+1}} : \mathbb{Z}_6\),

6. \(^2G_2(q_0), q = q_0^\alpha, \alpha \text{ prime.}\)

By Lemma 2.1.1 and the irreducibility of \(\varphi|_M\), we have \(\sqrt{|M|} \geq d_\ell(G)\), which is larger or equal to \(q(q-1)\) by [45]. Therefore, \(|M| \geq q^2(q-1)^2\). This inequality happens if and only if \(M\) is the maximal parabolic subgroup \(P\). Furthermore, from the complex character table of \(P\) is given in [46, p. 88], we have \(m_\ell(P) \leq m_\ell(C) = q(q-1)\).

Assume \(\ell \geq 5\), using the results about Brauer trees of \(G\) in [31], it is easy to check that \(d_\ell(G) = q^2 - q + 1\). Therefore, there is no example in this case since \(d_\ell(G) > m_\ell(P)\).

It remains to consider the case \(\ell = 2\). Now we have \(q(q-1) \geq m_2(P) \geq \varphi(1) \geq d_2(G) \geq q(q-1)\). So \(\varphi(1) = q(q-1)\). We will check all 2-blocks of \(G\) which are studied in [46] and [74]. We also use notation in these papers.

1) \(\xi_9, \xi_{10}, \eta_{i}^\pm\) are of 2-defect 0. Their degrees are all larger than \(q(q-1)\).

2) There is one 2-block of defect 1. All characters in this block are \(\eta_r\) and \(\eta'_r\) of degree \(q^3 + 1\). By Lemma 2.1.10, there is a unique 2-Brauer irreducible character of degree \(q^3 + 1\) in this block.

3) There are several 2-blocks of defect 2. Every character in these blocks has degree \((q-1)(q^2 - q + 1)\). Applying Lemma 2.1.10 again, all 2-Brauer irreducible characters in these blocks have degree \((q-1)(q^2 - q + 1)\).

4) The principal block and its decomposition matrix are described in [46]. The degrees of irreducible 2-Brauer characters in this block are: \(\varphi_1(1) = 1, \varphi_2(1) = q(q-1), \varphi_3(1) = (q-1)(q^2 - (q+1)(\sqrt{q/3} + 1)) > q(q-1), \varphi_4(1) = \varphi_5(1) = (q-1)(q/\sqrt{3})(q+1 - 3\sqrt{q/3})/2 > q(q-1)\).

We have shown that the unique possibility for \(\varphi\) is \(\varphi = \varphi_2\), the nontrivial constituent (of degree \(q^2 - q\)) of the reduction modulo 2 of the unique irreducible complex character of degree \(q^2 - q + 1\) in the principle block.
Now we will prove that $\varphi_2|_P$ is indeed irreducible. Assume the contrary that $\varphi_2|_P$ is reducible. Then it is the sum of more than one irreducible 2-Brauer characters of $P$. Since $P$ is solvable, these characters are liftable to complex characters. An easy inspection of the character table of $P$ shows that values of any irreducible character of degree less than $q^2 - q$ at the element $X$ of order 3 (which is a representative of a conjugacy class of $P$ given in [46, p. 88]) is positive. On the other hand, $\varphi_2(X) = -q$ which is negative. We get a contradiction. $\square$
CHAPTER 4
IRREDUCIBLE RESTRICTIONS FOR $3D_4(q)$

The main purpose of this chapter is to prove Theorem E. In other words, we show that the restriction of any irreducible representation of $3D_4(q)$ to any its proper subgroup is reducible.

4.1 Basic Reduction

**Theorem 4.1.1 (Reduction Theorem).** Let $G = 3D_4(q)$ and let $\varphi$ be an irreducible representation of $G$ in characteristic $\ell$ coprime to $q$. Suppose $\varphi(1) > 1$ and $M$ is a maximal subgroup of $G$ such that $\varphi|M$ is irreducible. Then $M$ is $G$-conjugate to one of the following groups:

(i) $P$, a maximal parabolic subgroup of order $q^{12}(q^6 - 1)(q - 1)$,
(ii) $Q$, a maximal parabolic subgroup of order $q^{12}(q^3 - 1)(q^2 - 1)$,
(iii) $G_2(q)$,
(iv) $3D_4(q_0)$ with $q = q_0^2$.

**Proof.** By [51, Theorem 4.1], $\bar{d}_\ell(3D_4(q)) \geq q^5 - q^3 + q - 1$ for every $\ell$ coprime to $q$. Next, according to [38], if $M$ is a maximal subgroup of $G$, but $M$ is not a maximal parabolic subgroup, then $M$ is $G$-conjugate to one of the following groups:

1. $G_2(q)$,
2. $PGL_3^\epsilon(q)$, where $4 \leq q \equiv \epsilon 1(\text{mod } 3)$, $\epsilon = \pm$,
3. $3D_4(q_0)$ with $q = q_0^\alpha$, $\alpha$ prime, $\alpha \neq 3$,
4. $L_2(q^3) \times L_2(q)$, where $2 \mid q$, a fundamental subgroup,
5. $C_G(s) = (SL_2(q^3) \circ SL_2(q)).2$, $q$ odd, involution centralizer,
6. $((Z_{q^2 + q + 1}) \circ SL_3(q)).f_+ .2$, where $f_+ = (3, q^2 + q + 1)$,
7. $((Z_{q^2 - q + 1}) \circ SU_3(q)).f_- .2$, where $f_- = (3, q^2 - q + 1)$,
8. $(Z_{q^2 \pm q + 1})^2 .SL_2(3)$,
9. $Z_{q^4 - q^2 + 1}.4$.

We need to consider the following cases:
\[ M = PGL_3'(q), \text{ where } 4 \leq q \equiv \epsilon 1 (\text{mod } 3), \epsilon = \pm. \text{ We have } |PGL_3'(q)| = q^3(q^2 - 1)(q^3 \pm 1). \text{ So } m_c(M) \leq \sqrt{q^3(q^2 - 1)(q^3 + 1)} < q^5 - q^3 + q - 1 \text{ for every } q \geq 4. \]

Therefore \( m_c(M) < \vartheta \ell(3D_4(q)) \), contradicting Lemma 2.1.1.

\( M = 3D_4(q_0) \) with \( q = q_0^\alpha \), \( \alpha \) prime, \( \alpha \neq 2, 3 \). We have \( |3D_4(q_0)| = q_0^{12}(q_0^6 - 1)^2(q_0^4 - q_0^2 + 1) < q_0^{28}. \text{ Hence, } m_c(M) \leq \sqrt{q_0^{28}} = q_0^{14}. \text{ Since } \alpha \text{ is prime and } \alpha \neq 2, 3, \alpha \geq 5. \text{ It follows that } m_c(M) < q_0^{14/5} < q^5 - q^3 + q - 1 \leq \vartheta \ell(3D_4(q)). \]

\( M = L_2(q^3) \times L_2(q) \), where \( 2 \mid q \). It is well known that \( m_c(L_2(q)) = q + 1 \) except that \( m_c(L_2(2)) = 2, m_c(L_2(3)) = 3 \) and \( m_c(L_2(5)) = 5. \) So we have \( m_c(L_2(q^3)) = q^3 + 1 \) for every \( q \). Hence \( m_c(M) = (q + 1)(q^3 + 1) \) for \( q \geq 4 \) and \( m_c(M) = 18 \) for \( q = 2 \). It is easy to see that \((q + 1)(q^3 + 1) < q^5 - q^3 + q - 1 \) for every \( q \geq 4 \). When \( q = 2 \), we also have \( m_c(M) = 18 < 25 = 2^5 - 2^3 + 2 - 1 \). Therefore, \( m_c(M) < \vartheta \ell(3D_4(q)) \) for every \( q \).

\( M = C_G(s) = (SL_2(q^3) \circ SL_2(q)).2, q \text{ odd, } s \text{ an involution. Here } C_G(M) \ni s \neq 1, \) contradicting Lemma 2.1.3.

\( M = ((Z_{q^2+q+1}) \circ SL_3(q)).f_+.2, \text{ where } f_+ = (3, q^2 + q + 1). \) By Lemma 2.1.9,
\[
m_c(M) \leq 2.f_+.m_c((Z_{q^2+q+1}) \circ SL_3(q)) \leq 2.f_+.m_c(SL_3(q)). \text{ From } [64], \text{ we have}
\[
m_c(SL_3(q)) = \begin{cases} 8, & q = 2, \\ 39, & q = 3, \\ 84, & q = 4, \\ (q + 1)(q^2 + q + 1), & q \geq 5. \end{cases}
\]

It is easy to check that \( 2.f_+.m_c(SL_3(q)) < q^5 - q^3 + q - 1 \) for every \( q \geq 2 \). Therefore \( m_c(M) < q^5 - q^3 + q - 1 \).

\( M = ((Z_{q^2-q+1}) \circ SU_3(q)).f_-2, \text{ where } f_- = (3, q^2 - q + 1). \) By Lemma 2.1.9,
\[
m_c(M) \leq 2.f_-m_c((Z_{q^2-q+1}) \circ SU_3(q)) \leq 2.f_-m_c(SU_3(q)). \text{ From } [64], \text{ we have}
\[
m_c(SU_3(q)) = \begin{cases} 8, & q = 2, \\ (q + 1)^2(q - 1), & q \geq 3. \end{cases}
\]

It is easy to check that \( 2.f_-m_c(SU_3(q)) < q^5 - q^3 + q - 1 \) for every \( q \geq 3 \). Therefore \( m_c(M) < q^5 - q^3 + q - 1 \) for every \( q \geq 3 \).
When \( q = 2 \), we have \( m_C(M) \leq \sqrt{|M|} = 36 \). Therefore if \( \varphi|_M \) is irreducible then \( \deg(\varphi) \leq 36 \). Inspecting the character tables of \( ^3D_4(2) \) in [12] and [36], we see that \( \deg(\varphi) = 25 \) for \( \ell = 3 \) or \( \deg(\varphi) = 26 \) for \( \ell \neq 3 \). Moreover, when \( \ell \neq 3 \), \( \varphi \) is the reduction modulo \( \ell \) of the unique irreducible complex representation \( \rho \) of degree 26. Since \( 26 \nmid |M| = 1296 \), \( M \) does not have any irreducible complex representation of degree 26, whence \( \rho|_M \) and \( \varphi|_M \) must be reducible. When \( \ell = 3 \), \( M = ((\mathbb{Z}_3 \circ SU_3(2)))\cdot 3.2 \simeq 3^{1+2} \cdot 2S_4 \).

So \( m_3(M) = m_3(3^{1+2} \cdot 2S_4) = m_3(2S_4) \leq m_C(2S_4) \leq \sqrt{|2S_4|} < 7 \). Therefore if \( \deg(\varphi) = 25 \) then \( \varphi|_M \) is reducible.

\[ • \] \( M = (\mathbb{Z}_{q^2+q+1})^2 \cdot SL_2(3) \). We have \( m_C(M) \leq |SL_2(3)| = 24 \). Since \( q^5 - q^3 + q - 1 > 24 \) for every \( q \geq 2 \), \( m_C(M) < \vartheta_\ell(3D_4(q)) \).

\[ • \] \( M = \mathbb{Z}_{q^4-q^2+1}.4 \). We have \( m_C(M) \leq 4 < \vartheta_\ell(3D_4(q)) \). \( \square \)

### 4.2 Restrictions to \( G_2(q) \) and to \( ^3D_4(\sqrt{q}) \)

In this section we handle two of the maximal subgroups singled out in Theorem 4.1.1.

**Theorem 4.2.1.** Let \( M \simeq G_2(q) \) be a subgroup of \( G = ^3D_4(q) \) and \( \varphi \in \text{IBr}_\ell(G) \) be of degree \( > 1 \). Then \( \varphi|_M \) is reducible.

**Proof.** Assume the contrary: \( \varphi|_M \) is irreducible. By Lemma 2.1.1, \( \varphi(1) < \sqrt{|M|} < q^7 \). We will identify the dual group \( G^* \) with \( G \). By the fundamental result of Broué and Michel [4], \( \varphi \) belongs to a union \( \mathcal{E}_\ell(G, s) \) of \( \ell \)-blocks, labeled by a semi-simple \( \ell' \)-element \( s \in G \).

Moreover, by [32], \( \varphi(1) \) is divisible by \( (G : C_G(s))_{\ell'} \). Assume \( s \neq 1 \). Then it is easy to check, using [14] for instance, that \( (G : C_G(s))_{\ell'} \geq q^8 + q^4 + 1 \). Since \( \varphi(1) < q^7 \), it follows that \( s = 1 \), i.e. \( \varphi \) belongs to a unipotent block.

According to [38], \( M = C_G(\tau) \) for some (outer) automorphism \( \tau \) of order 3 of \( G \).

Furthermore, the degrees of all complex irreducible characters of \( G \) are listed in [14]. An easy inspection reveals that \( G \) has a unique irreducible character of degree \( \psi(1) \) for every unipotent character \( \psi \in \text{Irr}(G) \). It follows that every unipotent (complex) character of \( G \) is \( \tau \)-invariant.
Next we show that $\varphi$ is also $\tau$-invariant. First consider the case where $q$ is odd. Then Corollary 6.9 of [21] states that the $\ell$-modular decomposition matrix of $G$ has a lower unitriangular shape. In particular, this implies that $\varphi$ is an integral linear combination of $\hat{\psi}$, with $\psi \in \mathcal{E}(G, 1)$, the set of unipotent characters of $G$. But each such $\psi$ is $\tau$-invariant, whence $\varphi$ is $\tau$-invariant. Now assume that $q$ is even. Then $\ell \neq 2$, and so it is a good prime for $R$, and $\ell$ does not divide $|Z(R)|$, where $R$ is the simple, simply connected algebraic group of type $D_4$. Hence, by the main result of [22], $\{\hat{\psi} \mid \psi \in \mathcal{E}(G, 1)\}$ is a basic set of Brauer characters of $\mathcal{E}_\ell(G, 1)$. It follows that $\varphi$ is an integral linear combination of $\hat{\psi}$, with $\psi \in \mathcal{E}(G, 1)$, and so it is $\tau$-invariant as above.

Consider the semidirect product $\tilde{G} = G \cdot \langle \tau \rangle$. Then $G \triangleleft \tilde{G}$, and $\tilde{G}/G$ is cyclic. Since $\varphi$ is $\tilde{G}$-invariant, it extends to $\tilde{G}$ by [18, Theorem III.2.14]. But $C_{\tilde{G}}(M) \ni \tau \neq 1$, hence $\varphi|_M$ cannot be irreducible by Lemma 2.1.3.

**Theorem 4.2.2.** Let $H \simeq ^3D_4(q)$ be a maximal subgroup of $G = ^3D_4(q^2)$ and $V \in \text{IBr}_\ell(G)$ be of dimension $> 1$. Then $V|_H$ is reducible.

**Proof.** Again assume the contrary. We consider a long-root parabolic subgroup $P = q^{2+16} \cdot SL_2(q^6) \cdot \mathbb{Z}_{q^2-1}$ of $G$, which also contains a long-root parabolic subgroup $P_H = q^{1+8} \cdot SL_2(q^3) \cdot \mathbb{Z}_{q-1}$ of $H$.

It is well known that $V|_Z$ affords all the nontrivial linear characters $\lambda$ of the long-root subgroup $Z := Z(P')$ (which is elementary abelian of order $q^2$), and the corresponding eigenspaces $V_\lambda$ are permuted regularly by the torus $\mathbb{Z}_{q^2-1}$. Let $U = q^{2+16}$ denote the unipotent radical of $P$ and consider any such $\lambda$. Then $\text{IBr}_\ell(U)$ contains a unique representation (of degree $q^8$), on which $Z$ acts via the character $\lambda$. Moreover, since $P'/U \simeq SL_2(q^6)$ has trivial Schur multiplier and is perfect, this representation of $U$ extends to a unique representation of $P'$, which we denote by $E_\lambda$. By Clifford theory, the $P'$-module $V_\lambda$ is isomorphic to $E_\lambda \otimes A$ for some $A \in \text{IBr}_\ell(P'/U)$. Suppose that $A$ contains a nontrivial composition factor, as a $SL_2(q^6)$-module. Then $\dim(A) \geq (q^6 - 1)/2$. It follows
that
\[ \dim(V) \geq (q^2 - 1)q^8(q^6 - 1)/2. \] (2.1)

On the other hand, the irreducibility of \( V|_H \) implies that
\[ \dim(V) < \sqrt{|H|} < q^{14}, \]
contradicting (2.1). Thus all composition factors of \( A \) are trivial. In particular, the
\( P' \)-module \( V_\lambda \) contains a simple submodule which is isomorphic to \( E_\lambda \).

Notice that we can embed \( P_H \) in \( P \) in such a way that \( Z \) contains \( Z := Z(P'_H) \)
(a long-root subgroup in \( H \), which is elementary abelian of order \( q \)), and \( U \) contains the
unipotent radical \( U_H = q^{1+8} \) of \( P_H \). Now choose \( \lambda \) such that \( Z \leq \text{Ker}(\lambda) \). Then it is easy
to see that \( E_\lambda|_{U_H} \) is just the regular representation, whence the subspace \( L \) of \( U_H \)-fixed
points in it is one-dimensional, and, since \( U_H \not< P'_H \), this subspace is acted on by \( P'_H \). But
\( P'_H/U_H \simeq SL_2(q^3) \) is perfect, hence \( P'_H \) acts trivially on \( L \).

We have shown that, for the given choice of \( \lambda \), \( P'_H \) has nonzero fixed points in \( V_\lambda \). Let
\( W \) be the subspace consisting of all \( P'_H \)-fixed points in \( V \). Then \( P_H/P'_H \simeq \mathbb{Z}_{q-1} \) acts on
\( W \) and so \( W \) contains a one-dimensional \( P_H \)-submodule \( T \). By the Frobenius reciprocity,
\[ 0 \neq \dim\text{Hom}_{P_H}(T, V|_{P_H}) = \dim\text{Hom}_H(\text{Ind}^H_{P_H}(T), V|_H). \]
But \( V|_H \) is irreducible, hence it is a quotient of \( \text{Ind}^H_{P_H}(T) \). In particular,
\[ \dim(V) \leq (H : P_H) \cdot \dim(T) = (q + 1)(q^8 + q^4 + 1). \] (2.2)

On the other hand, Theorem 4.1 of [51] implies that
\[ \dim(V) \geq q^2(q^8 - q^4 + 1) - 1, \]
contradicting (2.2).

\[ \Box \]

4.3 Restrictions to Maximal Parabolic Subgroups

Lemma 4.3.1. Let \( Q \) denote the maximal parabolic subgroup of order \( q^{12}(q^3 - 1)(q^2 - 1) \)
of \( G = ^3D_4(q) \), and let \( U := Op(Q) \). Then
(i) For any prime \( r \neq p \), \( O_r(Q) = 1 \).

(ii) Let \( \varphi \in \text{IBr}_\ell(Q) \) be an irreducible Brauer character of \( Q \) whose kernel does not contain \( U \). If \( q \) is odd, assume in addition that \( \varphi \) is faithful. Then \( \varphi \) lifts to a complex character \( \chi \) of \( Q \). Moreover, \( \chi \) is also faithful if \( \varphi \) is faithful.

Proof. (i) Since \( O_r(Q), U \triangleleft Q \) and \( O_r(Q) \cap U = 1 \), any element \( g \in O_r(Q) \) is centralized by \( U \), which has order \( q^{11} \). Thus \( q^{11} \) divides \( |C_Q(g)| \). Assuming \( g \neq 1 \), we see by [26] and [28] that \( g \) is \( Q \)-conjugate to the long-root element \( u = x_{3\alpha+2\beta}(1) \). But then \( g \) is a \( p \)-element, a contradiction. Hence \( O_r(Q) = 1 \).

(ii) Let \( \lambda \) be an irreducible constituent of \( \varphi|_U \), and let \( I \) denote the inertia group of \( \lambda \) in \( Q \). By Clifford theory, \( \varphi = \text{Ind}_I^Q(\psi) \) for some \( \psi \in \text{IBr}_\ell(I) \) whose restriction to \( U \) contains \( \lambda \). Since \( p \neq \ell \), we may view \( \lambda \) as an ordinary character of \( U \). By our assumption, \( \lambda \neq 1_U \). The structure of \( I/U \) is described in [26], [28]. In particular, if \( 2|q \), then \( I/U \) is always solvable. On the other hand, if \( q \) is odd, then \( I/U \) is solvable, except for one orbit, the kernel of any character in which however contains a long-root element \( x_{3\alpha+2\beta}(1) \) (in the notation of [26]). Recall we are assuming that \( \varphi \) is faithful if \( q \) is odd. It follows that in either case \( I/U \) is solvable, and so \( I \) is solvable. By the Fong-Swan Theorem, \( \psi \) lifts to a complex character \( \rho \) of \( I \). Hence \( \varphi \) lifts to the complex character \( \chi := \text{Ind}_I^Q(\rho) \).

Now assume that \( \varphi \) is faithful but \( K := \text{Ker}(\chi) \) is non-trivial; in particular, \( \ell \neq 0 \).

If \( K \) is not an \( \ell \)-group, then \( K \) contains a non-trivial \( \ell' \)-element \( g \). Since \( \varphi(g) = \chi(g) = \chi(1) = \varphi(1) \), we see that \( \varphi \) is not faithful, a contradiction. Hence \( K \) is an \( \ell \)-group, and so \( O_{\ell}(Q) \neq 1 \), contradicting (i).

\[ \square \]

**Theorem 4.3.2.** Let \( M \) be a maximal parabolic subgroup of \( G = ^3D_4(q) \) and \( \varphi \in \text{IBr}_\ell(G) \) be of degree \( > 1 \). Then \( \varphi|_M \) is reducible.

**Proof.** First suppose that \( M = P \), the long-root parabolic subgroup of \( G \). Then the statement follows from Lemma 2.1.11. So we may assume that \( M = Q \), the other maximal parabolic subgroup of \( G \). Also assume the contrary: \( \varphi|_Q \) is irreducible.
We will consider two particular long-root elements \( u = x_{3\alpha + 2\beta}(1) \) and \( v = x_{\beta}(1) \) of \( Q \), in the notation of [26], [27], [28]. Clearly, they are conjugate in \( G \), so \( \varphi(u) = \varphi(v) \). By Lemma 4.3.1, \( \varphi|_Q \) lifts to a complex irreducible character \( \chi \) of \( Q \) which is also faithful. Since \( u \) and \( v \) are \( \ell' \)-elements, we have \( \varphi(u) = \chi(u) \) and \( \varphi(v) = \chi(v) \). It follows that

\[
\chi(u) = \chi(v).
\]  

(3.3)

Note that \( Z := Z(O_p(Q)) = X_{3\alpha + 2\beta}X_{3\alpha + 3\beta} \) has order \( q^2 \), and consists of the \( q^2 - 1 \) \( Q \)-conjugates of \( u \) and \( 1 \). Thus \( Q \) acts transitively on \( Z \setminus \{1\} \) and on Irr\((Z) \setminus \{1_Z\} \). Since \( \text{Ker}(\chi) = 1 \), we conclude that \( \chi(u) = -\chi(1)/(q^2 - 1) \).

First consider the case \( q \) is odd. Then \( u \), resp. \( v \), belongs to the \( Q \)-conjugacy class \( c_{1,1} \), resp. \( c_{1,2} \), in the notation of [26]. According to [26], the faithful character \( \chi \) must be one of \( \chi_j(k) \), \( 16 \leq j \leq 20 \). If \( j = 16 \) or \( 17 \), then \( \chi(v) \) is explicitly computed in [26], and one sees that (3.3) is violated. Now suppose that \( j = 18 \) or \( 19 \). Then \( \chi(u) = -q^3(q^3 - 1)/2 \).

On the other hand, according to Proposition 2.1 of the Appendix of [55], \( \chi(v) = mq(q^3 - 1) \) with \( m \geq -(q^2 - 1)/2 \). It follows that \( \chi(v) > \chi(u) \), violating (3.3). Finally, suppose that \( j = 20 \). Then \( \chi(u) = -q^3(q^3 - 1) \). Meanwhile, by Proposition 2.1 of the Appendix of [55], \( \chi(v) = mq(q^3 - 1) \) with \( m \geq -(q^2 - 1) \). It follows that \( \chi(v) > \chi(u) \), again violating (3.3).

Next we consider the case \( q \) is even. Then \( u \), resp. \( v \), belongs to the \( Q \)-conjugacy class \( c_{1,1} \), resp. \( c_{1,7} \), in the notation of [28]. According to [28], the faithful character \( \chi \) must be one of \( \chi_j(k) \), \( 14 \leq j \leq 16 \). If \( j = 14 \) or \( 15 \), then \( \chi(u) \) is explicitly computed in [28], and one sees that (3.3) is violated. Finally, suppose that \( j = 16 \). Then \( \chi(u) = -q^3(q^3 - 1) \).

On the other hand, by Proposition 1.1 of the Appendix of [55], \( \chi(v) = mq(q^3 - 1) \) with \( m \geq -(q^2 - 1) \). It follows that \( \chi(v) > \chi(u) \), again violating (3.3).

\[\square\]

**Proof of Theorem E.** Assume the contrary: \( \varphi|_M \) is irreducible. Without loss we may assume that \( \varphi \) is absolutely irreducible and that \( M \) is a maximal subgroup of \( G \). Now we can apply Theorem 4.1.1 to \( M \) to get four possibilities (i) – (iv) for \( M \). None of them cannot however occur by Theorems 4.2.1, 4.2.2, and 4.3.2.

\[\square\]
CHAPTER 5
LOW-DIMENSIONAL CHARACTERS OF THE SYMPLECTIC GROUPS

The purpose of this chapter is to classify irreducible complex characters of the symplectic groups \( Sp_{2n}(q) \), \( q \) odd, of degrees up to \((q^n - 1)q^{4n-10}/2\).

This chapter is organized as follows. In §1, we describe the strategy to determine the low-dimensional complex characters of finite classical groups and give structures of centralizers of semi-simple elements in the symplectic and orthogonal groups. Unipotent characters of both symplectic and orthogonal groups will be determined in §2. We must include the orthogonal groups in this chapter since their unipotent characters will be used to determine non-unipotent characters of the symplectic groups. In the last section §3, low-dimensional non-unipotent characters of \( Sp_{2n}(q) \) will be handled.

5.1 Preliminaries

5.1.1 Strategy of the Proofs

One of the main tools to approach the problem of low-dimensional characters is Lusztig’s classification on the complex characters of finite groups of Lie type.

Let \( G \) be either the symplectic groups \( Sp_{2n}(q) \) or the orthogonal groups \( Spin_n^\pm(q) \), where \( q \) is a power of a prime \( p \). Let \( G \) be the algebraic group and \( F \) the Frobenius endomorphism on \( G \) such that \( G = G^F \). Let \( G^* \) be the dual group of \( G \) and \( F^* \) the dual Frobenius endomorphism and denote \( G^* = G^{*F^*} \). Lusztig’s classification (see chapter 13 of [15]) says that the set of irreducible complex characters of \( G \) is partitioned into Lusztig series \( \mathcal{E}(G, (s)) \) associated to various geometric conjugacy classes \( (s) \) of semi-simple elements of \( G^* \). In fact, \( \mathcal{E}(G, (s)) \) is the set of irreducible constituents of a Deligne-Lusztig character \( R^G_T(\theta) \), where \((T, \theta)\) is of the geometric conjugacy class associated to \((s)\).

The elements of \( \mathcal{E}(G, (1)) \) are called unipotent characters of \( G \). When \( G \) is a connected reductive group, for any semi-simple element \( s \in G^* \), there is a bijection \( \chi \mapsto \psi \) from \( \mathcal{E}(G, (s)) \) to \( \mathcal{E}(C_{G^*}(s), (1)) \) such that

\[
\chi(1) = \frac{|G|_{p'}}{|C_{G^*}(s)|_{p'} \psi(1)}. \tag{1.1}
\]
In this situation, we are going to say that the irreducible complex character $\chi$ of $G$ is parametrized by the pair $((s), \psi)$.

Suppose that we are determining irreducible complex characters of $G$ of degrees up to a certain bound $D$. Unipotent characters of $G$ as well as $G^*$ are classified by Lusztig (see §13.8 of [9]) and we will use that to find the low-dimensional unipotent characters of $G$. When a character $\chi$ is not unipotent, i.e. $(s) \neq (1)$, based on the structure of $C_{G^*}(s)$, we estimate $(G^* : C_{G^*}(s))_{\nu'}$ and come up with some certain cases when $C_{G^*}(s)$ is large enough. More specifically, from formula 1.1, we see that if $\chi(1) < D$ then $|C_{G^*}(s)| > |G|_{\nu'}/D$. The following Proposition is used frequently to determine unipotent characters of $C_{G^*}(s)$.

**Proposition 5.1.1** (Proposition 13.20 of [15]). Let $G$ and $G_1$ be two reductive groups defined over $\mathbb{F}_q$, and let $f : G \rightarrow G_1$ be a morphism of algebraic groups with a central kernel, defined over $\mathbb{F}_q$ and such that $f(G)$ contains the derived group $G_1'$; then the unipotent characters of $G^F$ are the $\theta \circ f$, where $\theta$ runs over the unipotent characters of $G_1^F$.

### 5.1.2 Centralizers of Semi-simple Elements

We collect here some well-known results about the structures of semi-simple elements in finite classical groups (see more on [8], [19], and [71]). Since proofs of the following lemmas are similar, we omit most of their proofs except the last one’s.

**Lemma 5.1.2.** Let $q$ be odd and $s \in SO_{2n+1}(q)$ be semi-simple. Then

$$C_{SO_{2n+1}(q)}(s) \simeq SO_{2k+1}(q) \times GO_{2(m-k)}^\pm(q) \times \prod_{i=1}^t GL_{a_i}(q^{k_i})$$

and

$$C_{GO_{2n+1}(q)}(s) \simeq GO_{2k+1}(q) \times GO_{2(m-k)}^\pm(q) \times \prod_{i=1}^t GL_{a_i}(q^{k_i}),$$

where $0 \leq k \leq m \leq n$, $\alpha_i = \pm$, and $\sum_{i=1}^t k_ia_i = n - m$.

**Lemma 5.1.3.** Let $q$ be even and $s \in GO_{2n}^\pm(q)$ be semi-simple. Then

$$C_{GO_{2n}^\pm(q)}(s) \simeq GO_{2m}^\pm(q) \times \prod_{i=1}^t GL_{a_i}(q^{k_i}),$$
where \( \alpha_i = \pm \) and \( \sum_{i=1}^{t} k_i a_i = n - m \).

Let \((.,.)\) be a non-degenerate symplectic form on the space \( V = \mathbb{F}_q^{2n} \), then the conformal symplectic group \( CSp_{2n}(q) \) is defined to be

\[
\{ g \in GL(V) \mid \exists \tau(g) \in \mathbb{F}_q^*, \forall u, v \in V, (gu, gv) = \tau(g)(u, v) \}.
\]

**Lemma 5.1.4.** Let \( q \) be odd and \( s \in CSp_{2n}(q) \) be semi-simple. Then

\[
C_{CSp_{2n}(q)}(s) \simeq C_{Sp_{2n}(q)}(s) \cdot \mathbb{Z}_{q-1}.
\]

Moreover,

(i) If \( \tau(s) \) is not a square in \( \mathbb{F}_q \), then

\[
C_{Sp_{2n}(q)}(s) \simeq Sp_m(q^2) \times \prod_{i=1}^{t} GL^{\alpha_i}_{a_i}(q^{k_i}),
\]

where \( m \) is even, \( \alpha_i = \pm \), and \( \sum_{i=1}^{t} k_i a_i = n - m \).

(ii) If \( \tau(s) \) is a square in \( \mathbb{F}_q \), then

\[
C_{Sp_{2n}(q)}(s) \simeq Sp_{2k}(q) \times Sp_{2(m-k)}(q) \times \prod_{i=1}^{t} GL^{\alpha_i}_{a_i}(q^{k_i}),
\]

where \( 0 \leq k \leq m \leq n \), \( \alpha_i = \pm \), and \( \sum_{i=1}^{t} k_i a_i = n - m \).

Let \( Q(.) \) be a non-degenerate quadratic form on the space \( V = \mathbb{F}_q^{2n} \). Then the conformal orthogonal group \( CO_{2n}^\pm(q) \) is defined to be

\[
\{ g \in GL(V) \mid \exists \tau(g) \in \mathbb{F}_q^*, \forall v \in V, Q(g(v)) = \tau(g)Q(v) \}.
\]

The definition of \( CO_{2n}^\pm(q) \) goes back to Remark 7.3 of [70]. When \( q \) is odd, it is also shown in [70] that \( CO_{2n}^\pm(q) = \{ g \in CO_{2n}^\pm(q) \mid \det(g) = \tau(g)^n \} \) and \( CO_{2n}^\pm(q) \) is actually a subgroup of index 2 of \( CO_{2n}^\pm(q) \) such that \( CO_{2n}^\pm(q)/SO_{2n}^\pm(q) \simeq CO_{2n}^\pm(q)/GO_{2n}^\pm(q) \simeq \mathbb{Z}_{q-1} \).

**Lemma 5.1.5.** Let \( q \) be odd and \( s \in CO_{2n}^\pm(q) \) be semi-simple. Then

\[
C_{CO_{2n}^\pm(q)}(s) \simeq C_{GO_{2n}^\pm(q)}(s) \cdot \mathbb{Z}_{q-1},
\]
and

\[ C_{\text{GO}_{2n}(q)^0}(s) \simeq C_{\text{SO}_{2n}(q)}(s) \cdot \mathbb{Z}_{q-1}. \]

Moreover,

(i) If \( \tau(s) \) is not a square in \( \mathbb{F}_q \), then

\[ C_{\text{GO}_{2n}(q)}(s) \simeq \text{GO}^\pm_m(q^2) \times \prod_{i=1}^t GL^{\alpha_i}_{a_i}(q^{k_i}), \]

where \( m \) is even, \( \alpha_i = \pm \), and \( \sum_{i=1}^t k_i a_i = n - m \).

(ii) If \( \tau(s) \) is a square in \( \mathbb{F}_q \), then

\[ C_{\text{GO}_{2n}(q)}(s) \simeq \text{GO}^\pm_{2k}(q) \times \text{GO}^\pm_{2(m-k)}(q) \times \prod_{i=1}^t GL^{\alpha_i}_{a_i}(q^{k_i}), \]

where \( 0 \leq k \leq m \leq n \), \( \alpha_i = \pm \), and \( \sum_{i=1}^t k_i a_i = n - m \).

**Proof.** Let \((.,.)\) be the non-degenerate bilinear form associated with \( Q(.) \). Fix a basis of \( V \) and let \( J \) be the Gram matrix of \((.,.)\) corresponding to this basis. Then \( s s J s = \tau J, \) where \( \tau := \tau(s) \). Hence \( \text{Spec}(s) = \text{Spec}(s) = \tau \text{Spec}(s J s^{-1} J^{-1}) = \tau \text{Spec}(s^{-1}) \). Denote the characteristic polynomial of \( s \) acting on \( V \) by \( P(x) \in \mathbb{F}_q[x] \). Then if \( \lambda \) is a root of \( P(x) \) then \( \tau \lambda^{-1} \) is also a root of \( P(x) \).

We consider two following cases:

**Case 1**: If \( \tau \) is not a square in \( \mathbb{F}_q \), suppose \( \tau = \lambda_1^2 \) where \( \lambda_1 \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \). Decompose \( P(x) \) into irreducible polynomials over \( \mathbb{F}_q \):

\[ P(x) = (x^2 - \tau)^m \prod_{i=1}^l f_i^{m_i}(x) \prod_{j=1}^{l'} g_j^{n_j}(x) \operatorname{hat}{g}_j^{n_j}(x), \]

where

- If \( \lambda \) is a root of \( f_i(x) \) then \( \tau \lambda^{-1} \) is also a root of \( f_i(x) \), \( \deg(f_i) \) is even, \( \pm \lambda_1 \) are not the roots of \( f_i \),
- If \( \lambda \) is a root of \( g_j(x) \) then \( \tau \lambda^{-1} \) is a root of \( \operatorname{hat}{g}_j(x) \), \( \deg(g_j) = \deg(\operatorname{hat}{g}_j) \), \( g_j \neq \operatorname{hat}{g}_j \),
- \( n = m + \sum_{i=1}^l m_i \deg(f_i)/2 + \sum_{j=1}^{l'} n_j \deg(g_j) \).
We have \( \tau^n = \det(s) = (-\tau)^{m-n} \). Therefore, \( m \) is even. Set

\[
V_{\tau} := \left( \frac{P(x)}{(x^2 - \tau)x=s} \right)_x=V, \quad V_i := \left( \frac{P(x)}{f_{i,x}(x)} \right)_x=s(V), \quad U_j := \left( \frac{P(x)}{g_{j,x}(x)} \right)_x=s(V).
\]

Then \( V = V_{\tau} \oplus V_1 \oplus \cdots \oplus V_l \oplus U_1 \oplus \cdots \oplus U_r \) is an orthogonal decomposition of \( V \). Let \( W \in \{V, V_{\tau}, V_1, \ldots, V_l, U_1, \ldots, U_r\} \). Then \( s|_W \) is semi-simple and \( (,\cdot) \) is non-degenerate on \( W \). For short notation, we denote \( C_{GO(W)}(s|_W) \) and \( C_{CO(W)}(s|_W) \) by \( C_{GO(W)}(s) \) and \( C_{CO(W)}(s) \), respectively.

1a) First, we will show that \( C_{GO(W)}(s) \cong GU_{m_0}(q^k) \) and \( C_{CO(W)}(s) \cong C_{GO(W)}(s) \cdot \mathbb{Z}_{q-1} \), where \( W = V_i \) for \( i = 1, \ldots, l \), \( m_0 = m_i \), and \( k = \deg(f_i)/2 \). Note that the characteristic polynomial of \( s \) acting on \( W \) is \( f_{i,m_0}(x) \). Let \( \lambda \in \mathbb{F}_{q^{2k}} \) be an eigenvalue of the action of \( s \) on \( W \). Then all eigenvalues of the action of \( s \) on \( W \) are \( \lambda, \lambda^q, \ldots, \lambda^{q^{2k-1}} \). Since \( \tau \lambda^{-1} \) is also a root of \( f_i(x) \) and \( \pm \lambda_1 \) are not, \( \tau \lambda^{-1} = \lambda^{q^i} \). Consider \( \tilde{W} = W \otimes_{\mathbb{F}_q} \mathbb{F}_{q^{2k}} \). Fix a basis \( (f_i) \) in \( W \), and define a Frobenius endomorphism \( \sigma : \Sigma_i x_i f_i \mapsto \Sigma_i x_i^q f_i \) on \( \tilde{W} \), where \( x_i \in \mathbb{F}_{q^{2k}} \). The simplicity of \( s \) implies that \( \tilde{W} = \tilde{W}_1 \oplus \cdots \oplus \tilde{W}_{2k+l} \), where \( \tilde{W}_i = \mathrm{Ker}(s - \lambda^{q^{i-1}}) \). We see that \( \sigma \) permutes the \( \tilde{W}_i \)’s cyclically: \( \sigma(\tilde{W}_i) = \tilde{W}_{i+1} \), where \( \tilde{W}_{2k+1} = \tilde{W}_1 \). Let \( g \in CO(W) \) be commuting with \( s|_W \). Then \( g \) preserves each \( \tilde{W}_i \). Moreover, it is easy to see that \( g \) also commutes with \( \sigma \). This implies that the action of \( g \) on \( \tilde{W} \) is completely determined by its action on \( \tilde{W}_1 \): \( g(\sigma^i w) = \sigma^i(g w) \) for \( w \in \tilde{W}_1 \). So \( C_{CO(W)}(s) \hookrightarrow GL_{m_0}(q^{2k}) \). If \( u \in \tilde{W}_1 \) and \( v \in \tilde{W}_j \) then \( \tau(u, v) = (su, sv) = \lambda^{q^{i-1}+q^{j-1}}(u, v) \). Therefore,

\[
\tilde{W}_1^\perp = \bigoplus_{i\neq 1+k} \tilde{W}_i \text{ and } \tilde{W}_{1+k} \cap \tilde{W}_1^\perp = 0.
\]

Choose a basis \( (u_1, \ldots, u_{m_0}) \) in \( \tilde{W}_1 \). Then \( (v_i) \) is a basis for \( \tilde{W}_{1+k} \), where \( v_i = \sigma^k(u_i) \). We see that \( (\sigma u_i, \sigma v_j) = (u_i, v_j)^q \). Hence, \( (u_i, v_j)^q = (\sigma^k u_i, \sigma^k v_j) = (v_i, u_j) = (u_j, v_i) \). In other words, \( U = U^q \). Thus, together with 1.2, \( U \) determines a non-degenerate Hermitian form on an \( m_0 \)-dimensional \( \mathbb{F}_{2k} \)-space.

If \( g \) acts on \( \tilde{W}_1 \) with matrix \( A = (a_{ij}) \) (with respect to the basis \( (u_i) \)), then \( g \) acts on \( \tilde{W}_{1+k} \) with matrix \( A^q = (a_{ij}^q) \) (with respect to the basis \( (v_i) \)). From 1.2, \( g \in C_{GO(W)}(s) \) if
and only if
\[(u_k, v_l) = (gu_k, gv_l) = (\sum_i a_{ik} u_i, \sum_j a_{jl}^q v_j) = \sum_{i,j} a_{ik} a_{jl}^q (u_i, v_j),\]
i.e. \(TAU A^q = U\). Therefore, \(C_{GO(W)}(s) \simeq GU_{na}(q^k)\). Similarly, \(g \in C_{CO(W)}(s)\) if and only if \(TAU A^q = \tau(g)U\). Note that if \(TAU A^q = U\) then \((\varepsilon A)U(\varepsilon A)^q = \tau(g)U\), where \(\varepsilon\) is a scalar in \(F_{q^{2k}}\) such that \(\varepsilon^{q^{k+1}} = \tau(g)\) (there always exists such an \(\varepsilon\) for any \(\tau(g) \in F_{q^k}^*\)). That means \(\tau : C_{CO(W)}(s) \to \mathbb{F}_q^*\) is an epimorphism and hence \(C_{CO(W)}(s) \simeq C_{GO(W)}(s) \cdot \mathbb{Z}_{q-1}\).

1b) Next, we will show that \(C_{GO(W)}(s) \simeq GL_{na}(q^k)\) and \(C_{CO(W)}(s) \simeq GL_{na}(q^k) : \mathbb{Z}_{q-1}\), where \(W = U_j\) for \(j = 1, \ldots, l', n_0 = n_j,\) and \(k = \deg(g_j)\). Note that the characteristic polynomial of \(s\) acting on \(W\) is \(g_j^{n_0}(x)g_j^{n_0}(x)\). Let \(\lambda \in \mathbb{F}_{q^k}\) be a root of \(g_j\). Then all the roots of \(g_j\) are \(\lambda, \lambda^q, \ldots, \lambda^{q^{k-1}}\) and all the roots of \(\tilde{g}_j\) are \(\tau \lambda^{-1}, \ldots, \tau \lambda^{-q^{k-1}}\). Let \((f_i)\) be a basis of \(W\), and define a Frobenius endomorphism \(\sigma : \sum_i x_i f_i \mapsto \sum_i x_i^q f_i\) on \(\tilde{W} := W \otimes_{\mathbb{F}_q} \mathbb{F}_{q^k}\), where \(x_i \in \mathbb{F}_{q^k}\). The simplicity of \(s\) implies that \(\tilde{W} = \tilde{W}_1 \oplus \cdots \oplus \tilde{W}_k \oplus \tilde{W}_1' \oplus \cdots \oplus \tilde{W}_k'\), where \(\tilde{W}_i = \text{Ker}(s - \lambda^{q^{i-1}})\) and \(\tilde{W}_j' = \text{Ker}(s - \tau \lambda^{-q^{j-1}})\). We see that \(\sigma\) permutes the \(\tilde{W}_i\)'s and \(\tilde{W}_j'\)'s cyclically: \(\sigma(\tilde{W}_i) = \tilde{W}_{i+1}, \sigma(\tilde{W}_j') = \tilde{W}_{j+1}'\), where \(\tilde{W}_{2k+1} = \tilde{W}_1\) and \(\tilde{W}_{2k+1}' = \tilde{W}_1'\). Let \(g \in CO(W)\) commuting with \(s|W\). Then \(g\) preserves each \(\tilde{W}_i, \tilde{W}_j'\). Again, the action of \(g\) on \(\tilde{W}\) is completely determined by its action on \(\tilde{W}_1\) and \(\tilde{W}_1'\). So \(C_{CO(W)}(s) \hookrightarrow GL_{2n_0}(q^k)\).

We also have
\[\tilde{W}_i^+ = \bigoplus_i \tilde{W}_i \bigoplus_{j \geq 2} \tilde{W}_j'\] and \(\tilde{W}_1' \cap \tilde{W}_1^+ = 0\). \(1.3\)

Choose basis \((u_i)\) and \((v_i)\) of \(\tilde{W}_1\) and \(\tilde{W}_1'\), respectively. Suppose that \(g\) acts on \(\tilde{W}_1\) with matrix \(A = (a_{ij})\) (with respect to the basis \((u_i)\)) and on \(\tilde{W}_1'\) with matrix \((b_{ij})\) (with respect to the basis \((v_i)\)). From \(1.3, g \in C_{GO(W)}(s)\) if and only if \((gu, gv) = (u, v)\) for any \(u \in \tilde{W}_1, v \in \tilde{W}_1'\), i.e.
\[(u_k, v_l) = (gu_k, gv_l) = (\sum_i a_{ik} u_i, \sum_j b_{jl} v_j) = \sum_{i,j} a_{ik} b_{jl} (u_i, v_j).\]
In other words, \( g \in C_{GO(W)}(s) \) if and only if \( ^tAUB = U \). Since \( U \) is non-degenerate, \( ^tAUB = U \) means that \( B \) is uniquely determined by \( A \). Therefore, \( C_{GO(W)}(s) \simeq GL_{n_0}(q^k) \).

Similarly, \( g \in C_{CO(W)}(s) \) if and only if \( (gu, gv) = \tau(g)(u, v) \) for any \( u \in \widetilde{W}_1, v \in \widetilde{W}_1' \) or equivalently \( ^tAUB = \tau(g)U \). Therefore, \( C_{CO(W)}(s) \simeq C_{GO(W)}(s) \cdot \mathbb{Z}_{q-1} \).

1c) Lastly, we will show that \( C_{GO(V_\tau)}(s) \simeq GO^\pm_m(q^2) \) and \( C_{CO(V_\tau)}(s) \simeq C_{GO(V_\tau)}(s) \cdot \mathbb{Z}_{q-1} \).

Again, we have \( \widetilde{V}_\tau = \widetilde{V}_1 \oplus \widetilde{V}_2 \), where \( \widetilde{V}_\tau = V_\tau \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2} \), \( \widetilde{V}_1 = \text{Ker}(s - \lambda_1) \), and \( \widetilde{V}_2 = \text{Ker}(s + \lambda_1) \). Furthermore, \( \widetilde{V}_1 \perp \widetilde{V}_2 \) and therefore \( \widetilde{V}_1, \widetilde{V}_2 \) are non-degenerate. Similar arguments as in 1a), we see that the action of an element \( g \in C_{CO(V_\tau)}(s) \) on \( \widetilde{V}_\tau \) is completely determined by its action on \( \widetilde{V}_1 \). Hence, \( C_{GO(V_\tau)}(s) \simeq GO(\widetilde{V}_1) = GO^\pm_m(q^2) \) and \( C_{CO(V_\tau)}(s) \simeq C_{GO(V_\tau)}(s) \cdot \mathbb{Z}_{q-1} \), since \( m \) is even.

Combining what we have proved in 1a), 1b), 1c), we have that

\[
C_{GO(V)}(s) \simeq C_{GO(V_\tau)}(s) \times \prod_i C_{GO(V_i)}(s) \times \prod_j C_{GO(U_j)}(s) \simeq GO^\pm_m(q^2) \times \prod_i GU_{m_i}(q^{\deg(f_i)/2}) \times \prod_j GL_{n_j}(q^{\deg(g_j)}).
\]

We have also proved that \( C_{CO(W)}(s) \simeq C_{GO(W)}(s) \cdot \mathbb{Z}_{q-1} \) for any \( W \in \{V_\tau, V_1, \ldots, V_l, U_1, \ldots, U_p\} \). More precisely, given any \( \alpha \in \mathbb{F}_q^* \), there exists \( g_W \in C_{CO(W)}(s) \) such that \( \tau(g_W) = \alpha \) for \( W \in \{V_\tau, V_1, \ldots, V_l, U_1, \ldots, U_p\} \). Set \( g = (g_{V_\tau}, g_{V_1}, \ldots, g_{V_l}, g_{U_1}, \ldots, g_{U_p}) \). Then \( g \in C_{CO(V)}(s) \) and \( \tau(g) = \alpha \). Hence, \( C_{CO(W)}(s) \simeq C_{GO(W)}(s) \cdot \mathbb{Z}_{q-1} \). The Lemma is proved in this case.

**Case 2**: If \( \tau \) is a square in \( \mathbb{F}_q \), suppose that \( \tau = \lambda_2^2 \) with \( \lambda_2 \in \mathbb{F}_q \). Replacing \( s \) by \( \lambda_2^{-1}s \) if necessary, we can assume that \( \tau = 1 \). Decompose \( P(x) \) into irreducible polynomials over \( \mathbb{F}_q \):

\[
P(x) = (x - 1)^k'(x + 1)^{2m - k'} \prod_{i=1}^l f_i^{m_i}(x) \prod_{j=1}^{l'} g_j^{n_j}(x) \tilde{g}_j^{n_j}(x),
\]

where

- If \( \lambda \) is a root of \( f_i(x) \) then \( \lambda^{-1} \) is also a root of \( f_i(x) \), \( \deg(f_i) \) is even, \( \pm 1 \) are not the roots of \( f_i \),
• If \( \lambda \) is a root of \( g_j(x) \) then \( \lambda^{-1} \) is a root of \( \hat{g}_j(x) \), \( \deg(g_j) = \deg(\hat{g}_j) \), \( g_j \neq \hat{g}_j \),
• \( n = m + \sum_{i=1}^{l} m_i \deg(f_i) / 2 + \sum_{j=1}^{l'} n_j \deg(g_j) \).

Since \( 1 = \tau^n = \det(s) = (-1)^{2m-k'} \), \( k' \) is even. Set \( k' = 2k \) and

\[
V' := \text{Ker}(s-1), V'' := \text{Ker}(s+1), V_i := \left( \frac{P(x)}{g_j^m(x)} \right)_{x=s}(V), U_j := \left( \frac{P(x)}{g_j^m(x)} \hat{g}_j^m(x) \right)_{x=s}(V).
\]

Then \( V = V' \oplus V'' \oplus V_1 \oplus \cdots V_i \oplus U_1 \oplus \cdots \oplus U_i \) is an orthogonal decomposition of \( V \). It is obvious that \( C_{GO(V')}(s) = GO(V') = GO_{2k}(q) \) and \( C_{GO(V''}) (s) = GO(V'') = GO_{2(m-k)}(q) \).

Furthermore, \( C_{CO(V')} (s) = CO(V') = GO_{2k}(q) \cdot \mathbb{Z}_{q-1} \) and \( C_{CO(V'')}(s) = CO(V'') = GO_{2(m-k)}(q) \cdot \mathbb{Z}_{q-1} \). Now we can repeat the arguments in Case 1 and complete the proof.

\[ \square \]

Remark. Lemma 5.1.5 is wrong when \( s \notin CO_{2n}^+(q)^0 \). Here is one example. Suppose that \( q \) is odd. Let \( V = \mathbb{F}_q^2 = \{(x, y) \mid x, y \in \mathbb{F}_q \} \) be the vector space of dimension 2. The quadratic form \( Q(x, y) = xy \) is non-degenerate of Witt index 1. The Gram matrix of \( Q \) corresponding to the basis \( \{(1, 0), (0, 1)\} \) is \( J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). By definition, \( GO_{2}^{+}(q) \) is the group of matrices \( A \in GL_2(q) \) such that \( \tau A J A = J \). Therefore,

\[
GO_{2}^{+}(q) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \mid a \in \mathbb{F}_q^* \right\}.
\]

Similarly,

\[
CO_{2}^{+}(q) = \left\{ A \in GL_2(q) \mid \exists \tau \in \mathbb{F}_q^*, \tau A J A = \tau J \right\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{F}_q \right\}.
\]

Take the element \( s = J \), which is semi-simple. Since \( \tau(s) = 1 \) and \( \det(s) = -1 \), \( s \notin CO_{2}^+(q)^0 \). Direct computation shows that

\[
C_{GO_{2}^+(q)}(s) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}.
\]

On the other hand,

\[
C_{CO_{2}^{+}(q)}(s) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \mid a \in \mathbb{F}_q^* \right\}.
\]

Therefore, \( C_{CO_{2}^{+}(q)}(s) \nsubseteq C_{GO_{2}^{+}(q)}(s) \cdot \mathbb{Z}_{q-1} \) by order comparison.
5.2 Unipotent Characters

5.2.1 Unipotent Characters of $SO_{2n+1}(q)$ and $PCSp_{2n}(q)$

Proposition 5.1 in [70] shows that $SO_{2n+1}(q)$ as well as $PCSp_{2n}(q)$ have a unique unipotent character of minimal degree $(q^n - 1)(q^n - q)/2(q + 1)$ and any other non-trivial unipotent character has degree greater than $q^{2n}/2(q + 1)$. We mimic its proof and get the following Proposition, which classifies unipotent characters of degrees up to $\approx q^{6n-15}/2$.

Proposition 5.2.1. Let $G^*$ be either $(B_n)_{ad}(q) = SO_{2n+1}(q)$ or $(C_n)_{ad}(q) = PCSp_{2n}(q)$.

Suppose that $n \geq 6$ and $\chi \in \text{Irr}(G^*)$ is unipotent. Then either $\chi$ is one of characters labeled by $(n), (0 \ 1, n), (0 1_n), (0 2_n-1), (0 2_n-1, 0 1_n), (2_n-1, 0 1_n), (1 n-1), (0 1 2 n), (0 1 2 n), (0 1 2 n), (0 1 2 n), (0 1 2 n), (0 1 2 n), or \chi(1) \geq (q^{2n-2} - 1)(q^{2n-1})(q^{n-2} - 1)(q^{n-3}/2 (q^2 - 1)(q^4 - 1)(q^3 + 1) \approx q^{6n-15}/2$, which is the degree of the unipotent character labeled by $(0 3_n-2)$.

Note: For reader’s convenience, we put these low-dimensional unipotent characters in Table 5-1, where degree of each character (labeled by a symbol) is calculated.

Proof. From [9, p. 466, 467], we know that the unipotent characters of $G^*$ are parametrized by symbols of the form

\[
\begin{pmatrix}
\lambda \\
\mu
\end{pmatrix} = \begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_a \\
\mu_1 & \mu_2 & \cdots & \mu_b
\end{pmatrix}
\]

where $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_a$, $0 \leq \mu_1 < \mu_2 < \cdots < \mu_b$, $a - b$ is odd and positive, $(\lambda_1, \mu_1) \neq (0, 0)$, and

\[
\sum_i \lambda_i + \sum_j \mu_j - \left(\frac{a + b - 1}{2}\right)^2 = n.
\]

The integer $n$ is called the rank of the symbol $(\lambda)$. The degree of the unipotent character $\chi_{\lambda, \mu}$ corresponding to the symbol $(\lambda)$ is equal to

\[
\frac{(q^2 - 1)(q^4 - 1) \cdots (q^{2n-1})(q^{2n-1}q^{\lambda_1} - q^{\lambda_1})(q^{\lambda_2} - q^{\lambda_2})(q^{\lambda_3} - q^{\lambda_3})}{2^{a+b-1}q^{(a+b-2)}(a+b-4)+(a+b-6)+\cdots} \prod_i \prod_{k=1}^{\lambda_i} (q^{2k-1} - 1) \prod_j \prod_{k=1}^{\mu_j} (q^{2k-1} - 1).
\]

1) Define

\[
D(n) = (q^{2n-2} - 1)(q^{2n-1})(q^{n-2} - 1)(q^{n-2} - q^3)/2(q^2 - 1)(q^4 - 1)(q^3 + 1).
\]
We have that
\[
\frac{D(n)}{D(n-1)} = \frac{(q^{2n} - 1)(q^{n-2} - 1)(q^{n-5} - 1)}{(q^{2n-4} - 1)(q^{n-3} - 1)(q^{n-6} - 1)}
\]
and \(q^6 < D(n)/D(n-1) < q^7\) for every \(n \geq 7\). Also, \(D(n) < q^{6n-14}/2\) for every \(n \geq 6\).

Let \(L_n\) be the set of 14 symbols in this Proposition. Note that if \((n, q) \neq (6, 2)\) then the degrees of characters corresponding to the symbols in \(L_n\) is smaller than \(D(n)\). We will prove by induction on \(n \geq 6\) that if \(\chi = \chi^{\lambda \mu}\) with \(\binom{\lambda}{\mu}\) is not in \(L_n\), then \(\chi(1) \geq D(n)\). The induction base \(n = 6\) can be checked easily by using [11, Table 27]. The rest of the proof establishes the induction step for \(n \geq 7\), by means of induction on \(b \geq 0\).

2) Now we consider the case \(b = 0\). Then \(a \geq 1\) is odd. We may assume \(a \geq 3\), and \(\lambda \neq (0, 1, n), (0, 2, n - 1), (0, 3, n - 2)\). First we assume \(a = 3\). If \(\lambda_1 = 0\) then \(\lambda = (0, k, n + 1 - k)\), where \(4 \leq k < (n + 1)/2\); in particular, \(n \geq 8\). In this case,
\[
\chi(1) = \frac{1}{2} \cdot \frac{(q^{2(k+1)} - 1) \cdots (q^{2n} - 1)}{(q^4 - 1) \cdots (q^{2(n+1-k)-1})} \cdot \frac{(q^k - 1)(q^{n-k+1} - 1)(q^{n-k+1} - q^k)}{q^2 - 1} >
\]
\[
> \frac{1}{2} q^{2(k-1)(n-k)} \cdot \frac{(q^k - 1)(q^{n-k+1} - 1)(q^{n-k+1} - q^k)}{q^2 - 1}.
\]
Since \(3 \leq k - 1 < n - k, 2(k-1)(n-k) \geq (n-4) = 6n - 24\). Moreover, \(n - k + 1 > k\). Therefore \((q^k - 1)(q^{n-k+1} - 1)(q^{n-k+1} - q^k)/(q^2 - 1) > q^{k-2} q^{2k} > q^{10}\). It follows that \(\chi(1) > \frac{1}{2} q^{6n-24} q^{10} = q^{6n-14}/2 > D(n)\) as required. If \(\lambda_1 \geq 1\), then \(\lambda = (k, l, n + 1 - k - l)\) with \(1 \leq k < l < n + 1 - k - l\). Then
\[
\chi(1) = \frac{1}{2} \cdot \frac{(q^{2(k+1)} - 1) \cdots (q^{2(k+l)} - 1)}{(q^2 - 1) \cdots (q^{2l} - 1)} \cdot \frac{(q^{2(k+l+1)} - 1) \cdots (q^{2n} - 1)}{(q^4 - 1) \cdots (q^{2(n+1-k-l)-1})} 
\]
\[
\cdot \frac{(q^{2k} - q^k)(q^{n+1-k-l} - q^k)(q^{n+1-k-l} - q^l)}{q^2 - 1} > \frac{1}{2} q^{2k} q^{2(k+l)(n-k-l)} q^{2k+l}.
\]
If \((k, l) = (1, 2)\) then \(\lambda = (1, 2, n - 2)\) and
\[
\chi(1) = \frac{(q^2 - q)(q^{2n-2} - 1)(q^{2n-1} - 1)(q^{n-2} - q)(q^{n-2} - q^2)}{2(q^2 - 1)^2(q^4 - 1)} > D(n)
\]
for every \(n \geq 6\). It remains to consider \((k, l) \neq (1, 2)\). Then \(k + l \geq 4\) and \(l \geq 3\). It follows that \(3 \leq k + l - 1\) and \(3 \leq l \leq n - k - l\). Therefore, \(2(k + l - 1)(n - k - l) \geq 6(n - 4)\).
So, in this case, \( \chi(1) > \frac{1}{2}q^{2kl}q^{6n-24}q^{2k+l} > \frac{1}{2}q^{6n-14} > D(n) \) and we are done. Now we may assume \( a \geq 5 \).

Suppose that \( \lambda_1 \geq 1 \). Consider the unipotent character \( \chi' \) labeled by the symbol \( (\lambda') \) of rank \( n-1 \), where \( \lambda' = (\lambda_1 - 1, \lambda_2, ..., \lambda_a) \). We have

\[
\frac{\chi(1)}{\chi'(1)} = \frac{(q^{\lambda_2} - q^{\lambda_1}) \cdots (q^{\lambda_a} - q^{\lambda_1})}{(q^{\lambda_2} - q^{\lambda_1-1}) \cdots (q^{\lambda_a} - q^{\lambda_1-1})} \cdot \frac{q^{2n} - 1}{q^{2\lambda_1} - 1} \geq \frac{q^{2n-\lambda_1}}{2}.
\]

Note that \( \sum_{i=1}^a \lambda_i - \left(\frac{a-1}{2}\right)^2 = n \) and hence \( n - \lambda_1 = \lambda_2 + \cdots + \lambda_a - \left(\frac{a-1}{2}\right)^2 \geq (a-1)\lambda_1 + \frac{a^2-1}{4} \geq 10 \). Therefore, \( \chi(1)/\chi'(1) \geq q^{20}/2 > q^7 \). It follows that \( \chi(1) \geq q^7D(n-1) > D(n) \). Now we may assume that \( \lambda_1 = 0 \).

Next we assume that \( \lambda_i - \lambda_{i-1} \geq 2 \) for some \( i \geq 2 \). Consider the unipotent character \( \chi' \) labeled by the symbol \( (\lambda) \) of rank \( n-1 \), where \( \lambda' = (\lambda_1, ..., \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, ..., \lambda_a) \). Similarly, we have \( \chi(1)/\chi'(1) > q^{2(n-\lambda_i)}/2 \). By induction hypotheses, \( \chi'(1) \geq D(n-1) \).

If \( n - \lambda_i \geq 4 \) then \( \chi(1) \geq q^7D(n-1) > D(n) \). Now suppose \( n - \lambda_i \leq 3 \). Note that \( \sum_i \lambda_i = \left(\frac{a-1}{2}\right)^2 + n \). If \( i \leq a-1 \) then \( \lambda_{a-1} \geq n-3 \) and \( \lambda_a \geq n-2 \). Then

\[
\left(\frac{a-1}{2}\right)^2 + n \geq 0 + 1 + \cdots + (a-3) + (n-3) + (n-2) \text{ or } a^2 - 8a - 9 + 4n \leq 0.
\]

This is a contradiction because \( a \geq 5 \) and \( n \geq 7 \). Therefore \( i = a \) and we have \( 0 + 1 + \cdots + (a-2) + \lambda_a = \left(\frac{a-1}{2}\right)^2 + n \).

Since \( \lambda_a \geq (n-3) \), \( a^2 - 4a - 9 \leq 0 \) and therefore \( a = 5 \). Then \( \lambda = (0, 1, 2, 3, n-2) \) and it is easy to check that \( \chi(1) > D(n) \). Now we may assume \( \lambda = (0, 1, 2, ..., a-2, a-1) \). Consider \( \chi' = \chi^{\lambda, \lambda'} \), where \( \lambda' = (0, 1, 2, ..., a-4, a-3) \), and \( \mu' = \mu \). Note that \( a^2 = 4n + 1 \) and the rank of \( \chi' \) is \( n - a + 1 \). We have

\[
\frac{\chi(1)}{\chi'(1)} = \frac{(q^{2n-1}) \cdots (q^{2n-1})}{2q^{(a-2)} \prod_{k=1}^{a-2} (q^{2k} - 1) \prod_{k=1}^{a-1} (q^{2k} - 1)} = \frac{(q^{2n-1}) \cdots (q^{2n-1})}{(q^2 - 1) \cdots (q^2 - 1)} \cdot \frac{q^{a-1} - q^{a-2} - 1}{2} \cdot \prod_{i=0}^{a-3} \frac{(q^{a-1} - q^i)(q^{a-2} - q^i)}{q^i(q^{2i+2} - 1)}.
\]

Remark that

\[
\prod_{i=0}^{a-3} \frac{(q^{a-1} - q^i)(q^{a-2} - q^i)}{q^i(q^{2i+1} + 1)} = \prod_{i=0}^{a-3} \frac{q^i(q^{i+2} - 1)}{q^{i+1} + 1} \geq \prod_{i=0}^{a-2} \frac{q^i(q^{i+2} - 1)}{q^{i+1} + 1} > q^4.
\]
Therefore,
\[
\frac{\chi(1)}{\chi'(1)} = \frac{q^{2n} - 1}{q^{2\lambda_1} - 1} \cdot \prod_{i=2}^{a} \frac{q^{\lambda_i} - q^{\lambda_i-1}}{q^{\lambda_i} - q^{\lambda_i-1} - 1} \cdot \prod_{j=1}^{b} \frac{q^{\mu_j} - q^{\mu_j-1} + q^{\mu_j-1}}{2(q^{2\mu_j} - 1)} > \frac{q^{2n} - 1}{2}.
\]

Since \( n \geq 7 \) and \( a^2 = 4n + 1, n \geq 12 \) and \( a \geq 7 \). Then \((n - a + 1)(a - 1) \geq 6(n - 6)\). Hence, \( \chi(1)/\chi'(1) > q^{12n-63}/2 > q^7 \) and we are done.

3) From now on we assume that \( b \geq 1 \). At this point we suppose that \( (\lambda_1, \mu_1) \neq (1, 0) \) and \( \lambda_1 \geq 1 \). Consider the character \( \chi' \) labeled by the symbol \( \left( \lambda_1 \right) \) of rank \( n - 1 \), where \( \lambda' = (\lambda_1 - 1, \lambda_2, ..., \lambda_a) \) and \( \mu' = \mu \). If \( \left( \lambda_1 \right) \in L_{n-1} \) then \( \left( \lambda_1 \right) \in \{(1 \frac{1}{2} 2), (3 \frac{n_2}{2} 1), (1 \frac{n_2}{2} \frac{1}{2}), (2 \frac{n_2}{2} 1)\} \). It is easy to check that the degrees of characters corresponding to these symbols are greater than \( D(n) \). Now we can assume \( \left( \lambda_1 \right) \) is not in \( L_{n-1} \). We have
\[
\frac{\chi(1)}{\chi'(1)} = \frac{q^{2n} - 1}{q^{2\lambda_1} - 1} \cdot \prod_{i=2}^{a} \frac{q^{\lambda_i} - q^{\lambda_i-1}}{q^{\lambda_i} - q^{\lambda_i-1} - 1} \cdot \prod_{j=1}^{b} \frac{q^{\mu_j} - q^{\mu_j-1} + q^{\mu_j-1}}{2(q^{2\mu_j} - 1)} > \frac{q^{2n} - 1}{2}.
\]

If \( n - \lambda_1 \leq 3 \) then \( \lambda_1 \geq n - 3 \) and we have \( n - 3 + n - 2 + ... + n + a - 4 + \frac{(b-1)b}{2} \leq \frac{(a+b-1)^2}{4} + n \).

This implies \( 4a(n - 3) + (a - b)^2 \leq 4n + 1 \) and therefore \( a(n - 3) \leq n \). Since \( n \geq 7, a = 1 \) which leads to a contradiction since \( b \geq 1 \). So \( n - \lambda_1 \geq 4 \). Then we have \( \chi(1)/\chi'(1) > q^7 \) and \( \chi(1) > q^7 D(n - 1) > D(n) \) as desired.

Similarly, for \( (\lambda_1, \mu_1) \neq (0, 1) \) and \( \mu_1 \geq 1 \), we consider the character \( \chi' \) labeled by the symbol \( \left( \lambda \right) \) of rank \( n - 1 \), where \( \lambda' = \lambda \) and \( \mu' = (\mu_1 - 1, \mu_2, ..., \mu_b) \). If \( \left( \lambda \right) \in L_{n-1} \) then \( \left( \lambda \right) \in \{(2 \frac{n-2}{2} 1), (0 \frac{n-2}{3} 3), (1 \frac{n-2}{2} 2), (0 \frac{1}{2} \frac{n-2}{2} ), (0 \frac{1}{2} 2), (0 \frac{1}{2} n-1) \} \). Again, it is easy to check that the degrees of characters corresponding to these symbols are greater than \( D(n) \). Now we can assume \( \left( \lambda \right) \) is not in \( L_{n-1} \). We have
\[
\frac{\chi(1)}{\chi'(1)} = \frac{q^{2n} - 1}{q^{2\mu_1} - 1} \cdot \prod_{j=2}^{b} \frac{q^{\mu_j} - q^{\mu_j-1}}{q^{\mu_j} - q^{\mu_j-1} - 1} \cdot \prod_{i=1}^{a} \frac{q^{\lambda_i} + q^{\mu_1}}{2(q^{2\mu_j} - 1)} > \frac{q^{2n} - 1}{2}.
\]

If \( n - \mu_1 \geq 4 \) then \( \chi(1)/\chi'(1) > q^7 \) and \( \chi(1) > q^7 D(n - 1) > D(n) \) as desired. If \( n - \mu_1 \leq 3 \) then \( \mu_1 \geq n - 3 \) and we have \( \frac{(a-1)a}{2} + n - 3 + n - 2 + ... + n + b - 4 \leq \frac{(a+b-1)^2}{4} + n \). This implies \( 4b(n - 3) + (a - b)^2 \leq 4n + 1 \). So \( b(n - 3) \leq n \). Since \( n \geq 7, b = 1 \). Then we have \( (a - 1)^2 \leq 13 \) and therefore \( a = 2 \) or \( a = 4 \).
Case a = 4, b = 1: Then $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_1 = n + 4$. This implies $\mu_1 \leq n - 2$. Hence $\binom{\lambda}{\mu} = \binom{0 \ 1 \ 2 \ 3}{n-2}$ or $\binom{0 \ 1 \ 2 \ 4}{n-3}$. The degrees of characters corresponding to both these symbols are greater than $D(n)$.

Case a = 2, b = 1: Since $n - 3 \leq \mu_1 \leq n$ and $\binom{\lambda}{\mu}$ is not in $L_n$, we have $\binom{\lambda}{\mu} = \binom{0 \ 4 \ 1}{n-3}$, $\binom{1 \ 3}{n-3}$, $\binom{0 \ 3}{n-2}$ or $\binom{1 \ 2}{n-2}$. Again, the degrees of characters corresponding to these symbols are greater than $D(n)$.

It remains to consider the case where $(\lambda_1, \mu_1) = (0, 1)$ or $(1, 0)$.

4) Here we suppose that $(\lambda_1, \mu_1) = (1, 0)$ but $\lambda \neq (1, 2, \ldots, a)$. Then there exists an $i \geq 2$ such that $\lambda_i \geq \lambda_{i-1} + 2$. We choose $i$ to be smallest possible. If $a = 2$ then $b = 1$ and $\binom{\lambda}{\mu} = \binom{1}{0} \in L_n$. Hence $a \geq 3$. Consider the character $\chi'$ labeled by the symbol $\binom{\lambda'}{\mu'}$ of rank $n - 1$, where $\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_a)$ and $\mu' = \mu$. If $\binom{\lambda'}{\mu'} \in L_{n-1}$ then $\binom{\lambda}{\mu}$ can only be $\binom{1 \ 3 \ n^{-1}}{n-1}$. The degree of the corresponding character is greater than $D(n)$. Now we can assume that $\binom{\lambda}{\mu'}$ is not in $L_{n-1}$. By induction hypothesis, $\chi'(1) \geq D_{n-1}$. Set

$$T_1 = \prod_{i' < i} \frac{q^{\lambda_i} - q^{\lambda_{i'}}}{q^{\lambda_{i-1}} - q^{\lambda_{i'}}}, T_2 = \prod_{i' > i} \frac{q^{\lambda_{i'}} - q^{\lambda_i}}{q^{\lambda_{i-1}} - q^{\lambda_{i'}}}, T_3 = \prod_{j} \frac{q^{\lambda_i} + q^{\mu_j}}{q^{\lambda_{i-1}} + q^{\mu_j}}.$$

Then $\chi(1)/\chi'(1) = [(q^{2n} - 1)/(q^{2\lambda_i} - 1)]T_1T_2T_3 > q^{2(n-\lambda_i+1)}/2$ (see [70, p. 2121]). If $n - \lambda_i \geq 3$ then $\chi(1)/\chi'(1) > q^7$ and therefore $\chi(1) \geq D(n)$ as desired. Now we assume $n - \lambda_i \leq 2$. There are two cases:

Case n - $\lambda_i = 2$: Then $i \geq a - 1$. If $i = a - 1$ then $a - b = 1$, $a = n - 2$ and $\binom{\lambda}{\mu} = \binom{1 \ldots \ n-4 \ n-2 \ n^{-1}}{0 \ldots n-4}$. Since $n \geq 7$, $a \geq 5$ and $i = b \geq 4$. It follows that $T_1 \geq q^3$, $T_3 \geq q^{b-1} \geq q^3$. Hence $\chi(1)/\chi'(1) > q^{10}/2 > q^7$ and we are done. If $i = a \geq 3$ then $T_1 \geq q^{a-2} \frac{q^{\lambda_i} - q}{q^{\lambda_{i-1}} - q}$, $T_2 = 1$ and $T_3 \geq q^{a^2+a+1} / q^{a-1+1}$. Then $\chi(1)/\chi'(1) > q^7$ and we are done.

Case n - $\lambda_i \leq 1$: Then $i = a$ and $\lambda_a \geq n - 1$. If $\lambda_a \geq n$ then $\lambda_a = n$, $a - b = 1$ and $\binom{\lambda}{\mu} = \binom{1 \ldots a-1 \ n}{0 \ldots a-2}$. Since $\binom{\lambda}{\mu}$ is not in $L_n$, $a \geq 4$. We have

$$\frac{\chi(1)}{\chi'(1)} = \prod_{j=1}^{a-1} \left( \frac{q^n - q^j}{q^{n-1} - q^j} \right) \frac{q^n + q^{j-1}}{q^{n-1} + q^{j-1}} = \frac{(q^n + 1)(q^{n-1} - 1)}{(q^{n-a} - 1)(q^{n-a+1} + 1)} > q^{2a-2}.$$
If \( a \geq 5 \) then \( \chi(1)/\chi'(1) > q^8 \) and therefore we are done. If \( a = 4 \) then \( (\lambda_\mu) = \binom{123}{012} \) and \( \chi(1) > D(n) \). If \( \lambda_a = n - 1 \) then we also have \( a - b = 1 \) and \( (\lambda_\mu) = \binom{12 ... a-1 n-1}{01 ... a-3 a-1} \).

If \( a = 3 \) then \( (\lambda_\mu) = \binom{12 n-1}{02} \). Check directly we see that \( \chi(1) > D(n) \). If \( a \geq 4 \) then \( T_1 > q^{a-1} > q^3 \), \( T_2 = 1 \) and \( T_3 > q^{b-1} \geq q^2 \). So \( \chi(1)/\chi'(1) > q^7 \) and we are done again.

Similarly, suppose that \( (\lambda_1, \mu_1) = (0, 1) \) but \( \mu 
eq (1, 2, ..., b) \); in particular, \( b \geq 2 \). Then there exists an index \( j \geq 2 \) such that \( \mu_j > \mu_j - 1 + 2 \). We choose \( j \) to be smallest possible. Consider the character \( \chi' \) labeled by the symbol \( (\lambda'_\mu) \) of rank \( n - 1 \), where \( \lambda' = \lambda \) and \( \mu' = (\mu_1, ..., \mu_{j-1}, \mu_j - 1, \mu_{j+1}, ..., \mu_b) \). If \( (\lambda'_\mu) \in L_{n-1} \) then \( (\lambda_\mu) \) can only be \( \binom{01 n-1}{13} \). The degree of the corresponding character is greater than \( D(n) \) and hence we are done in this case. Now we can assume \( (\lambda'_\mu) \) is not in \( L_{n-1} \). Set

\[
U_1 = \prod_{j' < j} \frac{q^{\mu_j} - q^{\mu_{j'}}}{q^{\mu_j - 1} - q^{\mu_{j'}}}, U_2 = \prod_{j > j'} \frac{q^{\mu_j} - q^{\mu_{j'}}}{q^{\mu_{j'} - 1} - q^{\mu_j}}, U_3 = \prod_i \frac{q^{\lambda_i} + q^{\mu_i}}{q^{\lambda_i - 1} + q^{\mu_i - 1}}.
\]

Then \( \chi(1)/\chi'(1) = [(q^{2n} - 1)/(q^{2\mu_j} - 1)]U_1U_2U_3 > q^{2(n-\mu_j + 1)}/2 \) (see [70, p. 2122]). If \( n - \mu_j \geq 3 \) then \( \chi(1)/\chi'(1) > q^7 \) and therefore \( \chi(1) \geq D_n \) as desired. Now we assume \( n - \mu_j \leq 2 \). There are two cases:

**Case n - \mu_j = 2**: Then \( j \geq b - 1 \). If \( j = b - 1 \) then \( a - b = 1 \), \( b = n - 2 \) and \( (\lambda_\mu) = \binom{01 ... n-2}{1 ... n-4 n-2 n-1} \). Since \( b \geq 7 \), \( b \geq 5 \) and \( j \geq 4 \). It follows that \( U_1 \geq q^2 \frac{q^{\mu_j - q}}{q^{\mu_j - 1} - q} \), \( T_3 \geq \frac{q^{\mu_j + 1}}{q^{\mu_j - 1} + 1} \). Hence \( \chi(1)/\chi'(1) > q^8/2 > q^7 \) and we are done. If \( j = b \geq 3 \) then \( V_1 \geq q^{\mu_j} / q^{\mu_j - 1} - q \), \( V_2 = 1 \) and \( V_3 \geq q^{\mu_j + 1} / q^{\mu_j - 1} + 1 \). Therefore, \( \chi(1)/\chi'(1) > q^7 \) and we are done again.

If \( j = b = 2 \) then \( a = 3 \) or \( a = 5 \). In any case, \( \lambda_a \leq 4 < n - 2 \). Therefore, in this case, \( U_1 > q \), \( U_2 = 1 \) and \( U_3 > q^2 \). Then \( \chi(1)/\chi'(1) > q^7 \) and \( \chi(1) \geq D_n \) as required.

**Case n - \mu_j = 1**: Then \( j = b \) and \( \mu_b \geq n - 1 \). If \( \mu_b \geq n \) then \( \mu_b = n \), \( a - b = 1 \) and \( (\lambda_\mu) = \binom{01 ... b}{12 ... b-1 n} \). Since \( (\lambda_\mu) \) is not in \( L_n \), \( b \geq 3 \). We have

\[
\frac{\chi(1)}{\chi'(1)} = \prod_{i=0}^b q^n + q^i, \prod_{i=1}^{b-1} q^n - q^i = \frac{(q^n + 1)(q^{n-1} - 1)}{(q^{n-b-1} + 1)(q^{n-b} - 1)} > q^{2b-1}.
\]

If \( b \geq 4 \) then \( \chi(1)/\chi'(1) > q^7 \) and therefore we are done. If \( b = 3 \) then \( (\lambda_\mu) = \binom{0123}{12 n} \). In this case, \( \chi(1) > D(n) \). If \( \mu_b = n - 1 \) then we also have \( a - b = 1 \) and \( (\lambda_\mu) = \binom{01 ... b-1 b+1}{12 ... b-1 n-1} \).
where \( n \geq b + 2 \). If \( b = 2 \) then \( (\lambda, n) = (1, 3) \). Check directly we see that \( \chi(1) > D(n) \). If \( b \geq 3 \) then \( U_1 > q^{b-1} \geq q^2 \), \( U_2 = 1 \) and \( U_3 \geq q^3 \). So \( \chi(1)/\chi'(1) > q^2 \) and we are done.

5) Here we suppose that \( \mu_1 = 0 \) and \( \lambda = (1, 2, \ldots, a) \). Consider the character \( \chi' \) labeled by the symbol

\[
\left( \begin{array}{c} \lambda' \\ \mu' \end{array} \right) = \left( \begin{array}{c} 0 & 1 & 2 & \ldots & a \\ \mu_2 & \mu_3 & \ldots & \mu_b \end{array} \right)
\]

of the same rank \( n \), but with the parameters \( a' = a + 1 \) and \( b' = b - 1 \). It is easy to see that \( (\lambda', \mu') \) is not in \( \mathcal{L}_n \). So \( \chi'(1) > D(n) \) by the induction hypothesis (on \( b \)). But

\[
\frac{\chi(1)}{\chi'(1)} = \frac{\prod_{i=1}^{\lambda_1} (1 + \frac{2}{q^{i-1}})}{\prod_{j=2}^{a'} (1 + \frac{2}{q^{j-1}})} \geq \frac{\prod_{i=1}^{\lambda_1} (1 + \frac{2}{q^{i-1}})}{\prod_{j=1}^{b-1} (1 + \frac{2}{q^{j-1}})} > 1,
\]

so we are done.

Similarly, suppose that \( \lambda_1 = 0 \) and \( \mu = (1, 2, \ldots, b) \) but \( \lambda \neq (0, 1, \ldots, a - 1) \). Then there exists an \( i \geq 2 \) such that \( \lambda_i \geq \lambda_{i-1} + 2 \). We choose \( i \) to be smallest possible. Consider the character \( \chi' \) labeled by the symbol

\[
\left( \begin{array}{c} \lambda' \\ \mu' \end{array} \right) = \left( \begin{array}{c} \lambda_1 & \ldots & \lambda_{i-1} & \lambda_i - 1 & \lambda_{i+1} & \ldots & \lambda_a \\ \mu_1 & \ldots & \mu_b \end{array} \right)
\]

of rank \( n' = n - 1 \). If \( (\lambda', \mu') \in \mathcal{L}_{n-1} \), then \( (\lambda, \mu) \) is one of the symbols \( (0, 1, 3, n-1), (0, 2, n-1) \). In this case, we can check that \( \chi(1) > D(n) \). Now we can suppose that \( (\lambda', \mu') \) is not in \( \mathcal{L}_{n-1} \). We have \( \frac{\chi(1)}{\chi'(1)} = q^\frac{2n-1}{q^{n+1}} \cdot V_1 V_2 V_3 \), where

\[
V_1 = \prod_{i < i'} \frac{q^{\lambda_i} - q^{\lambda_{i'}}}{q^{\lambda_i - 1} - q^{\lambda_{i'}}}, \quad V_2 = \prod_{i > i'} \frac{q^{\lambda_i} - q^{\lambda_{i'}}}{q^{\lambda_i} - q^{\lambda_{i'-1}}}, \quad V_3 = \prod_j \frac{q^{\lambda_i} + q^{\mu_j}}{q^{\lambda_i} + q^{\mu_j}}.
\]

Note that \( V_1 \geq (q^{\lambda_i} - 1)/(q^{\lambda_{i'-1}} - 1) > q \), \( V_2 > 1/2 \) and \( V_3 > 1 \). So \( \chi(1)/\chi'(1) > q^{2(n-\lambda_i)+1}/2 \). If \( n - \lambda_i \geq 4 \), then \( \chi(1)/\chi'(1) > q^3/2 \) and we are done. Now we assume that \( n - \lambda_i \leq 3 \). Then \( i \geq a - 1 \). There are two cases:

**Case i = a - 1**: We have \((a - b)^2 + 4b + 4\lambda_{a-1} + 4\lambda_a = 4n + 8a - 11\). Since \((a - b)^2 + 4b \geq 4a - 3\) and \( \lambda_a \geq a \), \( \lambda_{a-1} \leq n - 2 \). If \( \lambda_{a-1} = n - 2 \) then \( \lambda_a = a \). It follows
that \( a \geq n - 1 \geq 6 \). Hence, \( V_1 \geq q^{a-2} \geq q^4 \) and we have

\[
\frac{\chi(1)}{\chi'(1)} > \frac{1}{2}q^4 \frac{q^{2n} - 1}{q^{2n-4} - 1} \geq q^7,
\]

so \( \chi(1) > D_n \). If \( \lambda_{a-1} = n - 3 \) then \( \lambda_a \leq a + 1 \). Then \( a + 1 \geq n - 2 \) and therefore \( a \geq 4 \). Hence \( V_1 \geq q^{a-2} \geq q^2 \). Again, we have

\[
\frac{\chi(1)}{\chi'(1)} > \frac{1}{2}q^2 \frac{q^{2n} - 1}{q^{2n-6} - 1} \geq q^7
\]

and we are done.

**Case i = a** : Then \((a-b)^2 + 4b + 4\lambda_a = 4n + 4a - 3\). Therefore, \((a-b-2)^2 = 4(n-\lambda_a)+1\).

Since \( n-\lambda_a \leq 3 \), \( n-\lambda_a = 0 \) or \( 2 \). If \( n-\lambda_a = 2 \) then \((a-b-2)^2 = 9\) and we have \( a - b = 5 \).

In particular, \( a \geq 6 \). Then \( V_1 > q^{a-1} \geq q^5 \). Moreover, in this case, \( V_2 = 1 \) and \( V_3 > 1 \). So \( \chi(1)/\chi'(1) > q^9 \) and we are done. If \( n-\lambda_a = 0 \) then \( a - b = 1 \) or \( 3 \). In this case, \( V_1 > q^{a-1}, V_2 = 1 \) and \( V_3 > q^{b-1} \). So \( \chi(1)/\chi'(1) > q^{a+b-2} \). If \( (a, b) = (4, 3) \) or \( (5, 2) \) then \( \left( ^{\lambda}_{\mu} \right) = \left( ^{0 \ 1 \ 2 \ 3 \ n}_{0 \ 1 \ 2 \ 3 \ n} \right) \) or \( \left( ^{0 \ 1 \ 2 \ 3 \ n}_{0 \ 1 \ 2 \ 3 \ n} \right) \).

Checking directly we have \( \chi(1) > D(n) \). Otherwise, \( a + b \geq 9 \) and therefore \( \chi(1)/\chi'(1) \geq q^7 \) and we are done again.

6) Finally, we consider the case where \( \left( ^{\lambda}_{\mu} \right) = \left( ^{0 \ 1 \ 2 \ \ldots \ a-1}_{1 \ 2 \ \ldots \ b} \right) \). Consider the character \( \chi' \) labeled by the symbol \( \left( ^{\lambda'}_{\mu'} \right) = \left( ^{0 \ 1 \ 2 \ \ldots \ a-2}_{1 \ 2 \ \ldots \ b-1} \right) \) of rank \( n - 1 \). Then we have

\[
\frac{\chi(1)}{\chi'(1)} = \frac{q^{a+b-2}(q^{2n} - 1)}{(q^b - 1)(q^{a-1} + 1)} > \frac{q^{2n} - 1}{q} > q^7.
\]

Note that \( \left( ^{\lambda'}_{\mu'} \right) \) is not in \( \mathcal{L}_{n-1} \). Therefore, \( \chi(1) > q^7 \chi'(1) \geq q^7 D(n-1) > D(n) \) as desired. \( \square \)

**Corollary 5.2.2.** Let \( G^* \) be either \((B_n)_{ad}(q) = SO_{2n+1}(q)\) or \((C_n)_{ad}(q) = PCSp_{2n}(q)\).

Suppose that \( n \geq 5 \) and \( \chi \in \text{Irr}(G^*) \) is unipotent. Then either \( \chi \) is one of characters labeled by \( \left( ^{n}_{\ n} \right) \), \( \left( ^{0 \ 1 \ n}_{\ n} \right) \), \( \left( ^{0 \ 1 \ n}_{\ 0} \right) \), \( \left( ^{1 \ n}_{1 \ 0} \right) \), \( \left( ^{0 \ n}_{0 \ 1} \right) \), with degrees \( 1 \), \( (q^n - 1)(q^n - q)/2(q + 1), (q^n + 1)(q^n + q)/2(q + 1), (q^n + 1)(q^n - q)/2(q - 1), (q^n - 1)(q^n + q)/2(q - 1), \) respectively, or \( \chi(1) > q^{4n-8} \).
Proof. If \( n \geq 6 \), Corollary comes from Proposition 5.2.1. The case \( n = 5 \) can be verified easily by using Table 26 of [11].

5.2.2 Unipotent Characters of \( P(CO_{2n}^-(q))^0 \)

Proposition 7.1 in [70] shows that the projective conformal orthogonal group of type -, \( P(CO_{2n}^-(q))^0 \), has a unique unipotent character of minimal degree \((q^n+1)(q^{n-1}-q)/(q^2-1)\) and any other non-trivial unipotent character has degree greater than \(q^{2n-2}\). We mimic its proof and get the following Proposition, which classifies unipotent characters of degrees up to \(q^{4n-10}\).

**Proposition 5.2.3.** Let \( G^* = (2D_n)_{ad}(q) = P(CO_{2n}^-(q))^0 \). Suppose that \( n \geq 6 \) and \( \chi \in \text{Irr}(G^*) \) is unipotent. Then either \( \chi \) is one of the characters labeled by \((0,n),(1,n-1),(0,1)\) with degrees \(1, (q^n+1)(q^{n-1}-q)/(q^2-1), (q^{2n}-q^2)/(q^2-1)\), respectively, or \( \chi(1) > q^{4n-10} \). Furthermore, when \( n = 5 \), \( G^* \) has one more character of degree \( q^2(q^4+1)(q^5+1)/(q+1) \) (corresponding to the symbol \((2,3)\)), which is less than \(q^{4n-10}\).

**Proof.** From [9, p. 475, 476], we know that the unipotent characters of \( G \) are parametrized by symbols of the form

\[
\left(\begin{array}{c}
\lambda \\
\mu
\end{array}\right) = \left(\begin{array}{c}
\lambda_1 \lambda_2 \lambda_3 \cdots \lambda_a \\
\mu_1 \mu_2 \cdots \mu_b
\end{array}\right),
\]

where \(0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_a, 0 \leq \mu_1 < \mu_2 < \cdots < \mu_b, a > b, a - b \equiv 2(\text{mod} 4), (\lambda_1, \mu_1) \neq (0,0),\) and

\[
\sum_i \lambda_i + \sum_j \mu_j - \left[\left(\frac{a + b - 1}{2}\right)^2\right] = n.
\]

The integer \( n \) is called the rank of the symbol \((\lambda, \mu)\). The degree of the unipotent character \( \chi^{\lambda, \mu} \) corresponding to the symbol \((\lambda, \mu)\) is equal to

\[
\frac{(q^n+1)\prod_{i=1}^{n-1}(q^{2i} - 1)\prod_{i' < i}(q^{\lambda_i} - q^{\lambda_{i'}})\prod_{i' < j}(q^{\lambda_j} - q^{\lambda_{i'}})\prod_{i,j}(q^{\lambda_i} + q^{\mu_j})}{2^{2a+b-2}q^{(a+b-2)^2 + (a+b-4)^2 + \cdots} \prod_{i=1}^{a} \prod_{k=1}^{a} (q^{2k} - 1) \prod_{i,j} \prod_{k=1}^{b} (q^{2k} - 1)}.
\]

Denote \( \mathcal{L}_n = (0,n), (1,n-1), (0,1) \) if \( n \geq 6 \) and \( \mathcal{L}_n = (0,n), (1,n-1), (0,1), (2,n-2) \) if \( n = 5 \). We will prove by induction on \( n \geq 5 \) that if \( \chi = \chi^{\lambda, \mu} \) and \((\lambda, \mu) \notin \mathcal{L}_n\), then
\(\chi(1) > q^{4n-10}\). The induction base \(n = 5\) can be checked easily by using [11, Table 31].

The rest of the proof establishes the induction step for \(n \geq 6\).

1) At this point we suppose that \((\lambda_1, \mu_1) \neq (1, 0)\) and \(\lambda_1 \geq 1\) (eventually \(b\) may be zero). Consider the unipotent character \(\chi\) labeled by the symbol \((\lambda, \mu)\) of rank \(n - 1\), where \(\lambda' = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_a)\) and \(\mu' = \mu\). If \((\lambda', \mu') \in L_{n-1}\) then \((\lambda, \mu) = (2, n-2)\) and therefore \(\chi(1) = (q^{2n-2} - 1)(q^n + 1)(q^{n-2} - q^2)/(q^2 - 1)(q^4 - 1) > q^{4n-10}\). So we can suppose that \((\lambda, \mu)\) is not in \(L_{n-1}\). By induction hypothesis, \(\chi'(1) > q^{4(n-1)-10}\). Observe that \(n = \sum_i \lambda_i + \sum_j \mu_j - \frac{(a+b)^2 - 2(a+b)}{4} \geq a \lambda_1 + \frac{(a-b)^2}{4}\). Since \(a \geq 2\) and \(a - b \geq 2\), \(2 \lambda_1 < n\). Note that \(n \geq 6\). It follows that \(\lambda_1 \leq n - 4\). We have

\[
\frac{\chi(1)}{\chi'(1)} = \prod_{i=2}^{a} \frac{q^{\lambda_i} - q^{\lambda_i - 1}}{q^{2\lambda_i} - 1} \prod_{j=1}^{b} \frac{q^{\lambda_1} + q^{\mu_j}}{q^{\lambda_1} - q^{\lambda_1 - 1} + q^{\mu_j}} > \frac{(q^n + 1)(q^{n-1} - 1)}{2(q^{2\lambda_1} - 1)} \geq \frac{(q^n + 1)(q^{n-1} - 1)}{2(q^{2n-8} - 1)} > q^4,
\]

and therefore \(\chi(1) > q^{4n-10}\).

2) Similarly, supposing \((\lambda_1, \mu_1) \neq (0, 1)\) and \(\mu_1 \geq 1\), consider the unipotent character \(\chi\) labeled by the symbol \((\lambda, \mu)\) of rank \(n - 1\), where \(\lambda' = \lambda\) and \(\mu' = (\mu_1 - 1, \mu_2, \ldots, \lambda_b)\).

If \((\lambda', \mu') \in L_{n-1}\) then \((\lambda, \mu) = (0, 1, n-1)\) and therefore \(\chi(1) = (q^n + 1)(q^{n-1} - 1)(q^{n-1} - q)(q^{n-1} + q^2)/(q^2 - 1)^2 > q^{4n-10}\). So we can suppose that \((\lambda, \mu)\) is not in \(L_{n-1}\). By induction hypothesis, \(\chi'(1) > q^{4(n-1)-10}\). Set

\[
T_1 = \prod_{j=2}^{b} \frac{q^{\mu_j} - q^{\mu_1}}{q^{\mu_j} - q^{\mu_1 - 1}}, T_2 = \prod_{i=1}^{a} \frac{q^{\lambda_i} + q^{\mu_1}}{q^{\lambda_i} + q^{\mu_1 - 1}}.
\]

Then, if \(n - \mu_1 \geq 4\),

\[
\frac{\chi(1)}{\chi'(1)} = \frac{(q^n + 1)(q^{n-1} - 1)}{q^{2\mu_1} - 1} T_1 T_2 > \frac{(q^n + 1)(q^{n-1} - 1)}{2(q^{2\mu_1} - 1)} > q^4.
\]

If \(n - \mu_1 = 3\) then \(b = 1\) since \(b \mu_1 < n\) and \(n \geq 6\). Then \(T_1 = 1\) and we still have

\[
\frac{\chi(1)}{\chi'(1)} > \frac{(q^n + 1)(q^{n-1} - 1)}{q^{\mu_1 - 1}} > q^4.
\]

It follows that \(\chi(1) > q^{4n-10}\) when \(n - \mu_1 \geq 3\). If \(n - \mu_1 \leq 2\), then \((\lambda, \mu) = (0, 1, 2), (0, 1, 3)\). Checking directly, we again have \(\chi^{\lambda, \mu}(1) > q^{4n-10}\).
3) Now we consider the case where $b = 0$ and $\lambda_1 = 0$. If $a = 2$ then we can suppose $\lambda = (k, n - k)$ with $3 \leq k < n - k$ ($k = 2$ is considered already in 1)). Then

$$\chi(1) = \frac{(q^n + 1)(q^{n-k} - q^k) \prod_{i=k+1}^{n-1}(q^{2i} - 1)}{\prod_{i=1}^{n-k}(q^{2i} - 1)}$$

$$> q^{n+k-2}q^{2(k-1)(n-k-1)} \geq q^{n+1}q^{4n-16} > q^{4n-10}.$$  

Now we may assume $a \geq 6$. First we suppose that $\lambda \neq (0, 1, \ldots, a - 1)$. Then there exists $i \geq 2$ such that $\lambda_i - \lambda_{i-1} \geq 2$. Consider the character $\chi'$ corresponding to the symbol $\left(\chi'\right)$, where $\lambda' = (\ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots)$. Note that $n - \lambda_i \geq (a - 2)^2/4 \geq 4$. Therefore, $\chi(1)/\chi'(1) > (q^n + 1)(q^{n-1} - 1)/2(q^{2\lambda_i} - 1) > q^4$. The induction hypothesis $\chi'(1) > q^{4n-14}$ implies that $\chi(1) > q^{4n-10}$.

Next we suppose that $\lambda = (0, 1, \ldots, a - 1)$. Then $\chi(1) = (q^n + 1)f(a)$ and $n = a^2/4$, where

$$f(a) = \frac{(q^2 - 1)(q^4 - 1) \cdots (q^{(a^2-4)/2} - 1) \prod_{0 \leq \ell \leq i \leq a-1}(q^i - q^{i+1})}{2^{a^2}q^{(a^2-4)/2}(q^{a^2} - 1) \prod_{i=0}^{a-1} \prod_{k=1}^{q^i}(q^{2k} - 1)}$$  \hspace{1cm} (2.4)

for any even $a$. It is easy to show that $f(a)/f(a - 2) > q^{a(a-1)(a-3)/2}/2^6 > q^{a^2/4} = q^{3a}$ for $a \geq 6$. Furthermore, $f(4) > 1$. It follows that $\chi(1) > q^{4n-10}$ and we are done.

It remains to consider the case where $b \geq 1$ and $(\lambda_1, \mu_1) = (0, 1)$ or $(1, 0)$.

4) Here we suppose that $(\lambda_1, \mu_1) = (1, 0)$ but $\lambda \neq (1, 2, \ldots, a)$. Then there exists $i \geq 2$ such that $\lambda_i \geq \lambda_{i-1} + 2$. Choose $i$ to be smallest possible. Consider the unipotent character $\chi'$ labeled by $\left(\chi'\right)$ of rank $n - 1$, with $\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_a)$ and $\mu' = \mu$. By the induction hypothesis, $\chi'(1) > q^{4n-14}$. Set

$$U_1 = \prod_{i < 1} q^{\lambda_i} - q^{\lambda_i'}, U_2 = \prod_{i > 1} q^{\lambda_i} - q^{\lambda_i'}, U_3 = \prod_{i} q^{\lambda_i} + q^{\mu_i}.$$ 

If $n - \lambda_i \geq 3$ then

$$\frac{\chi(1)}{\chi'(1)} = \frac{(q^n + 1)(q^{n-1} - 1)}{q^{2\lambda_i} - 1} U_1 U_2 U_3 > \frac{(q^n + 1)(q^{n-1} - 1)}{q^{2\lambda_i} - 1} \cdot \frac{q^{\lambda_i} - q}{q^{\lambda_i} - q} \cdot \frac{1}{2} \cdot \frac{q^{\lambda_i} + 1}{q^{\lambda_i} - 1} >$$

$$> \frac{(q^n + 1)(q^{n-1} - 1)q^2}{2(q^{2\lambda_i} - 1)} \geq \frac{(q^n + 1)(q^{n-1} - 1)q^2}{2(q^{2n-6} - 1)} > q^4.$$
so $\chi(1) > q^{4n-10}$. If $n - \lambda_i \leq 2$ then $i = a$. Note that $n - \lambda_i \geq 1$. As $b \geq 1$, we have $a \geq 3$.

Therefore

$$U_1 \geq \frac{(q^{\lambda_a} - q)(q^{\lambda_a} - q^2)}{(q^{\lambda_a-1} - q)(q^{\lambda_a-1} - q^2)}, \quad U_2 = 1, \quad U_3 \geq \frac{q^{\lambda_a} + 1}{q^{\lambda_a-1} + 1}.$$ 

It follows that

$$\frac{\chi(1)}{\chi'(1)} \geq \frac{q^n + 1}{q^{n-1} + 1} \cdot \frac{(q^{\lambda_a} - q)(q^{\lambda_a} - q^2)}{(q^{\lambda_a-1} - q)(q^{\lambda_a-1} - q^2)} \cdot \frac{q^{\lambda_a} + 1}{q^{\lambda_a-1} + 1} > q^4,$$

so $\chi(1) > q^{4n-10}$ as required.

5) Similarly, suppose that $(\lambda_1, \mu_1) = (0, 1)$ but $\mu \neq (1, 2, \ldots, b)$. Then there exists $j \geq 2$ such that $\mu_j \geq \mu_{j-1} + 2$. Consider the unipotent character $\chi'$ labeled by $(\lambda'_{\mu'})$ of rank $n - 1$, with $\lambda' = \lambda$ and $\mu' = (\mu_1, \mu_{j-1}, \mu_j - 1, \mu_{j+1}, \ldots, \mu_b)$. By the induction hypothesis, $\chi'(1) > q^{4n-14}$. Set

$$V_1 = \prod_{j' < j} \frac{q^{\mu_j} - q^{\mu_{j'}}}{q^{\mu_j} - q^{\mu_{j'}} - 1}, \quad V_2 = \prod_{j' > j} \frac{q^{\mu_j} - q^{\mu_{j'}}}{q^{\mu_j} - q^{\mu_{j'}} - 1}, \quad V_3 = \prod_i \frac{q^{\lambda_i} + q^{\mu_j}}{q^{\mu_j} - q^{\mu_{j'}} - 1}.$$ 

If $n - \mu_j \geq 3$ then

$$\frac{\chi(1)}{\chi'(1)} = \frac{(q^n + 1)(q^{n-1} - 1)}{q^{2\mu_j} - 1} \cdot \frac{(q^n + 1)(q^{n-1} - 1)}{q^{2\mu_j} - 1} \cdot \frac{q^{\mu_j} - q}{q^{\mu_j} - q^{\mu_j - 1}} \cdot \frac{1}{2} \cdot \frac{q^{\mu_j} + 1}{q^{\mu_j - 1} + 1} >$$

$$> \frac{(q^n + 1)(q^{n-1} - 1)q^2}{2(q^{2\mu_j} - 1)} > q^4,$$

so $\chi(1) > q^{4n-10}$ as desired. If $n - \mu_j = 2$ then $j = b$ and therefore $V_2 = 1$. We have $\chi(1) / \chi'(1) > (q^n + 1)(q^{n-1} - 1)q^2 / (q^{2\mu_j} - 1) > q^4$ and we are done. Now suppose that $n - \mu_j \leq 1$. Then $n - \mu_j = 1$, $a - b = 2$, and

$$\left(\begin{array}{c} \lambda \\ \mu \end{array} \right) = \left(\begin{array}{c} 0 \ 1 \ \ldots \ a - 2 \ a - 1 \\ 1 \ 2 \ \ldots \ b - 1 \ n - 1 \end{array} \right).$$

As $b \geq 2$, we have $a \geq 4$. It follows that $V_1 > q$, $V_3 \geq (q^{n-1} + 1) / (q^{n-5} + 1) > q^3$. Hence $\chi(1) / \chi'(1) > q^4$ and we are done again.

6) Here we suppose that $\mu_1 = 0$ and $\lambda = (1, 2, \ldots, a)$. First we consider the case where $\mu \neq (0, 1, \ldots, b - 1)$. Then there exists an index $j \geq 2$ such that $\mu_j \geq \mu_{j-1} + 2$. 

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Consider the unipotent character $\chi'$ labeled by $\binom{\lambda'}{\mu'}$ of rank $n - 1$, with $\lambda' = \lambda$ and $\mu' = (\mu_1, ..., \mu_{j-1}, \mu_j - 1, \mu_{j+1}, ..., \mu_b)$. Since $\binom{\lambda'}{\mu'}$ is not in $L_{n-1}$, by the induction hypothesis, we have $\chi'(1) > q^{4n-14}$. Observe that $n - \mu_j \geq (a - b + 2)^2/4 \geq 4$. So we have
\[
\frac{\chi(1)}{\chi'(1)} = \frac{(q^n + 1)(q^{n-1} - 1)}{q^{2\mu_j} - 1} \cdot \prod_{j' < j} \frac{q^{\mu_j} - q^{\mu_j'}}{q^{\mu_j} - q^{\mu_j'}} \cdot \prod_{j' > j} \frac{q^{\mu_j'} - q^{\mu_j}}{q^{\mu_j'} - q^{\mu_j}} \cdot \prod_i \frac{q^{\lambda_i} + q^{\mu_j}}{q^{\lambda_i} + q^{\mu_j}} >
\]
\[
> \frac{(q^n + 1)(q^{n-1} - 1)}{2(q^{2\mu_j} - 1)} > q^4,
\]
and hence $\chi(1) > q^{4n-10}$.

Next we consider the case where $\mu = (0, 1, ..., b - 1)$. Observe that $n = a + (a - b)^2/4$. If $b = 1$ then $n = (a + 1)^2/4$ and $\chi(1) > (q^n + 1)f(a + 1)$ where $f$ is the function defined in formula (2.4). Since $n \geq 6$, it follows that $a \geq 7$. We have already proved that $f(a + 1) > q^{3n-10}$ for $a \geq 5$. Therefore $\chi(1) > q^{4n-10}$ as required. Now we can assume $b \geq 2$. Consider the unipotent character $\chi'$ labeled by $\binom{\lambda'}{\mu'}$ of rank $n - 1$, with $\lambda' = (1, 2, ..., a - 1)$ and $\mu' = (0, 1, ..., b - 2)$. Since $\binom{\lambda'}{\mu'}$ is not in $L_{n-1}$, again by the induction hypothesis, $\chi'(1) > q^{4n-14}$. Furthermore,
\[
\frac{\chi(1)}{\chi'(1)} = \frac{q^{a+b-2}(q^n + 1)(q^{n-1} - 1)}{(q^a - 1)(q^{b-1} + 1)} > q^{2n-3} > q^4.
\]
Consequently, $\chi(1) > q^{4n-10}$.

7) Finally, we suppose that $\lambda_1 = 0$ and $\mu = (1, 2, ..., b)$. First we consider the case where $\lambda \neq (0, 1, ..., a - 1)$. Then there exists $i \geq 2$ such that $\lambda_i \geq \lambda_{i-1} + 2$. Choose $i$ to be smallest possible. Consider the unipotent character $\chi'$ labeled by $\binom{\lambda'}{\mu'}$ of rank $n - 1$, with $\lambda' = (\lambda_1, ..., \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, ..., \lambda_a)$ and $\mu' = \mu$. If $\binom{\lambda'}{\mu'} \in L_{n-1}$ then $\binom{\lambda}{\mu} = \binom{0 1 2}{1 2 n-1}$ and by checking directly we have $\chi(1) > q^{4n-10}$. Therefore we can suppose that $\binom{\lambda'}{\mu'}$ is not in $L_{n-1}$. In other words, by the induction hypothesis, $\chi'(1) > q^{4n-14}$. Remark that $n - \lambda_i \geq 0$. Set
\[
W_1 = \prod_{\lambda' < \lambda} \frac{q^{\lambda} - q^{\lambda'}}{q^{\lambda} - q^{\lambda'}}, \quad W_2 = \prod_{\lambda' > \lambda} \frac{q^{\lambda'} - q^{\lambda}}{q^{\lambda'} - q^{\lambda}}., \quad W_3 = \prod_{j} \frac{q^{\lambda_j} + q^{\mu_j}}{q^{\lambda_j} + q^{\mu_j}}.
\]
Note that $W_1 \geq (q^{\lambda_i} - 1)/(q^{\lambda_i - 1} - 1) > q$, $W_2 > 1/2$ and $W_3 > 1$. Therefore, if $n - \lambda_i \geq 3$,

$$\frac{\chi(1)}{\chi'(1)} = \frac{(q^n + 1)(q^{n-1} - 1)}{q^{2\lambda_i} - 1} W_1 W_2 W_3 > \frac{(q^n + 1)(q^{n-1} - 1)q}{2(q^{2\lambda_i} - 1)} > q^4,$$

so $\chi(1) > q^{4n-10}$. If $1 \leq n - \lambda_i \leq 2$ then either $i = a$ or

$$\left( \begin{array}{c} \lambda \\ \mu \end{array} \right) = \left( \begin{array}{cccc} 0 & \ldots & n-4 & n-2 & n-1 \\ 1 & \ldots & n-3 \end{array} \right).$$

In the former case, $W_2 = 1$ and we have

$$\frac{\chi(1)}{\chi'(1)} > \frac{(q^n + 1)(q^{n-1} - 1)}{q^{2\lambda_i} - 1} \cdot \frac{(q^{\lambda_i} - 1)(q^{\lambda_i} - q)}{(q^{\lambda_i - 1} - 1)(q^{\lambda_i - 1} - q)} \cdot \frac{q^{\lambda_i} + q}{q^{\lambda_i} - q} >$$

$$> \frac{(q^n + 1)(q^{n-1} - 1)}{(q^{n-1} + 1)(q^{n-2} - 1)} \cdot \frac{q^{2\lambda_i} - q^2}{q^{2\lambda_i - 2} - q^2} > q^4,$$

and therefore we are done. In the latter case, since $n \geq 6$, $W_1 \geq q^4$ and we are done also.

If $n = \lambda_i$, then $i = a$, $a - b = 2$ and $\lambda = (0, 1, \ldots, a - 2, n)$. Since $\left( \begin{array}{c} \lambda \\ \mu \end{array} \right)$ is not in $\mathcal{L}_n$, $a \geq 4$.

Then we have

$$\frac{\chi(1)}{\chi'(1)} = \prod_{i=1}^{a-2} \frac{q^n - q^i}{q^{n-1} - q^i} \cdot \prod_{j=1}^{a-2} \frac{q^n + q^j}{q^{n-1} + q^j} > q^{2(a-2)} \geq q^4,$$

so $\chi(1) > q^{4n-10}$.

The last configuration we have to handle is that $\lambda = (0, 1, \ldots, a - 1)$ and $\mu = (1, 2, \ldots, b)$. Consider the unipotent character $\chi'$ labeled by $\left( \begin{array}{c} \lambda' \\ \mu' \end{array} \right)$ of rank $n - 1$, with $\lambda' = (0, 1, \ldots, a - 2)$ and $\mu' = (1, 2, \ldots, b - 1)$ (if $b = 1$, $\mu'$ is just empty). By the induction hypothesis, $\chi'(1) > q^{4n-14}$. Furthermore,

$$\frac{\chi(1)}{\chi'(1)} = \frac{q^{n+b-2}(q^n + 1)(q^{n-1} - 1)}{(q^n - 1)(q^{b-1} + 1)} > q^{2n-3} > q^4,$$

Consequently, $\chi(1) > q^{4n-10}$.

\[\square\]

### 5.2.3 Unipotent Characters of $P(CO_{2n}^+(q)^0)$

Proposition 7.2 in [70] shows that the projective conformal orthogonal group of type $+$, $P(CO_{2n}^+(q)^0)$, has a unique unipotent character of minimal degree $(q^n - 1)(q^{n-1} + q)/(q^2 - 1)$ and any other non-trivial unipotent character has degree greater than $q^{2n-2}$. 

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We mimic its proof and get the following Proposition, which classifies unipotent characters of degrees up to $q^{4n-10}$.

**Proposition 5.2.4.** Let $G^* = (D_n)_{ad}(q) = P(CO_{2n}^+(q^0))$. Suppose that $n \geq 5$, $(n, q) \neq (5, 2)$, and $\chi \in \text{Irr}(G^*)$ is unipotent. Then either $\chi$ is one of characters labeled by $\binom{\lambda}{\mu}$, $\binom{(n-1)}{1}$, $\binom{0 1}{1 0}$ with degrees 1, $(q^n - 1)(q^{n-1} + q)/(q^2 - 1)$, $(q^{2n} - q^2)/(q^2 - 1)$, respectively, or $\chi(1) > q^{4n-10}$. Furthermore, when $(n, q) = (5, 2)$, $G^*$ has one more character of degree 868 (corresponding to the symbol $\binom{0 1 2 4}{1 0}$), which is less than $q^{4n-10}$.

**Proof.** From [9, p. 471, 472], we know that the unipotent characters of $G$ are parametrized by symbols of the form

$$\binom{\lambda}{\mu} = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_a, \mu_1, \mu_2, \ldots, \mu_b),$$

where $0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_a$, $0 \leq \mu_1 < \mu_2 < \ldots < \mu_b$, $a - b \equiv 0(\text{mod } 4)$, $(\lambda_1, \mu_1) \neq (0, 0)$, and

$$\sum_i \lambda_i + \sum_j \mu_j - \left(\frac{a + b - 1}{2}\right)^2 = n.$$ 

The integer $n$ is called the rank of the symbol $\binom{\lambda}{\mu}$. The degree of the unipotent character $\chi^{\lambda, \mu}$ corresponding to the symbol $\binom{\lambda}{\mu}$ is equal to

$$\frac{(q^n - 1) \prod_{i=1}^{n-1} (q^{2i} - 1) \prod_{\lambda' \prec \lambda} (q^{\lambda_i} - q^{\lambda'_i}) \prod_{\lambda' \prec \lambda} (q^{\lambda_j} - q^{\lambda'_j}) \prod_{\lambda_i \prec \lambda_j} (q^{\lambda_i} + q^{\mu_j})}{2c q^{(a+b-2)+(a+b-4)+\ldots+(n-1)} \prod_{i=1}^{n} q^{\lambda_i} \prod_{i=1}^{\mu_j} q^{\mu_j} (q^{2k} - 1)}.$$ 

Here $c = (a + b - 2)/2$ if $\lambda \neq \mu$, and $c = a$ if $\lambda = \mu$. In the former case, $\binom{\lambda}{\mu}$ and $\binom{\mu}{\lambda}$ determine the same unipotent character. In the latter case, there are two unipotent characters with the same symbols.

When $n = 5$, Proposition can be verified directly by using [11, Table 30]. Denote $\mathcal{L}_n = \{(n, 0), (n - 1, 1), (0, 1), (1, n)\}$. We will prove by induction on $n \geq 5$ that if $\chi = \chi^{\lambda, \mu}$ and $\{\lambda, \mu\} \notin \mathcal{L}_n$, then $\chi(1) > q^{4n-10}$, provided $(n, q) \neq (5, 2)$. The rest of the proof establishes the induction step for $n \geq 6$.

1) At this point we suppose that $(\lambda_1, \mu_1) \neq (1, 0)$ and $\lambda_1 \geq 1$ (eventually $b$ may be zero). Consider the unipotent character $\chi'$ labeled by the symbol $\binom{\lambda'}{\mu'}$ of rank $n - 1$,
where $\lambda' = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_a)$ and $\mu' = \mu$. If $\{\lambda, \mu\} \in \mathcal{L}_{n-1}$ then \((\lambda, \mu) = \binom{2}{n-2}\) or \(\binom{2n-1}{2}\), and therefore $\chi(1) > q^{4n-10}$. So we can suppose that \((\lambda', \mu')\) is not in $\mathcal{L}_{n-1}$. By induction hypothesis, $\chi'(1) > q^{4(n-1)-10}$. The condition $n = \sum_i \lambda_i + \sum_j \mu_j - \frac{(a+b)^2 - 2(a+b)}{4} \geq a\lambda_1 + \frac{(a-b)^2}{4}$ implies that $n - \lambda_1 \geq 0$. If $\lambda_1 \leq n - 4$ then

$$\frac{\chi(1)}{\chi'(1)} \geq \frac{(q^n - 1)(q^{n-1} + 1)}{2(q^{2\lambda_1} - 1)} \cdot \prod_{i=2}^a \frac{q^{\lambda_i} - q^{\lambda_1}}{q^{\lambda_i} - q^{\lambda_1-1}} \cdot \prod_{j=1}^b \frac{q^{\lambda_1} + q^{\mu_j}}{q^{\lambda_1-1} + q^{\mu_j}} > \frac{(q^n - 1)(q^{n-1} + 1)}{4(q^{2\lambda_1} - 1)} > q^4,$$

so $\chi(1) > q^{4n-10}$. If $n - \lambda_1 \leq 3$ then $a = 1$ and $(a-b)^2/4 \leq 3$. Since $a - b \equiv 0 \pmod{4}$, $a = b = 1$. Note that $\{\lambda, \mu\}$ is not in $\mathcal{L}_n$. So \((\lambda', \mu') = \binom{n-2}{2}\) or \(\binom{n-3}{3}\). Checking directly, we have $\chi(1) > q^{4n-10}$ as required.

2) Now we consider the case where $b = 0$ and $\lambda_1 = 0$. Then $a \equiv 0 \pmod{4}$. First we suppose that $\lambda \neq (0, 1, \ldots, a - 1)$. Then there exists $i \geq 2$ such that $\lambda_i - \lambda_{i-1} \geq 2$. We choose $i$ to be smallest possible. Consider the character $\chi'$ corresponding to the symbol \((\lambda')\), where $\lambda' = (\ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots)$. Note that $n - \lambda_i \geq (a - 2)^2/4$. Therefore, $\chi(1)/\chi'(1) > (q^n + 1)(q^{n-1} - 1)/2(q^{2\lambda_i} - 1) > q^4$ if $a \geq 8$. It is obvious that \((\lambda') \notin \mathcal{L}_{n-1}\). If $(n, q) = (6, 2)$ and $\lambda' = (0, 1, 2, 5)$ or $(0, 1, 3, 4)$, we can check directly that $\chi(1) > q^{4n-10}$. Otherwise, the induction hypothesis $\chi'(1) > q^{4n-14}$ implies that $\chi(1) > q^{4n-10}$. If $a = 4$ and $n - \lambda_i \geq 3$, then again $\chi(1)/\chi'(1) > q^4$, so $\chi(1) > q^{4n-10}$. If $a = 4$ and $n - \lambda_i \leq 2$, then $i = a$ and $\lambda = (0, 1, 2, n - 1)$. Direct computation shows that $\chi(1) > q^{4n-10}$ and we are done. Next we suppose that $\lambda = (0, 1, \ldots, a - 1)$. Then $\chi(1) = (q^n - 1)f(a)$ and $n = a^2/4$, where $f(a)$ is the function defined in (2.4). Note that $n \geq 6$, so $a \geq 8$. We already showed that $f(a) > q^{3n}$ for $a \geq 6$. Hence, $\chi(1) > q^{4n-10}$ as desired.

Again, the case $a = 0$ and $\mu_1 = 0$ can be reduced to 2) by interchanging $\lambda$ and $\mu$. Therefore, it remains to consider the case where $a, b \geq 1$ and $(\lambda_1, \mu_1) = (0, 1)$ or $(1, 0)$. By a similar reason as above, it is enough to suppose that $(\lambda_1, \mu_1) = (1, 0)$.
3) Here we suppose that $\lambda \neq (1, 2, \ldots, a)$. Then there exists $i \geq 2$ such that $\lambda_i \geq \lambda_{i-1} + 2$. Choose $i$ to be smallest possible. Consider $\chi'$ labeled by $^{(\lambda)}_{\mu}$ of rank $n - 1$, with $\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_a)$ and $\mu' = \mu$. By induction hypothesis, $\chi'(1) > q^{4n-14}$.

Set

$$U_1 = \prod_{i' < i} \frac{q^{\lambda_i} - q^{\lambda_{i'}}}{q^{\lambda_{i+1}} - q^{\lambda_{i'}}}, U_2 = \prod_{i' > i} \frac{q^{\lambda_{i'}} - q^{\lambda_i}}{q^{\lambda_{i+1}} - q^{\lambda_i}}, U_3 = \prod_j \frac{q^{\lambda_1 + q^{\mu_j}}}{q^{\lambda_{i+1}} + q^{\mu_j}}.$$ 

If $n - \lambda_i \geq 2$ then

$$\frac{\chi(1)}{\chi'(1)} = \frac{(q^n - 1)(q^{n-1} + 1)}{q^{2\lambda_i} - 1} U_1 U_2 U_3 > \frac{(q^n - 1)(q^{n-1} + 1)}{q^{2\lambda_i} - 1} \cdot \frac{q^{\lambda_i} - q}{q^{\lambda_i} - q} \cdot \frac{1}{2} \cdot \frac{q^{\lambda_i} + 1}{q^{\lambda_i} - q} > \frac{(q^n - 1)(q^{n-1} + 1)q}{2(q^{2n} - 1)} > q^4,$$

so $\chi(1) > q^{4n-10}$. If $n - \lambda_i \leq 1$ then $i = a = b$. Note that $n - \lambda_i \geq 0$. If $n - \lambda_i = 1$ then

$$\left( \begin{array}{c} \lambda \\ \mu \end{array} \right) = \left( \begin{array}{c} 1 & 2 & \ldots & a - 1 & n - 1 \\ 0 & 1 & \ldots & a - 2 & a \end{array} \right).$$

The case $a = 2$ can be checked directly. So we can suppose $a \geq 3$. Then

$$U_1 \geq \frac{(q^{n-1} - q)(q^{n-1} - q^2)}{(q^{n-2} - q)(q^{n-2} - q^2)}, U_2 = 1, U_3 \geq \frac{q^{n-1} + 1}{q^{n-2} + 1}.$$

It follows that

$$\frac{\chi(1)}{\chi'(1)} \geq \frac{q^n - 1}{q^{n-1} - 1} \cdot \frac{(q^{n-1} - q)(q^{n-1} - q^2)}{(q^{n-2} - q)(q^{n-2} - q^2)} \cdot \frac{q^{n-1} + 1}{q^{n-2} + 1} > q^4,$$

so $\chi(1) > q^{4n-10}$ as required. If $n - \lambda_i = 0$ then

$$\left( \begin{array}{c} \lambda \\ \mu \end{array} \right) = \left( \begin{array}{c} 1 & 2 & \ldots & a - 1 & n \\ 0 & 1 & \ldots & a - 2 & a - 1 \end{array} \right).$$

Since $\{\lambda, \mu\}$ is not in $\mathcal{L}_n$, $a \geq 3$. Then

$$\frac{\chi(1)}{\chi'(1)} \geq \frac{(q^n - 1)(q^{n-1} + 1)}{q^{2n} - 1} \cdot \frac{(q^n - q)(q^n - q^2)}{(q^{n-1} - q)(q^{n-1} - q^2)} \cdot \frac{(q^n + 1)(q^n + q)(q^n + q^2)}{(q^{n-1} + 1)(q^{n-1} + q)(q^{n-1} + q^2)}$$

$$= \frac{(q^n - q)(q^n - q^2)}{(q^{n-1} - q)(q^{n-1} - q^2)} \cdot \frac{(q^n + q)(q^n + q^2)}{(q^{n-1} + q)(q^{n-1} + q^2)} > q^4,$$

and therefore $\chi(1) > q^{4n-10}$. 
4) Finally, we suppose that \( \mu_1 = 0 \) and \( \lambda = (1, 2, \ldots, a) \). First we consider the case where \( \mu \neq (0, 1, \ldots, b - 1) \). Then there exists an index \( j \geq 2 \) such that \( \mu_j \geq \mu_{j-1} + 2 \).

Consider the unipotent character \( \chi' \) labeled by \( \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} \) of rank \( n - 1 \), with \( \lambda' = \lambda \) and \( \mu' = (\mu_1, \ldots, \mu_{j-1}, \mu_j - 1, \mu_{j+1}, \ldots, \mu_b) \). If \( \{\lambda', \mu'\} \in \mathcal{L}_{n-1} \) then \( \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} = \begin{pmatrix} 0 & n-1 \\ 1 & 0 \end{pmatrix} \) and hence \( \chi(1) > q^{4n-10} \) by direct computation. Therefore we can assume \( \{\lambda', \mu'\} \) is not in \( \mathcal{L}_{n-1} \). By the induction hypothesis, \( \chi'(1) > q^{4n-14} \).

Set \[
V_1 = \prod_{j' < j} \frac{q^{\mu_{j'}} - q^{\mu_{j'}}}{q^{\mu_{j'}} - q^{\lambda'}}, \quad V_2 = \prod_{j' \geq j} \frac{q^{\mu_{j'}} - q^{\mu_{j'}}}{q^{\mu_{j'}} - q^{\lambda'}}, \quad V_3 = \prod_i \frac{q^{\lambda_i} + q^{\mu_j}}{q^{\lambda_i} + q^{\mu_j}}.
\]

Remark that \( V_1 > q, V_2 > 1/2, V_3 > 1 \) and \( n - \mu_j \geq 1 \). If \( n - \mu_j \geq 3 \), we have

\[
\frac{\chi(1)}{\chi'(1)} = \frac{(q^n - 1)(q^{n-1} + 1)}{q^{2\mu_j} - 1} V_1 V_2 V_3 > \frac{(q^n - 1)(q^{n-1} + 1)q}{2(q^{2\mu_j} - 1)} > q^4,
\]

so \( \chi(1) > q^{4n-10} \). If \( n - \mu_j = 2 \) then \( j = b \) and \( V_2 = 1 \). We still have \( \chi(1)/\chi'(1) > (q^n - 1)(q^{n-1} + 1)q/(q^{2n-4} - 1) > q^4 \) and we are done. The last case we need to handle is \( n - \mu_j = 1 \). Note that \( j = b \) and \( V_2 = 1 \) in this case. If \( b \geq 3 \) then

\[
V_1 \geq \frac{(q^{n-1} - 1)(q^{n-1} - q)}{(q^{n-2} - 1)(q^{n-2} - q)} > \frac{q(q^{n-1} - q)}{(q^{n-2} - q)}, \quad V_3 \geq \frac{q^{n-1} + q}{q^{n-2} + q}.
\]

It follows that

\[
\frac{\chi(1)}{\chi'(1)} > \frac{(q^n - 1)(q^{n-1} + 1)}{q^{2n - 1}}, \quad \frac{q(q^{n-1} - q)}{(q^{n-2} - q)}, \quad \frac{q^{n-1} + q}{q^{n-2} + q} > q^4,
\]

so \( \chi(1) > q^{4n-10} \). If \( b = 2 \) then \( a = b = 2 \) and \( \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & n-1 \end{pmatrix} \), hence \( \chi(1) > q^{4n-10} \).

Next we consider the case where \( \mu = (0, 1, \ldots, b - 1) \). Consider the unipotent character \( \chi' \) labeled by \( \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} \) of rank \( n - 1 \), with \( \lambda' = (1, 2, \ldots, a - 1) \) and \( \mu' = (0, 1, \ldots, b - 2) \) (if \( a = 1 \), \( \lambda' \) is just empty; if \( b = 1 \), \( \mu' \) is just empty). It is obvious that \( \{\lambda', \mu'\} \) is not in \( \mathcal{L}_{n-1} \) and therefore \( \chi'(1) > q^{4n-14} \) by the induction hypothesis. Furthermore,

\[
\frac{\chi(1)}{\chi'(1)} = \frac{q^{a+b-2}(q^n - 1)(q^{n-1} + 1)}{(q^n - 1)(q^{b-1} + 1)} > q^{2n-3} > q^4.
\]

Consequently, \( \chi(1) > q^{4n-10} \).\]
5.3 Non-unipotent Characters

The low-dimensional unipotent characters of $G = Sp_{2n}(q)$, $q$ odd, are already classified in Proposition 5.2.1 up to degree $(q^{2n-2} - 1)(q^{2n} - 1)\frac{(q^{n-2} - q^3)}{2}
(q^2 - 1)(q^4 - 1)(q^3 + 1)$, which is greater than $q^{4n-10}(q^n - 1)/2$ when $n \geq 6$. Therefore, Theorem F will be proved if we can classify non-unipotent characters of $G$ of degrees up to $q^{4n-10}(q^n - 1)/2$. We know that the dual group of $G$ is $G^* = SO_{2n+1}(q)$ and every non-unipotent character $\chi \in \text{Irr}(G)$ is parametrized by a pair $((s), \psi)$ where $(s)$ is a nontrivial geometric conjugacy class of a semi-simple element $s \in G^*$ and $\psi$ is an irreducible unipotent character of the centralizer $C := C_{G^*}(s)$. Moreover, $\chi(1) = (G^*:C)\psi'(1)$.

**Notation:** In order to label the eigenvalues of semi-simple elements, as well as conjugacy classes of finite classical groups, we follow the notation of [66] and [75]. Let $\kappa$ denote a generator of the field of $q^4$ elements, $\zeta = \kappa^{q^2 - 1}$, $\theta = \kappa^{q^2 + 1}$, $\eta = \theta^{q - 1}$, $\gamma = \theta^{q + 1}$.

Let $T_1 = \{1, \ldots, (q - 3)/2\}$, $T_2 = \{1, \ldots, (q - 1)/2\}$ if $q$ is odd and $T_1 = \{1, \ldots, (q - 2)/2\}$, $T_2 = \{1, \ldots, q/2\}$ if $q$ is even. Furthermore, when $q$ is odd, let $R_1^* = \{j \in \mathbb{Z} : 1 \leq j < q^2 + 1, j \neq (q^2 + 1)/2\}$ and define $R_1$ to be a complete set of class representatives of the equivalence relation $\sim$ on $R_1^*$:

$$i \sim j \iff i \equiv \pm j \text{ or } \pm qj \pmod{q^2 + 1}.$$ 

Similarly, let $R_2^* = \{j \in \mathbb{Z} : 1 \leq j \leq q^2 - 1, q - 1 \nmid j, q + 1 \nmid j\}$ and define $R_2$ to be a complete set of class representatives of the equivalence relation $\sim$ on $R_2^*$:

$$i \sim j \iff i \equiv \pm j \text{ or } \pm qj \pmod{q^2 - 1}.$$ 

Note that $|R_1| = (q^2 - 1)/4$ and $|R_2| = (q - 1)^2/4$.

**Proposition 5.3.1.** Let $G = Sp_{2n}(q)$ where $n \geq 6$ and $q$ is an odd prime power. Suppose that $\alpha, \alpha_1, \alpha_2 = \pm$ and $\chi \in \text{Irr}(G)$ is not unipotent. Then one of the following holds:

1) $\chi(1) > q^{4n-10}(q^n - 1)/2$. 

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2) \(\chi(1) = (q^{2n-2} - 1)(q^{2n} - 1)/|A|\), where \(A = GL_2^n(q)\), \(\mathbb{Z}_{q^2 - \alpha}\), or \(\mathbb{Z}_{q - \alpha_1} \times \mathbb{Z}_{q - \alpha_2}\), and \(\chi\) is parametrized by \((s), 1_C\), where \(C \simeq SO_{2n-3}(q) \times A\) or \(\chi(1) = q(q^{2n-2} - 1)(q^{2n} - 1)/(q - \alpha)(q^2 - 1)\), and \(\chi\) is parametrized by \((s), 1_{SO_{2n-3}(q)} \otimes \lambda\), where \(C \simeq SO_{2n-3}(q) \times GL_2^n(q)\) and \(\lambda\) is the unique nontrivial unipotent character of degree \(q\) of \(GL_2^n(q)\).

3) \(\chi(1) = (q^{n-1} + \alpha_1)(q^{2n} - 1)/2(q - \alpha_2)\), and \(\chi\) is parametrized by \((s), \lambda \otimes 1_{\mathbb{Z}_{q - \alpha_2}}\), where \(C \simeq GO_{2n-2}^\alpha(q) \times \mathbb{Z}_{q - \alpha_2}\) and \(\lambda\) is one of two extensions of \(1_{SO_{2n-2}^\alpha(q)}\) to \(GO_{2n-2}^\alpha(q)\).

4) \(\chi(1) = (q^{2n} - 1)\omega(1)/(q - \alpha)\), and \(\chi\) is parametrized by \((s), \omega \otimes 1_{\mathbb{Z}_{q - \alpha}}\), where \(C \simeq SO_{2n-1}(q) \times \mathbb{Z}_{q - \alpha}\) and \(\omega\) is one of unipotent characters of \(SO_{2n-1}(q)\) of degrees 1, \((q^{n-1} - 1)(q^{n-1} - q)/2(q + 1)\), \((q^{n-1} + 1)(q^{n-1} + q)/2(q + 1)\), \((q^{n-1} + 1)(q^{n-1} - q)/2(q - 1)\), or \((q^{n-1} - 1)(q^{n-1} + q)/2(q - 1)\).

5) \(\chi(1) = (q^{2n-2} - 1)(q^{2n} - 1)/2(q - \alpha_1)(q - \alpha_2)\), and \(\chi\) is parametrized by \((s), \lambda \otimes 1_{SO_{2n-3}(q) \times \mathbb{Z}_{q - \alpha_2}}\), where \(C \simeq GO_{2n}^\alpha(q) \times SO_{2n-3}(q) \times \mathbb{Z}_{q - \alpha_2}\) and \(\lambda\) is one of two extensions of \(1_{SO_{2n}^\alpha(q)}\) to \(GO_{2n}^\alpha(q)\).

6) \(\chi(1) = (q^{n} - 1)\omega(1)/2(q - \alpha)\), and \(\chi\) is parametrized by \((s), \omega \otimes \lambda\), where \(C \simeq SO_{2n-1}(q) \times GO_{2n}^\alpha(q)\), \(\omega\) is one of unipotent characters of \(SO_{2n-1}(q)\) of degrees 1, \((q^{n-1} - 1)(q^{n-1} - q)/2(q + 1)\), \((q^{n-1} + 1)(q^{n-1} + q)/2(q + 1)\), \((q^{n-1} + 1)(q^{n-1} - q)/2(q - 1)\), or \((q^{n-1} - 1)(q^{n-1} + q)/2(q - 1)\), and \(\lambda\) is one of two extensions of \(1_{SO_{2n}^\alpha(q)}\) to \(GO_{2n}^\alpha(q)\).

7) \(\chi(1) = (q^n + 1)\lambda(1)/2\), and \(\chi\) is parametrized by \((s), \lambda\), where \(C \simeq GO_{2n}^\pm(q)\) and \(\lambda\) is an extension of one of three unipotent characters of \(SO_{2n}^\pm(q)\) of degrees 1, \((q^n - 1)(q^{n-1} + q)/(q^2 - 1)\) or \((q^n - 2)(q^2 - 1)\) to \(GO_{2n}^\pm(q)\).

8) \(\chi(1) = (q^n - 1)\lambda(1)/2\), and \(\chi\) is parametrized by \((s), \lambda\), where \(C \simeq GO_{2n}^{-}(q)\) and \(\lambda\) is an extension of one of three unipotent characters of \(SO_{2n}^{-}(q)\) of degrees 1, \((q^n + 1)(q^{n-1} - q)/(q^2 - 1)\) or \((q^n - 2)(q^2 - 1)\) to \(GO_{2n}^{-}(q)\).

9) \(\chi(1) = (q^{n-1} + \alpha)(q^{2n} - 1)\omega(1)/2(q^2 - 1)\), and \(\chi\) is parametrized by \((s), \omega \otimes \lambda\), where \(C \simeq SO_3(q) \times GO_{2n-2}^{\alpha}(q)\), \(\omega = 1_{SO_3(q)}\) or the unique nontrivial unipotent character of degree \(q\) of \(SO_3(q)\), and \(\lambda\) is one of two extensions of \(1_{SO_{2n-2}^{\alpha}(q)}\) to \(GO_{2n-2}^{\alpha}(q)\).
10) \( \chi(1) = (q^{2n-2} - 1)(q^{2n} - 1)\lambda(1)/2(q^2 - 1)(q^2 - \alpha) \), and \( \chi \) is parametrized by 

\((s, 1_{SO_{2n-3}}(q) \otimes \lambda)\), where \( C \simeq SO_{2n-3}(q) \times GO^\alpha_4(q) \) and \( \lambda \) is one of two extensions of \( 1_{SO^\alpha_4(q)} \) or the unique unipotent character of degree \( q^2 \) of \( SO^\alpha_4(q) \) to \( GO^\alpha_4(q) \) or \( \lambda \) is the unique unipotent character of degree \( 2q \) of \( GO^\alpha_4(q) \).

**Proof.** For short notation, we set \( D(n) := (q^n - 1)q^{4n-10}/2 \). By Lemma 5.1.2, \( C \simeq A \times B \) where \( A \simeq \prod_{i=1}^t GL^\alpha_i(q^{k_i}), a_i, k_i \in \mathbb{N}, \alpha_i = \pm 1, \sum_{i=1}^t k_i a_i = n - m \) and \( B = SO_{2k+1}(q) \times GO^\pm_{2m-2k}(q), 0 \leq k \leq m \leq n \). It is easy to show that \( |B|_{\psi} \leq \prod_{i=1}^m (q^{2i} - 1) \) and \( |A|_{\psi} \leq \prod_{i=1}^{n-m} (q^n - (-1)^i) \). Therefore,

\[
\frac{|G|_{\psi}}{|C_{G^*}(s)|_{\psi}} \geq \frac{(q^{2m+1} - 1) \cdots (q^{2n} - 1)}{(q + 1)(q^2 - 1) \cdots (q^{n-m} - (-1)^{n-m})} := f(m, n).
\]

1) Case \( m \leq n - 3 \). Since the function \( f(x, n) \) is first increasing and then decreasing in the interval \([0, n]\), we have \( f(m, n) \geq \min\{f(0, n), f(n - 3, n)\} \). It is easy to check that both \( f(0, n) \) and \( f(n - 3, n) \) are greater than \( D(n) \) and therefore 1) holds.

2) Case \( m = n - 2 \). First we suppose that \( k = n - 2 \). Then \( \chi(1) = \frac{(q^{2n-2} - 1)(q^{2n-1})}{|A|_{\psi}} \psi(1) \), where \( A = GL^\alpha_2(q), GL^\alpha_1(q^2) \) or \( GL^\alpha_1(q) \times GL^\alpha_2(q) \), \( \alpha, \alpha_1, \alpha_2 := \pm 1 \) and \( \psi \) is a unipotent character of \( C = SO_{2n-3}(q) \times A \). Suppose that \( \psi = \omega \otimes \lambda \) where \( \omega, \lambda \) are unipotent characters of \( SO_{2n-3}(q), A \), respectively. If \( \omega \) is nontrivial then \( \omega(1) \geq \frac{(q^{n-2} - 1)(q^{n-2} - q)}{2(q + 1)} \) by [70, Proposition 5.1]. Then

\[
\chi(1) \geq \frac{(q^{2n-2} - 1)(q^{2n} - 1)(q^{2n-2} - 1)(q^{2n} - q)}{2(q + 1)^2(q^2 - 1)} > D(n).
\]

If \( \omega \) is trivial then 2) holds since \( GL^\alpha_1(q^2) \) or \( GL^\alpha_1(q) \times GL^\alpha_2(q) \) has only one unipotent character, which is trivial, while \( A = GL^\alpha_2(q) \) has two unipotent characters, the trivial one and the one of degree \( q \).

Next, we consider \( k = 0 \) then

\[
\chi(1) \geq \frac{(q^{n-2} - 1)(q^{2n-2} - 1)(q^{2n} - 1)}{2(q + 1)(q^2 - 1)} > D(n).
\]
Finally, if $1 \leq k \leq n - 3$, we have
\[
\chi(1) \geq \frac{(q^{2n-2} - 1)(q^{2n} - 1)}{(q + 1)(q^2 - 1)} \cdot \frac{\prod_{i=k+1}^{n-2} (q^{2i} - 1)}{2 \prod_{i=1}^{n-2-k} (q^{2i} - 1)} \frac{1}{2^k q^{2(k-n-2-k)} (q^{n-2-k} - 1)} \cdot \frac{1}{2} q^{2(n-3)} > D(n).
\]

3) Case $m = n - 1$. Assume $k = 0$. Then \( \chi(1) = \frac{(q^{n-1+\alpha_1})(q^{n-1})}{2(q - \alpha_2)} \psi(1) \), where \( \psi \) is a unipotent character of \( C \simeq \text{GO}^{\alpha_1}_{2n-2}(q) \times \text{GL}^{\alpha_2}(q) = (\text{SO}^{\alpha_1}_{2n-2}(q) \cdot 2) \times \text{GL}^{\alpha_2}(q) \), \( \alpha_1, \alpha_2 = \pm 1 \). Note that \( \text{GL}^{\alpha_2}(q) \simeq \mathbb{Z}_{q-\alpha_2} \) has only one unipotent character, which is the trivial one. Suppose that \( \psi \) is an irreducible constituent of \( (\omega)^{\text{SO}^{\alpha_1}_{2n-2}(q)} \cdot 2 \otimes 1_{\text{GL}^{\alpha_2}(q)} \) where \( \omega \) is a unipotent character of \( \text{SO}^{\alpha_1}_{2n-2}(q) \). By Proposition 5.1.1, the unipotent characters of \( \text{SO}^{\pm}_{2n-2}(q) \) are of the form \( \theta \circ f \), where \( \theta \) runs over the unipotent characters of \( P(\text{CO}^{\pm}_{2n-2}(q)^0) \) and \( f \) is the canonical homomorphism \( f : \text{SO}^{\pm}_{2n-2} \to P(\text{CO}^{\pm}_{2n-2}(q)^0) \).

In particular, degrees of the unipotent characters of \( \text{SO}^{\pm}_{2n-2}(q) \) are the same as those of \( P(\text{CO}^{\pm}_{2n-2}(q)^0) \). Therefore, by Propositions 7.1, 7.2 of [70], if \( \omega \) is nontrivial then \( \omega(1) \geq (q^{n-1} + 1)(q^{n-2} - q)/(q^2 - 1) \). Then
\[
\chi(1) \geq \frac{(q^{n-1} + \alpha_1)(q^{2n} - 1)}{2(q - \alpha_2)} \cdot \frac{(q^{n-1} + 1)(q^{n-2} - q)}{q^2 - 1} > D(n).
\]

Suppose that \( \omega \) is trivial. Since \( (1_{SO^{\alpha_1}_{2n-2}(q)} \cdot 2_{\text{GO}^{\alpha_1}_{2n-2}(q)}) \) is the sum of two different irreducible characters of degree 1, \( \text{GO}^{\alpha_1}_{2n-2}(q) \) has two unipotent characters of degree 1. So, in this case, \( \psi = \lambda \otimes 1_{\text{GL}^{\alpha_2}(q)} \) where \( \lambda \) is one of two extensions of \( 1_{SO^{\alpha_1}_{2n-2}(q)} \) to \( \text{GO}^{\alpha_1}_{2n-2}(q) \) and therefore 3) holds.

Assume \( k = n - 1 \). Then \( \chi(1) = \frac{q^{2n-1}}{q - \alpha} \psi(1), \alpha = \pm 1 \), where \( \psi \) is a unipotent character of \( \text{SO}_{2n-1}(q) \times \text{GL}^{\alpha}(q) \). Again, we have \( \psi = \omega \otimes 1_{\text{GL}^{\alpha}(q)} \) where \( \omega \) is a unipotent character of \( \text{SO}_{2n-1}(q) \). By Corollary 5.2.2, \( \omega \) is either one of characters of degrees 1, \( (q^{n-1} - 1)(q^{n-1} - q)/2(q + 1), (q^{n-1} - 1)(q^{n-1} + q)/2(q + 1), (q^{n-1} + 1)(q^{n-1} - q)/2(q - 1), \), (q^{n-1} - 1)(q^{n-1} + q)/2(q - 1), \)
\((q^{n-1} - 1)(q^{n-1} + q)/2(q-1)\) or \(\omega(1) > q^{4n-12}\). In the former case, 4) holds. In the latter case, \(\chi(1) \geq \frac{2^{n-1}}{q-\alpha} q^{4n-12} > D(n)\).

Assume \(k = 1\). Then

\[
\chi(1) \geq (G^\ast : C)_{\psi'} \geq \frac{(q^{n-2} - 1)(q^{2n-2} - 1)(q^{2n} - 1)}{2(q+1)(q^2 - 1)} > D(n).
\]

Assume \(k = n - 2\). Then

\[
\chi(1) = \frac{(q^{2n-2} - 1)(q^{2n} - 1)}{2(q - \alpha_1)(q - \alpha_2)} \psi(1),
\]

where \(\psi\) is an irreducible unipotent character of \(C \simeq SO_{2n-3}(q) \times GO_2^{\alpha_1}(q) \times GL_{1}^{\alpha_2}(q)\), \(\alpha_1, \alpha_2 = \pm\). Note that both \(SO_2^{\alpha_1}(q)\) and \(GL_1^{\alpha_2}(q)\) have only one unipotent character, which is the trivial one. Therefore, \(\psi = \omega \otimes \lambda \otimes 1_{GL_1^{\alpha_2}(q)}\), where \(\omega\) is a unipotent character of \(SO_{2n-3}(q)\) and \(\lambda\) is one of two extensions of \(1_{SO_2^{\alpha_1}(q)}\) to \(GO_2^{\alpha_1}(q)\). If \(\omega\) is trivial then 5) holds. If \(\omega\) is nontrivial then \(\omega(1) \geq (q^{n-2} - 1)(q^{n-2} - q)/2(q+1)\) by Proposition 5.1 of [70]. Then,

\[
\chi(1) \geq \frac{(q^{2n} - 1)(q^{2n-2} - 1)}{2(q+1)^2} \cdot \frac{(q^{n-2} - 1)(q^{n-2} - q)}{2(q+1)} > D(n).
\]

Finally, assume \(2 \leq k \leq n - 3\). We have

\[
\chi(1) \geq \frac{q^{2n} - 1}{q + 1} \cdot \frac{(q^{2(k+1)} - 1) \cdots (q^{2(n-1)} - 1)}{2(q^2 - 1) \cdots (q^{2(n-k)} - 1)} \cdot (q^{n-1-k} - 1)
\]

\[
\geq \frac{q^{2n} - 1}{q + 1} \cdot \frac{1}{2} q^{2(k-1)} \cdot (q^{n-1-k} - 1) \geq \frac{q^{2n} - 1}{q + 1} \cdot \frac{1}{2} q^{4(n-3)} \cdot (q^2 - 1) > D(n).
\]

4) Case \(m = n\). Since \(\chi\) is not unipotent, \(k < n\). We have

\[
\chi(1) = \frac{\prod_{i=k+1}^{n}(q^{2i} - 1) \cdot (q^{n-k} + \alpha)}{2\prod_{i=k}^{n-k}(q^{2i} - 1)} \psi(1),
\]

where \(\psi\) is a unipotent character of \(C \simeq SO_{2k+1}(q) \times GO_2^{\alpha_1}(q)\), \(\alpha = \pm\). So, \(\chi(1) > \frac{1}{2} q^{2k(n-k)}(q^{n-k} - 1)\).

Assume \(k = n - 1\). Then \(\chi(1) = \frac{q^{2n-1}}{2(q-\alpha)} \psi(1)\). We have \(\psi = \omega \otimes \lambda\), where \(\omega\) is a unipotent characters of \(SO_{2n-1}(q)\) and \(\lambda\) is one of two extensions of \(1_{SO_2^{\alpha_1}(q)}\) to \(GO_2^{\alpha_1}(q)\).
By Corollary 5.2.2, \( \omega \) is either one of characters of degrees 1, \( (q^n - 1)(q^{n-1} - q)/2(q + 1) \), \( (q^n + 1)(q^{n-1} + q)/2(q + 1) \), \( (q^n - 1)(q^{n-1} - q)/2(q - 1) \), \( (q^n - 1)(q^{n-1} + q)/2(q - 1) \) or \( \omega(1) > q^{4n-12} \). In the former case, 6) holds. In the latter case, \( \chi(1) > \frac{q^{2n-1}}{2(q^2-1)}q^{4n-12} > D(n) \).

Assume \( k = 0 \). Then \( \chi(1) = \frac{q^{n+1}}{2} \psi(1) \), where \( \psi \) is a unipotent character of \( C \simeq \text{GO}^\alpha_{2n}(q) = \text{SO}^\alpha_{2n}(q) \cdot 2 \). Suppose that \( \psi \) is an irreducible constituent of \( \omega^C \), where \( \psi \) is a unipotent character of \( \text{SO}^\alpha_{2n}(q) \). We have \( \omega = \theta \circ f \) where \( f \) is the canonical homomorphism \( f : \text{SO}^\pm_{2n}(q) \to P(\text{CO}^\pm_{2n}(q)^0) \) and \( \theta \) is a unipotent character of \( P(\text{CO}^\pm_{2n}(q)^0) \). There are two cases:

**Case** \( \alpha = + \): By Proposition 5.2.4, \( \theta \) is either one of characters of degrees 1, \( (q^n - 1)(q^{n-1} + q)/(q^2 - 1) \), \( (q^n - q^2)/(q^2 - 1) \), or \( \theta(1) > q^{4n-10} \). In the latter case,

\[
\chi(1) = \frac{q^n + 1}{2} \psi(1) \geq \frac{q^n + 1}{2} \omega(1) > \frac{q^n + 1}{2} q^{4n-10} > D(n).
\]

In the former case, by [65], the character afforded by the rank 3 permutation module of \( \text{GO}^+_2(q) \) on the set of all singular 1-spaces of \( \mathbb{F}^2_q \) is the sum of three unipotent characters of degrees 1, \( (q^n - 1)(q^{n-1} + q)/(q^2 - 1) \), and \( (q^{2n} - q^2)/(q^2 - 1) \). Hence, 7) holds in this case.

**Case** \( \alpha = - \): By Proposition 5.2.3, \( \theta \) is either one of characters of degrees 1, \( (q^n + 1)(q^{n-1} - q)/(q^2 - 1) \), \( (q^n - q^2)/(q^2 - 1) \), or \( \theta(1) > q^{4n-10} \). In the latter case,

\[
\chi(1) = \frac{q^n - 1}{2} \psi(1) \geq \frac{q^n - 1}{2} \omega(1) > \frac{q^n - 1}{2} q^{4n-10} = D(n).
\]

In the former case, again by [65], the character afforded by the rank 3 permutation module of \( \text{GO}^-_{2n}(q) \) on the set of all singular 1-spaces of \( \mathbb{F}^2_q \) is the sum of three unipotent characters of degrees 1, \( (q^n + 1)(q^{n-1} - q)/(q^2 - 1) \), and \( (q^{2n} - q^2)/(q^2 - 1) \). Hence, 8) holds in this case.

Assume \( k = 1 \). Then \( \chi(1) = \frac{(q^{2n-1})(q^{n-1} + 1)}{2(q^2 - 1)} \psi(1) \), where \( \psi \) is a unipotent character of \( C \simeq \text{SO}_3(q) \times \text{GO}^\alpha_{2n-2}(q) \). Suppose that \( \psi \) is an irreducible constituent of \( \omega \otimes \varphi^C \) where \( \omega, \varphi \) are unipotent characters of \( \text{SO}_3(q), \text{SO}^\alpha_{2n-2}(q) \), respectively. Let us consider
the first case where \( \varphi \) is nontrivial. Then by Propositions 7.1, 7.2 of [70], we have \( \varphi(1) \geq (q^{n-1} + 1)(q^{n-2} - q)/(q^2 - 1) \). Then

\[
\chi(1) \geq \frac{(q^{2n} - 1)(q^{n-1} - 1)(q^{n-1} + 1)(q^{n-2} - q)}{2(q^2 - 1)^2} > D(n).
\]

Now suppose \( \varphi = 1_{SO_{2n-2}(q)} \). Since \( SO_3(q) \) has two unipotent characters, the trivial one and the unique nontrivial unipotent character of degree \( q, 9 \) holds in this case.

Assume \( k = 2 \). Then

\[
\chi(1) = \frac{(q^{2n-2} - 1)(q^{2n} - 1)(q^{n-2} \pm 1)}{2(q^2 - 1)(q^4 - 1)} \psi(1) > D(n).
\]

Assume \( k = n - 2 \). Then

\[
\chi(1) = \frac{(q^{2n-2} - 1)(q^{2n} - 1)}{2(q^2 - 1)(q^2 - \alpha)} \psi(1),
\]

where \( \psi \) is a unipotent character of \( C \simeq SO_{2n-3}(q) \times GO_4^\alpha(q), \alpha = \pm \). Suppose that \( \psi \) is an irreducible constituent of \( \omega \otimes \varphi^{GO_4^\alpha(q)} \), where \( \omega, \varphi \) are unipotent characters of \( SO_{2n-3}(q), SO_4^\alpha(q) \), respectively. If \( \omega \) is nontrivial, by Proposition 5.1 of [70], \( \omega(1) \geq (q^{n-2} - 1)(q^{n-2} - q)/2(q + 1) \) and therefore

\[
\chi(1) \geq \frac{(q^{2n-2} - 1)(q^{2n} - 1)(q^{n-2} - 1)(q^{n-2} - q)}{4(q + 1)(q^4 - 1)} > D(n).
\]

Now we assume \( \omega = 1_{SO_{2n-3}(q)} \). There are two cases:

**Case** \( \alpha = + \): Let \( V := M_{2 \times 2}(F_q) \) be the space of \( 2 \times 2 \) matrices over \( F_q \). Then \( V \) is a vector space over \( F_q \) of rank 4. The determinant function \( Q(M) = \det(M), \ M \in V \) is a quadratic form on \( V \) with Witt index 2. So the group of linear transformations \( V \rightarrow V \) preserving \( Q \) will be \( GO_4^+(q) \). Consider an action of group \( GL_2(q) \times GL_2(q) \) on \( V \) by \( \tau(A, B)(M) = A^{-1}MB \). We see that \( \tau(A, B) \) preserves \( Q \) if and only if \( \det(A) = \det(B) \) and \( \tau(A, B) \) is the identity if and only if \( A = B \) is a scalar matrix. Moreover, \( \tau(A, B) \) has determinant \( \det(A)^{-2}\det(B)^2 \), which is 1 when \( \det(A) = \det(B) \). Therefore, we have a
homomorphism
\[ \tau : (SL_2(q) \times SL_2(q)) \cdot \mathbb{Z}_2 \to SO_4^+(q), \]
where \( \mathbb{Z}_2 \) is generated by
\[ U = \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) \).

We have \( \text{Ker}(\rho) = \mathbb{Z}_2 \), which is generated by \((-I, -I)\). Therefore, \(((SL_2(q) \times SL_2(q)) \cdot \mathbb{Z}_2)/\mathbb{Z}_2 \cong 2 \cdot (PSL_2(q) \times PSL_2(q)) \cdot 2 \) is a subgroup of \( SO_4^+(q) \). Since \(|2 \cdot (PSL_2(q) \times PSL_2(q)) \cdot 2| = |SO_4^+(q)| = q^2(q^2 - 1)^2, 2 \cdot (PSL_2(q) \times PSL_2(q)) \cdot 2 \cong SO_4^+(q)\).

We denote \( L_1, L_2 \) the first and second terms respectively in \( PSL_2(q) \times PSL_2(q) \). Note that \( P(CO_4^+(q)) \) as well as \( SO_4^+(q) \) have four unipotent characters of degrees \( 1, q, q, q^2 \).

Since \( PSL_2(q) \) has only one character of degree \( q \) which is denoted by \( \mu_1 \) for \( L_1 \) and \( \mu_2 \) for \( L_2 \) and \( U \) fixes \( L_1 \) and \( L_2 \), the unipotent characters of \( SO_4^+(q) \) must be extensions of characters \( 1_{L_1 \times L_2}, \mu_1 \otimes 1_{L_2}, 1_{L_1} \otimes \mu_2 \) and \( \mu_1 \otimes \mu_2 \) which are considered as characters of \( 2 \cdot (L_1 \times L_2) \).

Consider the transformation \( T : V \to V \) defined by \( T(M) = M^T \), the transpose of \( M \). It is clear that \( T \) preserves \( Q \) and therefore \( T \in GO_4^+(q) \). Moreover \( \text{det}(T) = -1 \).

So \( GO_4^+(q) = SO_4^+(q) \rtimes < T > \). We have \( T^{-1} \tau(A, B)T(M) = B^T M(A^T)^{-1} \) for every \( A, B \in GL_2(q) \). So \( T \) fixes \( \mu_1 \otimes \mu_2 \) and maps one of \( \{ \mu_1 \otimes 1_{L_2}, 1_{L_1} \otimes \mu_2 \} \) to the other. In other words, the unipotent character of degree \( q^2 \) of \( SO_4^+(q) \) has two extensions to \( GO_4^+(q) \) and the inductions of two unipotent characters of degree \( q \) of \( SO_4^+(q) \) to \( GO_4^+(q) \) are equal and irreducible. So 10) holds in this case.

**Case** \( \alpha = - \). Note that \( SO_4^-(q) \) as well as \( P(CO_4^-(q)) \) have two unipotent characters of degrees \( 1, q^2 \). It is well known that \( SO_4^-(q) \cong PSL_2(q^2) \times 2_1 \) and 
\[ (GO_4^-(q) = (PSL_2(q^2) \times 2_1) \cdot 2_2 \cong 2_1 \times (PSL_2(q^2) \cdot 2_2), \]
where \( 2_2 \) acts trivially on \( 2_1 \). Since \( PSL_2(q^2) \) has only one irreducible character of degree \( q^2 \) which is unipotent, the unipotent character of degree \( q^2 \) of \( SO_4^-(q) \) is invariant in \( GO_4^+\). Therefore, \( GO_4^-(q) \) has two unipotent characters of degree \( q^2 \). So 10) also holds in this case.
Lastly, we assume $3 \leq k \leq n - 3$. Then
\[
\chi(1) \geq \frac{\prod_{i=k+1}^{n}(q^{2i} - 1)}{2 \prod_{i=1}^{n-k}(q^{2i} - 1)} \cdot (q^{n-k} \pm 1) > \\
> \frac{1}{2} q^{2k(n-k)}(q^{n-k} \pm 1) \geq \frac{1}{2} q^{6(n-3)}(q^3 - 1) > D(n).
\]

\[\square\]

**Counting semi-simple conjugacy classes in $G^* = SO_{2n+1}(q)$:** In Proposition 5.3.1 and its proof, we have not shown how to count the number of semi-simple conjugacy classes $(s)$ in $SO_{2n+1}(q)$ for a certain $C := C_{G^*}(s)$, which will imply the number of irreducible characters of $Sp_{2n}(q)$ at each degree. Actually, the way to count them is pretty similar in all the cases from 2) to 10) in Proposition 5.3.1. First, from the structure of $C_{G^*}(s)$, we know the form of the characteristic polynomial as well as the eigenvalues of $s$.

In general, we have
\[
\text{Spec}(s) = \{1, \ldots, 1, -1, \ldots, -1, \ldots\}. \\
\text{spec}_{2k+1}^{2(n-k)} \times \text{spec}_{2(n-m)}^{2(m-k)}.
\]

By [73, (2.6)], if $-1 \notin \text{Spec}(s)$, there is exactly one $GO_{2n+1}(q)$-conjugacy class of semi-simple elements $(s)$ for a given $\text{Spec}(s)$. This class is also an $SO_{2n+1}(q)$-conjugacy class since $(GO_{2n+1}(q) : C_{GO_{2n+1}(q)}(s)) = (SO_{2n+1}(q) : C_{SO_{2n+1}(q)}(s))$ (see Lemma 5.1.2). The situation is a little bit different when $-1 \in \text{Spec}(s)$. In that case, again by [73, (2.6)], there are exactly two $GO_{2n+1}(q)$-conjugacy class of semi-simple elements $(s)$ for a given $\text{Spec}(s)$, in which $C_{GO_{2n+1}(q)}(s) \simeq GO_{2k+1}(q) \times GO_{2(m-k)}^{\pm}(q) \times \prod_{i=1}^{t} GL_{\alpha_i}^{\alpha_i}(q^{k_i})$ (+ for one class and – for the other). These classes are also $SO_{2n+1}(q)$-conjugacy class by the same reason as before. So, in order to count the number of semi-simple conjugacy classes $(s)$ with a given centralizer $C$, we need to count the number of choices of $\text{Spec}(s)$. This is demonstrated in Tables 5-2, 5-3.

We finish this section by two following Corollaries of Theorem F.

**Corollary 5.3.2.** Let $\chi$ be an irreducible complex character of $G = Sp_{2n}(q)$, where $n \geq 6$ and $q$ is an odd prime power. Then $\chi(1) = 1$ (1 character), $(q^n \pm 1)/2$ (4 characters),

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\( (q^n + \alpha_1)(q^n + \alpha_2q)/2(q + \alpha_1\alpha_2) \) (4 characters) \((\alpha_{1,2} = \pm1)\), \( (q^{2n} - 1)/2(q \pm 1) \) (4 characters), 
\( (q^{2n} - 1)/(q \pm 1) \) (q - 2 characters), or \( \chi(1) \geq (q^{2n} - 1)(q^{n-1} - q)/2(q^2 - 1) \).

**Corollary 5.3.3.** Let \( \chi \) be an irreducible complex character of \( G = Sp_{2n}(q) \), where \( n \geq 6 \) and \( q \) is an odd prime power. Then \( \chi(1) = 1 \) (1 character), \( (q^n \pm 1)/2 \) (4 characters), 
\( (q^n + \alpha_1)(q^n + \alpha_2q)/2(q + \alpha_1\alpha_2) \) (4 characters) \((\alpha_{1,2} = \pm1)\), \( (q^{2n} - 1)/2(q \pm 1) \) (4 characters), 
\( (q^{2n} - 1)/(q \pm 1) \) (q - 2 characters), \( (q^{2n} - 1)(q^{n-1} \pm q)/2(q^2 - 1) \) (4 characters), 
\( (q^{2n} - 1)(q^{n-1} \pm 1)/2(q^2 - 1) \) (4 characters), \( (q^{2n} - 1)(q^{n-1} \pm 1)/2(q \pm 1) \) (4q - 8 characters), 
\( (q^{2n} - q^2)(q^n \pm 1)/2(q^2 - 1) \) (4 characters), \( q(q^{2n} - 1)(q^{n-1} \pm 1)/2(q^2 - 1) \) (4 characters), or \( \chi(1) \geq (q^{2n} - 1)(q^{n-1} - 1)(q^{n-1} - q^2)/2(q^4 - 1) \).
Table 5-1. Low-dimensional unipotent characters of $Sp_{2n}(q)$, $n \geq 6$, $q$ odd, I

<table>
<thead>
<tr>
<th>Characters</th>
<th>Symbols</th>
<th>Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1_G$</td>
<td>$\begin{pmatrix} n \ \end{pmatrix}$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; n \ - \end{pmatrix}$</td>
<td>$(q^n - 1)(q^n - q)$</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>$\begin{pmatrix} 0 &amp; 1 \ n \end{pmatrix}$</td>
<td>$\frac{2(q + 1)}{(q^n + 1)(q^n + q)}$</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>$\begin{pmatrix} 1 &amp; n \ 0 \end{pmatrix}$</td>
<td>$2(q + 1)$</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>$\begin{pmatrix} 2 &amp; n - 1 \ 0 \end{pmatrix}$</td>
<td>$(q^n - 1)(q^n + q)$</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>$\begin{pmatrix} 0 &amp; 2 &amp; n - 1 \ - \end{pmatrix}$</td>
<td>$\frac{2(q^4 - 1)}{(q^{2n} - 1)(q^{n-1} - 1)(q^{n-1} - q^2)}$</td>
</tr>
<tr>
<td>$\chi_7$</td>
<td>$\begin{pmatrix} 0 &amp; 2 \ n - 1 \end{pmatrix}$</td>
<td>$\frac{2(q^4 - 1)}{(q^{2n} - 1)(q^{n-1} + 1)(q^{n-1} + q^2)}$</td>
</tr>
<tr>
<td>$\chi_8$</td>
<td>$\begin{pmatrix} 2 &amp; n - 1 \ 0 \end{pmatrix}$</td>
<td>$\frac{2(q^4 - 1)}{(q^{2n} - 1)(q^{n-1} - 1)(q^{n-1} + q^2)}$</td>
</tr>
<tr>
<td>$\chi_9$</td>
<td>$\begin{pmatrix} 0 &amp; n - 1 \ 2 \end{pmatrix}$</td>
<td>$\frac{2(q^2 - 1)^2}{q(q^{2n} - 1)(q^{n-2} - q^2)}$</td>
</tr>
<tr>
<td>$\chi_{10}$</td>
<td>$\begin{pmatrix} 1 &amp; n - 1 \ 1 \end{pmatrix}$</td>
<td>$\frac{(q^2 - 1)^2}{(q^{2n} - q^2)(q^n - 1)(q^n - q^2)}$</td>
</tr>
<tr>
<td>$\chi_{11}$</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; 2 &amp; n \ 1 \end{pmatrix}$</td>
<td>$\frac{2(q^4 - 1)}{(q^{2n} - q^2)(q^n - 1)(q^n - q^2)}$</td>
</tr>
<tr>
<td>$\chi_{12}$</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; 2 \ 1 &amp; n \end{pmatrix}$</td>
<td>$\frac{2(q^4 - 1)}{(q^{2n} - q^2)(q^n + 1)(q^n + q^2)}$</td>
</tr>
<tr>
<td>$\chi_{13}$</td>
<td>$\begin{pmatrix} 1 &amp; 2 \ n &amp; 0 \end{pmatrix}$</td>
<td>$\frac{2(q^4 - 1)}{(q^{2n} - q^2)(q^n + 1)(q^n - q^2)}$</td>
</tr>
<tr>
<td>$\chi_{14}$</td>
<td>$\begin{pmatrix} 0 &amp; 1 \ 1 &amp; 2 \end{pmatrix}$</td>
<td>$\frac{2(q^2 - 1)^2}{(q^{2n} - q^2)(q^n - 1)(q^n + q^2)}$</td>
</tr>
</tbody>
</table>
Table 5-2. Low-dimensional irreducible characters of $Sp_{2n}(q)$, $n \geq 6$, $q$ odd, II

<table>
<thead>
<tr>
<th>Spec(s)</th>
<th>$C_G^*(s)$</th>
<th>$\psi(1)$</th>
<th>$\chi(1)$</th>
<th>$\sharp$ of characters</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1, \ldots, 1, \gamma^k, \gamma^{-k}, \gamma^k, \gamma^{-k}}$ $k \in T_1$</td>
<td>$SO_{2n-3}(q) \times GL_2(q)$</td>
<td>1</td>
<td>$\frac{(q^{2n-2} - 1)(q^{2n} - 1)}{(q-1)(q^2 - 1)}$</td>
<td>$(q - 3)/2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$q \frac{(q^{2n-2} - 1)(q^{2n} - 1)}{(q-1)(q^2 - 1)}$</td>
<td>$(q - 3)/2$</td>
</tr>
<tr>
<td>${1, \ldots, 1, \eta^k, \eta^{-k}, \eta^k, \eta^{-k}}$ $k \in T_2$</td>
<td>$SO_{2n-3}(q) \times GU_2(q)$</td>
<td>1</td>
<td>$\frac{(q^{2n-2} - 1)(q^{2n} - 1)}{(q-1)(q^2 - 1)}$</td>
<td>$(q - 1)/2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$q \frac{(q^{2n-2} - 1)(q^{2n} - 1)}{(q-1)(q^2 - 1)}$</td>
<td>$(q - 1)/2$</td>
</tr>
<tr>
<td>${1, \ldots, 1, \theta^j, \theta^{n-j}, \theta^{-j}, \theta^{-n-j}}$ $j \in R_2$</td>
<td>$SO_{2n-3}(q) \times GL_1(q^2)$</td>
<td>1</td>
<td>$\frac{q^2 - 1}{1}$</td>
<td>$(q - 1)^2/4$</td>
</tr>
<tr>
<td>${1, \ldots, 1, \zeta^j, \zeta^{n-j}, \zeta^{-j}, \zeta^{-n-j}}$ $j \in R_1$</td>
<td>$SO_{2n-3}(q) \times GU_1(q^2)$</td>
<td>1</td>
<td>$\frac{(q^{2n-2} - 1)(q^{2n} - 1)}{(q-1)(q^2 - 1)}$</td>
<td>$(q^2 - 1)/4$</td>
</tr>
<tr>
<td>${1, \ldots, 1, \gamma^k, \gamma^{-k}, \gamma^l, \gamma^{-l}}$ $k, l \in T_1, k \neq l$</td>
<td>$SO_{2n-3}(q) \times GL_1(q) \times GL_1(q)$</td>
<td>1</td>
<td>$\frac{(q^{2n-2} - 1)(q^{2n} - 1)}{(q-1)^2}$</td>
<td>$(q - 3)(q - 5)/8$</td>
</tr>
<tr>
<td>${1, \ldots, 1, \eta^k, \eta^{-k}, \eta^l, \eta^{-l}}$ $k, l \in T_2, k \neq l$</td>
<td>$SO_{2n-3}(q) \times GU_1(q) \times GU_1(q)$</td>
<td>1</td>
<td>$\frac{(q^{2n-2} - 1)(q^{2n} - 1)}{(q-1)^2}$</td>
<td>$(q - 1)(q - 3)/8$</td>
</tr>
<tr>
<td>${1, \ldots, 1, \gamma^k, \gamma^{-k}, \gamma^l, \gamma^{-l}}$ $k \in T_1, l \in T_2$</td>
<td>$SO_{2n-3}(q) \times GL_1(q) \times GU_1(q)$</td>
<td>1</td>
<td>$\frac{(q^{2n-2} - 1)(q^{2n} - 1)}{(q-1)^2}$</td>
<td>$(q - 1)(q - 3)/4$</td>
</tr>
<tr>
<td>${1, -1, \ldots, -1, \gamma^k, \gamma^{-k}}$, $k \in T_1$</td>
<td>$GO_{2n-2}^e(q) \times GL_1(q)$</td>
<td>1</td>
<td>$\frac{(q^{2n} - 1)(q^{n-1} + \alpha)}{2(q-1)}$</td>
<td>$2(q - 3)$</td>
</tr>
<tr>
<td>${1, -1, \ldots, -1, \eta^k, \eta^{-k}}$, $k \in T_2$</td>
<td>$GO_{2n-2}^e(q) \times GU_1(q)$</td>
<td>1</td>
<td>$\frac{(q^{2n} - 1)(q^{n-1} + \alpha)}{2(q+1)}$</td>
<td>$2(q - 1)$</td>
</tr>
<tr>
<td>${1, \ldots, 1, \gamma^k, \gamma^{-k}}$, $k \in T_1$</td>
<td>$SO_{2n-1}(q) \times GL_1(q)$</td>
<td>(⋆)</td>
<td>$\frac{q^{2n} - 1}{q-1} \psi(1)$</td>
<td>$(q - 3)/2$</td>
</tr>
<tr>
<td>${1, \ldots, 1, \eta^k, \eta^{-k}}$, $k \in T_2$</td>
<td>$SO_{2n-1}(q) \times GU_1(q)$</td>
<td>(⋆)</td>
<td>$\frac{q^{2n} - 1}{q+1} \psi(1)$</td>
<td>$(q - 1)/2$</td>
</tr>
<tr>
<td>Spec(s)</td>
<td>$C_{G^*}(s)$</td>
<td>$\psi(1)$</td>
<td>$\chi(1)$</td>
<td>$\sharp$ of chars</td>
</tr>
<tr>
<td>---------------------------</td>
<td>------------------------------------------------------------------------------</td>
<td>-----------</td>
<td>--------------------------------------------------------------------------</td>
<td>-------------------</td>
</tr>
</tbody>
</table>
| $\{1, ..., 1, -1, -1, -1, \gamma^k, \gamma^{-k}\}$  
$k \in T_1$       | $SO_{2n-3}(q) \times GO_2^\alpha(q)$  
$\times GL_1(q)$          | 1         | $(q^{2n-2} - 1)(q^{2n-1} - 1)$                                           | $2(q - 3)$       |
| $\{1, ..., 1, -1, -1, \eta^k, \eta^{-k}\}$  
$k \in T_2$       | $SO_{2n-3}(q) \times GO_2^\alpha(q)$  
$\times GU_1(q)$          | 1         | $(q^{2n-2} - 1)(q^{2n-1} - 1)$                                           | $2(q - 1)$       |
| $\{1, ..., 1, -1, -1\}$   | $SO_{2n-1}(q) \times GO_2^\alpha(q)$                                        | (*)       | $\frac{q^{2n-1}}{2(q - \alpha)}\psi(1)$                                | 4                 |
| $\{1, -1, ..., -1\}$      | $GO_{2n}^\alpha(q)$                                                          | 1         | $\frac{(q^n - \alpha)(q^{n-1} + \alpha)}{q^n - q^2}$                     | $2(q^2 - 1)$      |
|                           |                                                                              |           | $\frac{(q^{2n-1} - \alpha)(q^{n-1} + \alpha)}{q^n - q^2}$                | 4                 |
|                           |                                                                              |           | $\frac{q^n - q^2}{q^2 - 1}$                                              | 4                 |
| $\{1, 1, 1, -1, ..., -1\}$ | $SO_3(q) \times GO_{2n-2}^\alpha(q)$                                        | 1         | $\frac{(q^{2n-1} - 1)(q^{n-1} + \alpha)}{2(q^2 - 1)}$                    | 4                 |
| $\{1, ..., 1, -1, -1, -1, -1\}$ | $SO_{2n-3}(q) \times GO_4^\alpha(q)$                                        | 1         | $\frac{q^n - q^2}{q^2 - 1}$                                              | $2(q^2 - 1)$      |
|                           |                                                                              |           | $\frac{(q^{2n-2} - 1)(q^{2n-1} - 1)}{2(q^2 - 1)(q^2 - \alpha)}$           | 4                 |
| $\{1, ..., 1, -1, -1, -1, -1\}$ | $SO_{2n-3}(q) \times GO_4^+(q)$                                              | 2$q$      | $\frac{q^n - q^2}{q^2 - 1}$                                              | $2(q^2 - 1)$      |
|                           |                                                                              |           | $\frac{q^{2n-2} - 1)(q^{2n-1})}{2(q^2 - 1)(q^2 - \alpha)}$                | 4                 |
|                           |                                                                              |           | $\frac{(q^{2n-2} - 1)(q^{2n-1} - 1)}{(q^2 - 1)^2}$                       | 1                 |

Here, (*) is $1$, $(q^{n-1} - 1)(q^{n-1} - q)/2(q + 1)$, $(q^{n-1} + 1)(q^{n-1} + q)/2(q + 1)$, $(q^{n-1} + 1)(q^{n-1} - q)/2(q - 1)$, or $(q^{n-1} - 1)(q^{n-1} + q)/2(q - 1)$. 
LOW-DIMENSIONAL CHARACTERS OF THE ORTHOGONAL GROUPS

6.1 Odd Characteristic Orthogonal Groups in Odd Dimension

The aim of this section is to classify low-dimensional complex representations of groups $G = \text{Spin}_{2n+1}(q)$ where $q$ is a power of an odd prime $p$. The dual group $G^*$ is the projective conformal symplectic group $\text{PCSp}_{2n}(q)$, which is the quotient of $\tilde{G} = \text{CSp}_{2n}(q)$ by its center $\mathbb{Z} \tilde{G} \simeq \mathbb{Z}_{q-1}$.

Proof of Theorem G. If $\chi$ is unipotent, Theorem is done by Corollary 5.2.2. So we can assume $\chi$ is not unipotent such that $\chi(1) \leq q^{4n-8}$. Suppose that $\chi$ is parametrized by $((s^*), \psi)$, where $s^*$ is a non-trivial semi-simple element in $G^*$. Consider an inverse image $s$ of $s^*$ in $\tilde{G}$. Then $(s)$ is a non-trivial conjugacy class of semi-simple elements in $\tilde{G}$. Let $C^* = C_{G^*}(s^*)$ and $C = C_{\tilde{G}}(s)$. Let $g$ be any element in $\tilde{G}$ such that the image of $g$ in $G^*$ belongs to $C^*$. Then $gsg^{-1} = \mu(g)s$ for some $\mu(g) \in \mathbb{Z}_{q-1}$. Therefore, $\tau(s) = \tau(gsg^{-1}) = \tau(\mu(g)s) = \mu(g)^2 \tau(s)$. It follows that $\mu(g)$ is 1 or $-1$. Hence

$$(G^* : C^*)_{\nu'} \geq (\tilde{G} : C)_{\nu'}/2.$$  

Case 1: If $\tau(s)$ is not a square in $\mathbb{F}_q$, by Lemma 5.1.4, $C \simeq (\text{Sp}_m(q^2) \times \prod_{i=1}^t \text{GL}_{\alpha_i}(q^{k_i}))$. Therefore,

$$\chi(1) \geq (G^* : C^*)_{\nu'} \geq (\tilde{G} : C)_{\nu'}/2 > q^{4n-8}.$$
Case 2: If \( \tau(s) \) is a square in \( \mathbb{F}_q \), multiplying \( s \) with a suitable scalar, we can assume that \( \tau(s) = 1 \) or in other words \( s \in Sp_{2n}(q) \). By Lemma 5.1.4, \( C \simeq (Sp_{2k}(q) \times Sp_{2(m-k)}(q) \times \prod_{i=1}^{t} GL_{\alpha_i}^{\alpha_i}(q^{k^i})) \cdot Z_{q-1} \), where \( \alpha_i = \pm, \Sigma_{i=1}^{t} k_i \alpha_i = n - m \). It is easy to see that \( |Sp_{2k}(q) \times Sp_{2(m-k)}(q)|_{p'} \leq \prod_{i=1}^{m} (q^{2i} - 1) \) and \( |\prod_{i=1}^{t} GL_{\alpha_i}^{\alpha_i}(q^{k^i})|_{p'} \leq \prod_{i=1}^{n-m} (q^i - (-1)^i) \).

Therefore,

\[
(\tilde{G} : C)_{p'} \geq \frac{(q^{2(m+1)} - 1) \cdots (q^{2n} - 1)}{(q + 1)(q^2 - 1) \cdots (q^{n-m} - (-1)^{n-m})} =: f(m, n).
\]

1) When \( m \leq n - 2 \). We have \( (\tilde{G} : C)_{p'} \geq \min\{f(0, n), f(n - 2, n)\} \). Since both \( f(0, n) = \prod_{i=1}^{n} (q^i + (-1)^i) \) and \( f(n - 2, n) = (q^{2n-2} - 1)(q^{2n} - 1)/(q + 1)(q^2 - 1) \) are greater than \( 2q^{4n-8} \), \( \chi(1) \geq (\tilde{G} : C)_{p'}/2 > q^{4n-8} \).

2) When \( m = n - 1 \). If \( 1 \leq k \leq n - 2 \) then \( (\tilde{G} : C)_{p'} \geq (q^{2n-2} - 1)(q^{2n} - 1)/(q + 1)(q^2 - 1) \). Again, \( \chi(1) \geq (\tilde{G} : C)_{p'}/2 > q^{4n-8} \). So \( k = 0 \) or \( n - 1 \). Modulo \( Z(\tilde{G}) \), we can assume that \( k = n - 1 \). There are two cases:

- \( C = (Sp_{2n-2}(q) \times GL_1(q)) \cdot Z_{q-1} \), which is happened when \( \text{Spec}(s) = \{1, \ldots, 1, \lambda, \lambda^{-1}\} \), where \( \pm 1 \neq \lambda \in \mathbb{F}_q^\times \). Note that there are \( (q - 3)/2 \) choices for \( \lambda \), namely, \( \lambda = \gamma^k, k \in T_1 \). For each such \( \lambda \), there is exactly one semi-simple conjugacy class of such elements \( s \) in \( Sp_{2n}(q) \) by [73, (2.6)]. This conjugacy class is also the class of \( s \) in \( CSp_{2n}(q) \). Therefore, there is exactly one conjugacy class of semi-simple elements \( (s^*) \) in \( G^* \) such that \( \text{Spec}(s) = \{1, \ldots, 1, \lambda, \lambda^{-1}\} \) for each \( \lambda = \gamma^k, k \in T_1 \).

- \( C = (Sp_{2n-2}(q) \times GU_1(q)) \cdot Z_{q-1} \), which is happened when \( \text{Spec}(s) = \{1, \ldots, 1, \lambda, \lambda^{-1}\} \), where \( \pm 1 \neq \lambda \in \mathbb{F}_q^\times \) and \( \lambda^{-1} = \lambda^\eta \). Note that there are \( (q - 1)/2 \) choices for \( \lambda \), namely, \( \lambda = \eta^k, k \in T_2 \). Similarly as above, there is exactly one conjugacy class of semi-simple elements \( (s^*) \) in \( G^* \) such that \( \text{Spec}(s) = \{1, \ldots, 1, \lambda, \lambda^{-1}\} \) for each \( \lambda = \eta^k, k \in T_2 \).

In two above situations, if \( \{1, \ldots, 1, \lambda, \lambda^{-1}\} = \{\mu, \ldots, \mu, \mu\lambda, \mu\lambda^{-1}\} \) for some \( \mu \in \mathbb{F}_q^\times \), then \( \mu = 1 \) since \( n \geq 5 \). Therefore, \( C \) is the complete inverse image of \( C^* \) in \( \tilde{G} \). In other words, \( C^* = C/Z(\tilde{G}) \) and hence \( (G^* : C^*)_{p'} = (\tilde{G} : C)_{p'} \). Consider the canonical homomorphism \( f : Sp_{2n-2}(q) \times GL_1^\alpha(q) \hookrightarrow C \rightarrow C^*, \alpha = \pm \), whose kernel is contained in
the center of $Sp_{2n-2}(q) \times GL_1^\alpha(q)$ and image contains the commutator group of $C^*$ since $Sp_{2n-2}(q) \times GL_1^\alpha(q)$ contains the commutator group of $C \simeq (Sp_{2n-2}(q) \times GL_1^\alpha(q)) \cdot \mathbb{Z}_{q-1}$.

By Proposition 5.1.1, the unipotent characters of $Sp_{2n-2}(q) \times GL_1^\alpha(q)$ are of the form $\psi \circ f$, where $\psi$ runs over the unipotent characters of $C^*$. In particular, $\psi$ is trivial or $\psi(1) \geq (q^{n-1} - 1)(q^{n-1} - q)/2(q + 1)$ by Proposition 5.1 of [70]. In the latter case,

$$\chi(1) \geq \frac{q^{2n-1}-1}{q+1} \cdot \frac{(q^{n-1}-1)(q^{n-1}-q)}{2(q+1)} > q^{4n-8}.$$ 

Therefore, in this case, $\chi$ is one of $(q - 3)/2$ characters of degree $(q^{2n} - 1)/(q - 1)$ or $(q - 1)/2$ characters of degree $(q^{2n} - 1)/(q + 1)$.

3) When $m = n$. Since $(s^*)$ is non-trivial, we assume $1 \leq k \leq n - 1$.

First, if $k = 1$ or $n - 1$, modulo $Z(\tilde{G})$, we may assume $\text{Spec}(s) = \{-1, -1, 1, \ldots, 1\}$. Again, $C^* = C/Z(\tilde{G})$ and $(G^* : C^*)_p = (\tilde{G} : C)_p$. There is a unique possibility for $(s^*)$ in this case. We have $\chi(1) = \frac{q^{2n-1}}{q + 1} \psi(1)$ where $\psi$ is a unipotent character of $C^*$. Applying Proposition 5.1.1 again for the canonical homomorphism $f : Sp_2(q) \times Sp_{2n-2}(q) \xrightarrow{\text{can}} C \rightarrow C^*$, we see that either $\psi$ is trivial, or the unipotent character of degree $q$, or $\psi(1) \geq (q^{n-1} - 1)(q^{n-1} - q)/2(q + 1)$. In the last case,

$$\chi(1) \geq \frac{q^{2n-1}}{q + 1} \cdot \frac{(q^{n-1}-1)(q^{n-1}-q)}{2(q+1)} > q^{4n-8}.$$ 

The first two cases explain why $G$ has two characters of degrees $(q^{2n} - 1)/(q^2 - 1)$ and $q(q^{2n} - 1)/(q^2 - 1)$.

Next, if $k = 2$ or $n - 2$, then one again can show that $C^* = C/Z(\tilde{G})$ since $n \geq 5$. Therefore we have

$$\chi(1) \geq (\tilde{G} : C)_p \geq \frac{(q^{2n-2}-1)(q^{2n}-1)}{(q^2-1)(q^4-1)} > q^{4n-8}.$$ 

Finally, if $3 \leq k \leq n - 3$, in particular $n \geq 6$, then

$$\chi(1) \geq (\tilde{G} : C)_p/2 \geq \frac{(q^{2n-4})(q^{2n-2}-1)(q^{2n}-1)}{2(q^2-1)(q^4-1)(q^6-1)} > q^{4n-8}.$$
6.2 Even Characteristic Orthogonal Groups in Even Dimension

In this section we classify low-dimensional complex characters of the group $G = \Omega_{2n}^\alpha(q)$, where $\alpha = \pm$ and $q$ is a power of 2. Since $q$ is even, we can identify $G$ with its dual group. Recall that $\Omega_{2n}^\alpha(q) = [GO_{2n}^\alpha(q), GO_{2n}^\alpha(q)]$ and $GO_{2n}^\alpha(q) = \Omega_{2n}^\alpha(q) \cdot \mathbb{Z}_2$.

**Proof of Theorem H.** If $\chi$ is unipotent then we are done by Propositions 5.2.3, 5.2.4. Now we assume $\chi$ is not unipotent. Suppose that $\chi$ is parametrized by $((s), \psi)$, where $s$ is a non-trivial semi-simple element in $G$ and $\psi$ is a unipotent character of $C := C_G(s)$ such that $\chi(1) \leq q^{4n-10}$. Set $C' = C_{GO_{2n}^\alpha(q)}(s)$. Since $C$ is a subgroup of $C'$ of index 1 or 2, $(G : C)_{2'} = (GO_{2n}^\alpha(q) : C')_{2'}$.

Let $V = \mathbb{F}_q^{2n}$ be endowed with a non-degenerate quadratic form $Q(\cdot)$. Fix a basis of $V$ and let $J$ be the Gram matrix of $Q$ corresponding to this basis. Then $\mathbb{F}J = J$. Hence Spec$(s) = \text{Spec}(s) = \text{Spec}(Js^{-1}J^{-1}) = \text{Spec}(s^{-1})$. Denote the characteristic polynomial of $s$ acting on $V$ by $P(x) \in \mathbb{F}_q[x]$ and decompose $P(x)$ into distinct irreducible polynomials over $\mathbb{F}_q$:

$$P(x) = (x - 1)^{2m} \prod_{i=1}^{l} f_i^{m_i}(x) \prod_{j=1}^{l'} g_j^{n_j}(x) \tilde{g}_j^{n_j}(x),$$

where

- if $\lambda$ is a root of $f_i(x)$ then $\lambda^{-1}$ is also a root of $f_i(x)$, 1 is not a root of $f_i$,
- if $\lambda$ is a root of $g_j(x)$ then $\lambda^{-1}$ is a root of $\tilde{g}_j(x)$, $\deg(g_j) = \deg(\tilde{g}_j)$, $g_j \neq \tilde{g}_j$,
- $n = m + \sum_{i=1}^{l} m_i \deg(f_i)/2 + \sum_{j=1}^{l'} n_j \deg(g_j)$.

By Lemma 5.1.3, $C' \simeq GO_{2n}^\pm(q) \times \prod_{i=1}^{l} GL_{\alpha_i}(q^{k_i})$, where $\alpha_i = \pm$, $\sum_{i=1}^{l} k_i a_i = n - m$. Since $(s)$ is non-trivial, $m < n$. It is easy to see that $|\prod_{i=1}^{l} GL_{\alpha_i}(q^{k_i})|_{2'} \leq \prod_{i=1}^{n-m}(q^i - (-1)^i)$. Therefore, with convention that $q^0 = 1$, we have

$$(G : C)_{2'} \geq \frac{(q^{2(m+1)} - 1) \cdots (q^{2n} - 1)(q^m - 1)}{(q + 1)(q^2 - 1) \cdots (q^{n-m} - (-1)^{n-m})(q^n + 1)} =: f(m, n).$$
1) When $1 \leq m \leq n - 2$. We have

$$f(1, n) = (q - 1)(q^n - 1) \prod_{i=1}^{n-1} (q^i + (-1)^i) > q^{4n-10}$$

and

$$f(n-2, n) = \frac{(q^{2n-2} - 1)(q^n - 1)(q^{n-2} - 1)}{(q+1)(q^2 - 1)} > q^{4n-10}.$$ 

Therefore, $\chi(1) \geq \min\{f(1, n), f(n-2, n)\} > q^{4n-10}$.

2) When $m = n - 1$. Then $C \simeq GO_{2n-2}^\pm(q) \times GL_1^\beta(q)$ with $\beta = \pm$. Since $GL_1^\beta(q) \simeq \Omega_2^\beta(q)$, $C' \simeq GO_{2n-2}^{\alpha\beta}(q) \times GL_1^\beta(q)$. It is obvious that $C = \Omega_{2n-2}^{\alpha\beta}(q) \times GL_1^\beta(q)$. There are two cases:

- $C' = GO_{2n-2}^{\alpha\beta}(q) \times GL_1^\beta(q)$ corresponding to the case $\text{Spec}(s) = \{1, \ldots, 1, \lambda, \lambda^{-1}\}$, where $1 \neq \lambda \in \mathbb{F}_q^\times$. Note that there are $(q - 2)/2$ choices for $\lambda$, namely, $\lambda = \gamma^k, k \in T_1$.

  For each such $\lambda$, there is exactly one semi-simple conjugacy class of such elements $s$ in $GO_{2n}^\alpha(q)$ by [73, (3.7)]. Since $(G : C) = (GO_{2n}^\alpha(q) : C')$, $GO_{2n}^\alpha(q)$-conjugacy class of $s$ is also a $G$-conjugacy of $s$. Therefore, there is also exactly one conjugacy class of semi-simple elements $(s)$ in $G$ such that $\text{Spec}(s) = \{1, \ldots, 1, \lambda, \lambda^{-1}\}$ for each $\lambda = \gamma^k, k \in T_1$.

- $C' = GO_{2n-2}^\alpha(q) \times GU_1(q)$ corresponding to the case $\text{Spec}(s) = \{1, \ldots, 1, \lambda, \lambda^{-1}\}$, where $1 \neq \lambda \in \mathbb{F}_q^\times$ and $\lambda^{-1} = \lambda^q$. Similarly as above, there is exactly one conjugacy class of semi-simple elements $s$ in $G$ such that $\text{Spec}(s) = \{1, \ldots, 1, \lambda, \lambda^{-1}\}$ for each $\lambda = \eta^k, k \in T_2$.

We have $\chi(1) = \frac{(q^n - \alpha)(q^{n-1} + \alpha \beta)}{q - \beta} \psi(1)$, where $\psi$ is a unipotent character of $C$. If $\psi$ is non-trivial and $(n, q, \alpha) \neq (5, 2, -)$ then by Propositions 7.1 and 7.2 of [70],

$$\psi(1) \geq \frac{(q^{n-1} - \alpha \beta)(q^{n-2} + \alpha \beta q)}{q^2 - 1}.$$ 

In that case,

$$\chi(1) \geq \frac{(q^n - \alpha)(q^{n-1} + \alpha \beta)}{q - \beta} \cdot \frac{(q^{n-1} - \alpha \beta)(q^{n-2} + \alpha \beta q)}{q^2 - 1} > q^{4n-10}.$$
If ψ is non-trivial and (n, q, α) = (5, 2, −), then ψ(1) ≥ q^3(q - 1)^3(q^2 - 1)/2 = 28 and we still have χ(1) > q^{4n-10}. So ψ must be trivial. In summary, in this case, χ is one of (q - 2)/2 characters of degree (q^n - α)(q^{n-1} + α)/(q - 1) or q/2 characters of degree (q^n - α)(q^{n-1} - α)/(q + 1).

3) When m = 0. If n ≥ 6, we still have f(0, n) = \prod_{i=1}^{n}(q^i + (-1)^i)/q^n + 1 > q^{4n-10}. Now we consider the case n = 5. Since χ(1) ≤ q^{4n-10}, C' = GU_5(q). This forces G = Ω_10^{-}(q). Then (G : C')_2 = (q - 1)(q^2 + 1)(q^3 - 1)(q^4 + 1) and therefore ψ(1) = 1.

There are exactly q/2 conjugacy classes (s) in GO_10^{-}(q) of semi-simple elements such that C' ∼ GU_5(q), which is happened when Spec(s) = {λ, λ, λ, λ, λ, λ^{-1}, λ^{-1}, λ^{-1}, λ^{-1}, λ^{-1}}, λ = η^k, k ∈ T_2. Note that GU_5(q) ≤ Ω_10^{-}(q), so C = GU_5(q) and (GO_10^{-}(q) : C') = 2(Ω_10^{-}(q) : C).

In other words, |s^{GO_10^{-}(q)}| = 2|s^{Ω_10^{-}(q)}|. Therefore there are exactly q conjugacy classes (s) in Ω_10^{-}(q) such that C = GU_5(q). This gives q characters of degree (q - 1)(q^2 + 1)(q^3 - 1)(q^4 + 1) of Ω_10^{-}(q).

\[\square\]

6.3 Odd Characteristic Orthogonal Groups in Even Dimension

In this section, we classify low-dimensional complex representations of the group G = Spin_{2n}^0(q) where α = ± and q is an odd prime power. The dual group G∗ is the projective conformal orthogonal group P(CO_{2n}^0(q)^0), which is quotient of CO_{2n}^0(q)^0 by its center Z_{q-1}.

Proof of Theorem I. If χ is unipotent then we are done by Propositions 5.2.3, 5.2.4. So we can assume that χ is not unipotent. We denote \( \tilde{G} := CO_{2n}^0(q) \), \( \tilde{G}^0 := CO_{2n}^0(q)^0 \), and \( Z := Z(\tilde{G}) \simeq Z(\tilde{G}^0) \simeq Z_{q-1} \). Suppose that s∗ is a non-trivial semi-simple element in G∗. Consider an inverse image s of s∗ in \( \tilde{G}^0 \). Then (s) is a non-trivial conjugacy class of semi-simple elements in \( \tilde{G} \). Set C∗ := C_{G∗}(s∗), \( \tilde{C} := C_{\tilde{G}}(s) \), and \( \tilde{C}^0 := C_{\tilde{G}^0}(s) \). Suppose that χ is parametrized by ((s∗), ψ), ψ ∈ Irr(C∗), such that χ(1) ≤ q^{4n-10}. Let g be any element in \( \tilde{G}^0 \) such that the image of g in G∗ belongs to C∗. Then gsg^{-1} = μ(g)s for some μ(g) ∈ Z_{q-1}. Therefore, τ(s) = τ(gsg^{-1}) = τ(μ(g)s) = μ(g)^2τ(s). It follows that μ(g) is 1.
Therefore, if

\[ (G^* : C^*)_{p'} \geq (\tilde{G}^0 \colon \tilde{C}^0)_{p'}/2 \geq (\tilde{G} : \tilde{C})_{p'}/4. \]

**Case 1:** If \( \tau(s) \) is not a square in \( \mathbb{F}_q \), multiplying \( s \) by a suitable scalar in \( \mathbb{F}_q \), we can assume that \( \tau(s) \) is fixed. By Lemma 5.1.5, \( \tilde{C} \simeq (GO^\pm_m(q^2) \times \prod_{i=1}^t GL_{n_i}^{\alpha_i}(q^{k_i})) \cdot \mathbb{Z}_{q-1} \), where \( \alpha_i = \pm, \Sigma_{i=1}^t k_i \alpha_i = n - m \). Denote the group \( \prod_{i=1}^t GL_{n_i}^{\alpha_i}(q^{k_i}) \) by \( A \) for short. Now we will show that, when \( n \geq 6 \), \( (\tilde{G} : \tilde{C})_{p'} > 4q^{4n-10} \), and therefore \( \chi(1) \geq (G^* : C^*)_{p'} \geq (\tilde{G} : \tilde{C})_{p'}/4 > q^{4n-10} \). It is easy to see that \( |A|_{p'} \leq \prod_{i=1}^{n-m}(q^i - (-1)^i) \). Hence,

\[ (\tilde{G} : \tilde{C})_{p'} \geq \frac{2(q^2 - 1) \cdots (q^{2n} - 1)}{(q^n + 1)(q + 1)(q^2 - 1) \cdots (q^{n-m} - (-1)^{n-m})|GO^\pm_m(q^2)|} =: f(m, n). \]

We have \( f(0, n) = 2 \prod_{i=1}^n(q^i + (-1)^i)/(q^n + 1) > 4q^{4n-10} \) and \( f(n, n) > 4q^{4n-10} \) for every \( n \geq 6 \). Therefore, \( \chi(1) \geq (\tilde{G} : \tilde{C})_{p'}/4 \geq \min\{f(0, n), f(n, n)\}/4 > q^{4n-10} \) when \( n \geq 6 \).

Now let us consider the case \((n, \alpha) = (5, +)\). Note that \( GU_5(q) \) is not a subgroup of \( GO_{10}^+(q) \). Therefore, if \( A \neq GL_5(q) \), one can show that \( (\tilde{G} : \tilde{C})_{p'} > 4q^{10} \) and hence \( \chi(1) > q^{10} \). If \( A = GL_5(q) \), then the characteristic polynomial of \( s \) has the form

\[ P(x) = (x - \lambda)^5(x - \tau \lambda^{-1})^5 \]

with \( \lambda \in \mathbb{F}_q^* \) and \( \tau := \tau(s) \). Now we will show that

\[ (G^* : C^*)_{p'} = (\tilde{G}^0 : \tilde{C}^0)_{p'} \]

and therefore \( \chi(1) \geq (\tilde{G} : \tilde{C})_{p'}/2 = (q + 1)(q^2 + 1)(q^3 + 1)(q^4 + 1) > q^{10} \). Suppose that \( s_1, s_2 \in \tilde{G}^0 \) are \( \tilde{G}^0 \)-conjugate and their images in \( G^* \) are the same.

Then \( s_1 = \pm s_2 \). Let \( V_1 := \text{Ker}(s - \lambda) \) and \( V_2 := \text{Ker}(s - \tau \lambda^{-1}) \). Note that \( V_1 \) and \( V_2 \) are totally isotropic subspaces in \( V \) and \( V_1 \cap V_2 = \{0\} \). By definition, an element in \( \tilde{G}^0 \) cannot carry \( V_1 \) to \( V_2 \). This implies that \( s_1 = s_2 \). In other words, \( |s^*G^*| = |s\tilde{G}^0| \) and hence

\[ (G^* : C^*)_{p'} = (\tilde{G}^0 : \tilde{C}^0)_{p'} \]

as desired.

The next case is \((n, \alpha) = (5, -)\). Note that \( GL_5(q) \) is not a subgroup of \( GO_{10}^+(q) \). Therefore, if \( A \neq GU_5(q) \), one again can show that \( \chi(1) > q^{10} \).

If \( A = GU_5(q) \), then the characteristic polynomial of \( s \) has the form \( P(x) = [(x - \lambda)(x - \tau \lambda^{-1})]^5 \), where \( (x - \lambda)(x - \tau \lambda^{-1}) \) is an irreducible polynomial over \( \mathbb{F}_q \). Note that there are \( (q + 1)/2 \) choices for such a pair \((\lambda, \tau \lambda^{-1})\). So there are exactly \( (q + 1)/2 \) conjugacy classes \((s)\) in \( \tilde{G} \) such that \( \tilde{C} \simeq GU_5(q) \). That means there are exactly \( (q + 1) \) such conjugacy classes \((s)\) in

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\[ \tilde{G}^0 = \tilde{C} \] in this case. Now modulo \( Z \), we have exactly \((q + 1)/2\) conjugacy classes \((s^*)\) in \( G^* \). Arguing similarly as above, one has \((G^* : C^*)_{\nu'} = (\tilde{G}^0 : \tilde{C})_{\nu'} = (q - 1)(q^2 + 1)(q^3 - 1)(q^4 + 1)\) and therefore \( C^* = \tilde{C}^0 / Z \). The condition \( \chi(1) = (G^* : C^*)_{\nu'} \psi(1) \leq q^{10} \) implies that \( \psi(1) = 1 \). Since \( C^* \) has a unique unipotent character of degree 1, we get exactly \((q + 1)/2\) characters of degree \((q - 1)(q^2 + 1)(q^3 - 1)(q^4 + 1)\) in this case.

**Case 2:** If \( \tau(s) \) is a square in \( \mathbb{F}_q \), again, we can assume that \( \tau(s) = 1 \). By Lemma 5.1.5, \( \tilde{C} \cong (GO_{2k}^+(q) \times GO_{2m-2k}^+(q) \times \prod_{i=1}^{t} GL_{\alpha_i}^+(q^{k_i}))) / \mathbb{Z}_{q-1} \), where \( \alpha_i = \pm, \Sigma_{i=1}^{t} k_i a_i = n - m \).

It is easy to see that \(|GO_{2k}^+(q) \times GO_{2m-2k}^+(q)|_{\nu'} \leq 2 \prod_{i=1}^{m} (q^{2i} - 1)/(q^m - 1) \) if \( m \geq 1 \) and \(|\prod_{i=1}^{t} GL_{\alpha_i}^+(q^{k_i})|_{\nu'} \leq \prod_{i=1}^{n-m} (q^i - (-1)^i) \). Therefore, with convention that \( q^0 - 1 = 2 \), we have

\[
(\tilde{G} : \tilde{C})_{\nu'} \geq \frac{(q^{2n+1} - 1) \cdots (q^{2n} - 1)(q^m - 1)}{(q^n + 1)(q + 1)(q^2 - 1) \cdots (q^{n-m} - (-1)^{n-m})} =: g(m, n).
\]

1) When \( 1 \leq m \leq n - 2 \), \((\tilde{G} : C)_{\nu'} \geq \min\{g(1, n), g(n - 2, n)\} \). Direct computation shows that \( g(1, n) = (q^n - 1) \prod_{i=1}^{n-1} (q^i + (-1)^i)/(q + 1) > 4q^{n-10} \) and \( g(n - 2, n) = (q^{2n-2} - 1)(q^n - 1)(q^{n-2} - 1)/(q + 1)(q^2 - 1) > 4q^{n-10} \). Therefore, \( \chi(1) > q^{n-10} \).

2) When \( m = 0 \), then \( g(0, n) = 2 \prod_{i=1}^{n} (q^i + (-1)^i)/(q^n + 1) \). We still have \( g(0, n) = 4q^{4n-10} \) if \( n \geq 6 \). The case \((n, \alpha) = (5, +)\) can be argued similarly as when \( \tau(s) \) is not a square in \( \mathbb{F}_q \). So we only need to consider the case \((n, \alpha) = (5, -)\). Note that \( GL_5(q) \) is not a subgroup of \( GO_{10}^+(q) \). Therefore, if \( A \neq GU_5(q) \), one again can show that \( \chi(1) > q^{10} \). So we can assume \( A = GU_5(q) \). Then the characteristic polynomial of \( s \) has the form \( P(x) = [(x - \lambda)(x - \lambda^{-1})]^5 \), where \((x - \lambda)(x - \lambda^{-1})\) is an irreducible polynomial over \( \mathbb{F}_q \). Note that there are exactly \((q - 1)/2\) such a pair \((\lambda, \lambda^{-1})\). Repeating arguments as when \( \tau(s) \) is not a square in \( \mathbb{F}_q \), we get exactly \((q - 1)/2\) characters of degree \((q - 1)(q^2 + 1)(q^3 - 1)(q^4 + 1)\) in this case.

3) When \( m = n - 1 \). If \( 1 \leq k \leq n - 2 \) then \((\tilde{G} : \tilde{C})_{\nu'} \geq (q^{2n-2} - 1)(q^n - 1)(q^{n-2} - 1)/2(q + 1)^2 > 4q^{4n-10} \). Hence, \( \chi(1) \geq (G^* : C^*)_{\nu'} > q^{4n-10} \). So \( k = 0 \) or \( n - 1 \). With no loss, we can assume \( k = n - 1 \). Note that \( \text{Spec}(s) = \{1, ..., 1, \lambda, \lambda^{-1}\} \) in this case. If \( \{1, ..., 1, \lambda, \lambda^{-1}\} = \{\mu, ..., \mu, \mu \lambda, \mu \lambda^{-1}\} \) for some \( \mu \in \mathbb{F}_q^* \), then \( \mu = 1 \) since \( n \geq 5 \).
Therefore, \( \widetilde{C}^0 \) is the complete inverse image of \( C^* \) in \( \widetilde{G}^0 \). In other words, \( C^* = \widetilde{C}^0 / Z \) and hence \((G^* : C^*)_{\nu'} = (\widetilde{G}^0 : \widetilde{C}^0)_{\nu'} = (\widetilde{G} : \widetilde{C})_{\nu'} \). Also, since \( GL^\beta_2(q) \cong \Omega^\beta_2(q) \),

\[
\widetilde{C} \cong (GO_{2n-2}^\alpha(q) \times GL^\beta_1(q)) \cdot \mathbb{Z}_{q-1}.
\]

One can show that \( \widetilde{C}^0 \cong (SO_{2n-2}^\alpha(q) \times GL^\beta_1(q)) \cdot \mathbb{Z}_{q-1} \).

There are two cases:

- \( \widetilde{C} = (GO_{2n-2}^\alpha(q) \times GL^\beta_1(q)) \cdot \mathbb{Z}_{q-1} \) corresponding to the case \( \text{Spec}(s) = \{1, ..., 1, \lambda, \lambda^{-1}\} \), where \( \pm 1 \neq \lambda \in \mathbb{F}_q^\times \). Note that there are \((q - 3)/2\) choices for \( \lambda \), namely, \( \lambda = \gamma^k, k \in T_1 \). For each such \( \lambda \), there is exactly one semi-simple conjugacy class of such elements \( s \) in \( GO_{2n}^\alpha(q) \) by [73, (2.6)]. This conjugacy class is also the class of \( s \) in \( \widetilde{C}^0 \). Therefore, there is exactly one conjugacy class of semi-simple elements \( (s^*) \) in \( G^* \) such that \( \text{Spec}(s) = \{1, ..., 1, \lambda, \lambda^{-1}\} \) for each \( \lambda = \gamma^k, k \in T_1 \).

- \( \widetilde{C} = (GO_{2n-2}^\alpha(q) \times GU_1(q)) \cdot \mathbb{Z}_{q-1} \) corresponding to the case \( \text{Spec}(s) = \{1, ..., 1, \lambda, \lambda^{-1}\} \), where \( \pm 1 \neq \lambda \in \mathbb{F}_q^\times \) and \( \lambda^{-1} = \lambda^q \). Note that there are \((q - 1)/2\) choices for \( \lambda \), namely, \( \lambda = \eta^k, k \in T_2 \). Similarly as above, there is exactly one conjugacy class of semi-simple elements \( (s^*) \) in \( G^* \) such that \( \text{Spec}(s) = \{1, ..., 1, \lambda, \lambda^{-1}\} \) for each \( \lambda = \eta^k, k \in T_2 \).

Using Proposition 5.1.1 for the canonical homomorphism \( f : SO_{2n-2}^\alpha(q) \times GL^\beta_1(q) \hookrightarrow \widetilde{C}^0 \rightarrow C^*, \beta = \pm \), we see that the unipotent characters of \( SO_{2n-2}^\alpha(q) \times GL^\beta_1(q) \) are of the form \( \psi \circ f \), where \( \psi \) runs over the unipotent characters of \( C^* \). In particular, by Propositions 7.1, 7.2 of [70], \( \psi \) is trivial or \( \psi(1) \geq (q^{n-1} - \alpha \beta)(q^{n-2} + \alpha \beta q)/(q^2 - 1) \). In the latter case,

\[
\chi(1) \geq \frac{(q^n - \alpha)(q^{n-1} + \alpha \beta)}{q - \beta} \cdot \frac{(q^{n-1} - \alpha \beta)(q^{n-2} + \alpha \beta q)}{q^2 - 1} > q^{4n-10}.
\]

Therefore, in this case, \( \chi \) is one of \((q - 3)/2\) representations of degree \((q^n - \alpha)(q^{n-1} + \alpha)/(q - 1)\) or \((q - 1)/2\) representations of degree \((q^n - \alpha)(q^{n-1} - \alpha)/(q + 1)\).

4) When \( m = n \). Since \( (s^*) \) is non-trivial, we assume \( 1 \leq k \leq n - 1 \).

If \( k = 1 \) or \( n - 1 \), modulo \( Z \), we may assume \( \text{Spec}(s) = \{-1, -1, 1, ..., 1\} \). Then \( \widetilde{C} = (GO^\beta_2(q) \times GO_{2n-2}^\alpha(q)) \cdot \mathbb{Z}_{q-1} \) and \( \widetilde{C}^0 \cong (SO^\beta_2(q) \times SO_{2n-2}^\alpha(q)) \cdot 2 \cdot \mathbb{Z}_{q-1} \). Again, \( C^* = \widetilde{C}^0 / Z \) and \((G^* : C^*)_{\nu'} = (\widetilde{G} : \widetilde{C})_{\nu'} \). There is a unique possibility for \((s^*) \) for each
\[ \beta = \pm. \]  We have \[ \chi(1) = \frac{(q^n - \alpha)(q^{n-1} + \alpha \beta)}{2(q - \beta)} \psi(1) \] where \( \psi \) is a unipotent character of \( C^* \).

Consider the canonical homomorphism \( f : (SO_2^\beta(q) \times SO_{2n-2}^{\alpha\beta}(q)) \cdot 2 \hookrightarrow \tilde{C}^0 \to C^* \), whose kernel is contained in the center of \((SO_2^\beta(q) \times SO_{2n-2}^{\alpha\beta}(q)) \cdot 2\) and image contains the commutator group of \( C^* \) since \((SO_2^\beta(q) \times SO_{2n-2}^{\alpha\beta}(q)) \cdot 2\) contains the commutator group of \( \tilde{C}^0 \). By Proposition 5.1.1, the unipotent characters of \((SO_2^\beta(q) \times SO_{2n-2}^{\alpha\beta}(q)) \cdot 2\) are of the form \( \psi \circ f \), where \( \psi \) runs over the unipotent characters of \( C^* \). In particular, \( \psi \) is one of two linear unipotent characters or \( \psi(1) \geq (q^{n-1} - \alpha \beta)(q^{n-2} + \alpha \beta q)/(q^2 - 1) \). In the latter case,

\[ \chi(1) \geq \frac{(q^n - \alpha)(q^{n-1} + \alpha \beta)}{2(q - \beta)} \cdot \frac{(q^{n-1} - \alpha \beta)(q^{n-2} + \alpha \beta q)}{q^2 - 1} > q^{4n-10}. \]

If \( 2 \leq k \leq n - 2 \), then

\[
\chi(1) \geq \frac{\tilde{G} : \tilde{C}}{4} = \frac{|GO_{2n}^\alpha(q)|}{4|GO_4^\beta(q)| \cdot |GO_{2n-4}^{\alpha\beta}(q)|} = \frac{(q^{2n-2} - 1)(q^n - \alpha)(q^{n-2} + \alpha \beta)(q^2 + \beta)}{8(q^2 - 1)(q^4 - 1)} > q^{4n-10}.
\]

\[ \Box \]

### 6.4 Groups \( \text{Spin}_{12}^+(3) \)

In this section, we classify the irreducible complex characters of \( G = \text{Spin}_{12}^+(3) \), \( \alpha = \pm \), of degrees up to \( 4 \cdot 3^{15} \). The arguments in this section are similar to those in §6.3 and we will keep all notation from there. Two following Lemmas can be checked by direct computation.

**Lemma 6.4.1.** Suppose that \( \chi \in \text{Irr}(P(CO_{12}^-(3)^0)) \) is unipotent. Then either \( \chi \) is one of the characters labeled by \((0, 6), (1, 5), (2, 4), (0, 1, 6), (0, 1, 2, 5)\), or \( \chi(1) > 4 \cdot 3^{15} \).

**Lemma 6.4.2.** Suppose that \( \chi \in \text{Irr}(P(CO_{12}^+(3)^0)) \) is unipotent. Then either \( \chi \) is one of the characters labeled by \((0, 6), (1, 5), (4, 2), (3, 3) \) (2 characters), \((0, 1, 6), (0, 1, 2, 5)\), or \( \chi(1) > 4 \cdot 3^{15} \).

**Proposition 6.4.3.** \( \text{Spin}_{12}^+(3) \) has exactly 28 irreducible complex characters of degrees less than \( 4 \cdot 3^{15} \) and \( \text{Spin}_{12}^-(3) \) has exactly 16 irreducible complex characters of degrees less than \( 4 \cdot 3^{15} \).
Proof. Let $\chi$ be an irreducible complex character of $G$ of degree less than $4 \cdot 3^{15}$. Since we have already counted the number of unipotent characters in two above lemmas, we can assume that $\chi$ is not unipotent. Suppose that $\chi$ is parametrized by $((s^*), \psi)$, where $1 \neq s^* \in G^* := P(CO_{12}^0(3)^0)$ and $\psi \in \text{Irr}(C^*)$, $C^* = C_{G^*}(s^*)$. Recall that $\tilde{G} := CO_{12}^0(3)$ and $\tilde{G}^0 := CO_{12}^0(3)^0$. Denote $Z := Z(\tilde{G}) = Z(\tilde{G}^0)$ the center of $\tilde{G}$ as well as $\tilde{G}^0$. Let $s$ be an inverse image of $s^*$ in $\tilde{G}^0$. Set $\tilde{C} = C_{\tilde{G}}(s)$ and $\tilde{G}^0 = C_{\tilde{G}^0}(s)$.

**Case 1**: $\tau(s) = -1$. Then we have $\tilde{C} \simeq (GO_m^\pm(9) \times \prod_{i=1}^t GL_{a_i}^\pm(3^{k_i})) \cdot \mathbb{Z}_2$, where $\alpha_i = \pm, \Sigma_{i=1}^t k_i = m = 6$. Since $\chi(1) < 4 \cdot 3^{15}$, $(\tilde{C} : \tilde{C})_3 < 16 \cdot 3^{15}$. This inequality happens only when $\tilde{C} \simeq GL_6^\pm(3) \cdot \mathbb{Z}_2$. This forces $G = Spin^+_{12}(3)$. Let us consider the case $\tilde{C} \simeq GL_6(3) \cdot \mathbb{Z}_2$. Note that there is a unique conjugacy class $(s)$ of semi-simple elements in $\tilde{G}$ so that $\tilde{C} \simeq GL_6(3) \cdot \mathbb{Z}_2$, which is happened when the characteristic polynomial of $s$ is $(x^2 - 1)^6$. In this case, $\tilde{C}^0 = \tilde{C}$, $(\tilde{G} : \tilde{C}) = 2(\tilde{G}^0 : \tilde{C}^0)$, and therefore there are two conjugacy classes of semi-simple elements in $\tilde{G}^0$, as well as in $G^*$, such that $\tilde{C} \simeq GL_6(3) \cdot \mathbb{Z}_2$. Furthermore, $C^*$ is an extension by 2 of $\tilde{C}/Z$. Therefore, $\chi$ is one of 4 characters of degree $(G^* : C^*)_3 = \frac{1}{2} \prod_{i=1}^5 (3^i + 1)$, provided that $G = Spin^+_{12}(3)$.

Next, we consider the case $\tilde{C} \simeq GU_6(3) \cdot \mathbb{Z}_2$. Note that there are two semi-simple conjugacy classes $(s)$ in $\tilde{G}$ so that $\tilde{C} \simeq GL_6(3) \cdot \mathbb{Z}_2$, which is happened when the characteristic polynomial of $s$ is $(x^2 + x - 1)^6$ or $(x^2 - x - 1)^6$. In this case, $\tilde{C}^0 = \tilde{C}$, $(\tilde{G} : \tilde{C}) = 2(\tilde{G}^0 : \tilde{C}^0)$, and therefore there are 4 such conjugacy classes of semi-simple elements in $\tilde{G}^0$. Modulo $Z$, we get exactly two semi-simple conjugacy classes $(s^*)$ in $G^*$. Furthermore, $C^*$ is an extension by 2 of $\tilde{C}/Z$. Therefore, we get two characters of degree $\prod_{i=1}^5 (3^i + (-1)^i)$ of $G = Spin^+_{12}(3)$.

**Case 2**: $\tau(s) = 1$. We have $\tilde{C} \simeq (GO_{2k}^\pm(3) \times GO_{2m-2k}^\pm(3) \times \prod_{i=1}^t GL_{a_i}^\pm(3^{k_i})) \cdot \mathbb{Z}_2$, where $\alpha_i = \pm, \Sigma_{i=1}^t k_i = m = 6$. The inequality $\chi(1) < 4 \cdot 3^{15}$ implies that $m = 0, 5, 6$. If $m = 0$, similarly as in case 1, one can show that $\tilde{C} \simeq GU_6(3) \cdot \mathbb{Z}_2$ and $C^*$ is an extension by 2 of $\tilde{C}/Z$. Hence, $\chi$ is one of 4 characters of degree $\frac{1}{2} \prod_{i=1}^5 (3^i + (-1)^i)$, provided that $G = Spin^+_{12}(3)$.
When $m = 5$, it is easy to show that $(G^* : C^*)_\gamma = (\tilde{G} : \tilde{C})_\gamma$. If $k = 0$ or 5, we have already shown in the proof of Theorem I that either $\chi$ is the unique character of degree $(3^6 - \alpha)(3^5 - \alpha)/4$ or

$$\chi(1) \geq \frac{(3^6 - \alpha)(3^5 - \alpha)}{4} \cdot \frac{(3^5 + \alpha)(3^4 - \alpha 3)}{8} > 4 \cdot 3^{15}.$$  

If $1 \leq k \leq 4$, $\chi(1) \geq (\tilde{G} : \tilde{C})_\gamma \geq (3^{10} - 1)(3^6 - 1)(3^4 - 1)/32 > 4 \cdot 3^{15}$.

Now we suppose $m = 6$. First, if $k = 1$ or 5, we have $\chi = \psi(1)(3^6 - \alpha)(3^5 + \alpha \beta)/2(3 - \beta)$, where $\beta = \pm$ and $\psi$ is a unipotent character of $C^* = \tilde{C}^0/Z$, $\tilde{C}^0 \simeq (SO_2^\beta(3) \times SO_{10}^\alpha(3)) \cdot 2 \cdot \mathbb{Z}_2$. When $\beta = +$, the inequality $\chi(1) < 4 \cdot 3^{15}$ forces $\psi$ to be linear. Therefore, $\chi$ is one of 2 characters of degree $(3^6 - \alpha)(3^5 + \alpha)/4$. When $\beta = -$, note that the unipotent characters of $(SO_2^-(3) \times SO_{10}^{-\alpha}(3)) \cdot 2$ are of the form $\psi \circ f$, where $f$ is the canonical homomorphism $f : (SO_2^-(3) \times SO_{10}^{-\alpha}(3)) \cdot 2 \to C^*$. Since $SO_2^-(3)$ has a unique unipotent character which is trivial, the unipotent characters of $(SO_2^-(3) \times SO_{10}^{-\alpha}(3)) \cdot 2$ are actually unipotent characters of $GO_{10}^{-\alpha}(3)$. Therefore, $\psi$ is one of 2 unipotent characters of degree 1; 2 unipotent characters of degree $(3^5 + \alpha)(3^4 - \alpha 3)/8$; or $\psi(1) \geq (3^{10} - 3^2)/8$.

Hence there are 4 characters of degrees less than $4 \cdot 3^{15}$ in this case. Next, if $k = 2$ or 4, we have

$$\chi(1) = \frac{(3^4 + \alpha \beta)(3^{10} - 1)(3^6 - \alpha)}{2(3^2 - 1)(3^2 - \beta)} \cdot \psi(1),$$

where $\psi$ is a unipotent character of $C^* = \tilde{C}^0/Z(\tilde{G})$, $\tilde{C}^0 \simeq (SO_4^\beta(3) \times SO_8^\alpha(3)) \cdot 2 \cdot \mathbb{Z}_2$. Since $\chi(1) < 4 \cdot 3^{15}$, $\psi$ is one of two linear unipotent characters of $C^*$ for each $\beta = \pm$. This gives 4 more characters of degrees less than $4 \cdot 3^{15}$. Finally, if $k = 3$, then $\chi(1) \geq (G^* : C^*)_\gamma \geq (\tilde{G} : \tilde{C})_\gamma/2 > 4 \cdot 3^{15}$.  

□
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