BEREZIN-TOEPLITZ QUANTIZATION BY WIENER-REGULARIZED PATH INTEGRALS

By

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Dedicated to my parents, with love and gratitude.

La Nature est un temple où de vivants piliers
Laissent parfois sortir de confuses paroles;
L'homme y passe à travers des forêts de symboles
Qui l'observent avec des regards familiers.

Charles Baudelaire, Correspondances [Bau99]

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TABLE OF CONTENTS

Abstract of Dissertation Presented to the Graduate School
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This dissertation investigates a class of operators resulting from a quantization scheme attributed to Berezin. These so-called Berezin-Toeplitz operators are defined on a Hilbert space of square-integrable holomorphic sections in a line bundle over the classical phase space.

A first concern is the construction of self-adjoint Berezin-Toeplitz operators associated with semibounded quadratic forms. These forms are obtained from the inner product of the Hilbert space by multiplying the underlying measure with sufficiently regular real-valued functions. The semigroups generated by the associated self-adjoint Berezin-Toeplitz operators may under certain conditions be represented in the form of Wiener-regularized path integrals, according to a concept by Daubechies and Klauder. More explicitly, the integration is taken over Brownian-motion paths in phase space in the ultra-diffusive limit. Finally, the probabilistic representation and an invariance property of Brownian motion are combined to yield a relation between resolvents of different Berezin-Toeplitz operators.

All results are the consequence of a relation between Berezin-Toeplitz operators and Schrödinger operators defined via certain quadratic forms. The probabilistic representation is derived in conjunction with a version of the Feynman-Kac formula.

CHAPTER 1
INTRODUCTION

1.1 Historical Notes on Path-Integral Quantization

The development of quantum theory in the 20^{th} century had to bridge a gap between small and large scale physics. On one end was the empirically driven demand for models explaining processes on an atomic or even subatomic level, on the other was the requirement to be consistent with classical physics. After all, a fundamental theory should explain our immediate impression of the world around us and therefore it should allow for an effective classical description above a certain scale. Such a requirement is called a correspondence principle, and the art of constructing quantum models with a prescribed asymptotic behavior is henceforth referred to as quantization. Apart from being conceptually desirable, a correspondence principle addresses the practical need to successively calculate more and more corrections to the classical predictions as the relevant scale approaches the quantum regime. For this purpose, it is an important part of a quantization procedure to state precisely in which sense the prescribed asymptotics are satisfied.

Traditionally, quantization schemes promote classical observables to self-adjoint operators on a Hilbert space and interpret the one-dimensional subspaces as possible states of the quantum system. This determines the kinematical framework. The quantization of dynamics is then accomplished via the one-parameter unitary group generated by the self-adjoint operator that serves as a quantum analogue of the classical Hamiltonian. The combination of Schrödinger's and Heisenberg's approaches has in the course of time been molded into the concept of canonical quantization with its characteristic features: The underlying Hilbert space consists of square-integrable functions on the classical configuration space, and is interpreted as the spectral representation of the self-adjoint position operator. The dynamics are derived from a partial differential equation commonly known as Schrödinger's equation.

In 1948, Feynman [Fey48] proposed an alternative way to solve Schrödinger's equation, which did not require the explicit appearance of operators, but instead involved the concept of path integration. Unfortunately, despite its usefulness in physics and other disciplines [FH65, Das79, Sch81, Wit89, Roe94, Kle95, CGT$^+$99], the mathematical justification of the form originally suggested by Feynman is somewhat unsatisfactory. It involves a time-slicing regularization that realizes the unitary time evolution according to Schrödinger's equation in an expression related to the Lie-Trotter formula, see [Tob56, Nel64a, Che68, RS80]. However, this expression does not permit an interpretation as a bona fide integral over continuous paths in configuration space. It was Kac [Kac49] who established this goal in a seemingly insignificant but far-reaching modification of Feynman's formula. Instead of the unitary group, he represented the self-adjoint semigroup generated by a Schrödinger operator in terms of a well-defined integral with respect to a probability measure that had already been introduced by Wiener [Wie23] in the study of Brownian motion. Today, Feynman's approach and its modification by Kac are an integral part of the canonical description of quantum systems.

Despite the experimental verification of many models, it was puzzling that canonical quantization relied on a particular choice of coordinates for its realization [Dir56, footnote on page 114]. This situation called for a coordinate-independent reformulation of quantization prescriptions. A systematic program was proposed in the theory of geometric quantization founded by Souriau [Sou66] and Kostant [Kos70]. Following their approach, a coordinate-independent path-integral representation of quantum dynamics in the spirit of Feynman's original time-slicing procedure was developed by Blattner, Guillemin, and Sternberg [Bla75, GS77], see also [Śni80].

Yet another method to construct quantum models was introduced by Berezin [Ber72, Ber74]. He realized these models with the help of certain continuous representations in the sense of Klauder [Kla63a, Kla63b, Kla64, MK64, KM65, KMC65], more specifically by using spaces of holomorphic functions on phase-space manifolds with a Kähler structure. Rawnsley and others [CGR90, CGR93, CGR94, BMS94, CGR95] subsequently cast Berezin's construction in a manifestly coordinate-independent form by borrowing ideas from geometric quantization. In this form the underlying Hilbert space consists of square-integrable, holomorphic sections in a holomorphic line bundle. A Berezin-Toeplitz operator T_f on such a Hilbert space is characterized by its associated sesquilinear form, which is obtained by multiplying the measure in the inner product of the Hilbert space with a sufficiently regular real-valued function f. The quantization context arises from interpreting this function as a classical observable that is in some sense in correspondence with T_f. Indeed, one may prove that the precise notion of a correspondence principle applies in the case of homogeneous or compact Kähler manifolds, see [Ber74, Per86] or [BMS94, Sch98]. Reformulating ideas of Berezin [Ber72] in Rawnsley's fashion, one arrives at a coordinate-independent path integral similar to the construction of Blattner, Kostant, and Sternberg.

1.2 Scope of This Work

The main objective of this dissertation is to generalize an approach to path-integral quantization proposed by Daubechies and Klauder [DK82, KD82, KD84, DK85, DK86]; see also [DKP87]. Superficially, it is a phase-space version of Feynman's path integral that has been rendered mathematically well-defined by a Wiener-measure regularization. However, a closer look shows that the construction by Daubechies and Klauder can be understood as a path-integral formulation of Berezin-Toeplitz quantization on certain homogeneous Kähler manifolds. Indeed, a generalization to arbitrary Kähler manifolds has been advocated in several publications [KO89, AKL93, Kla94, AK96] and carried out for the compact case by Charles [Cha99]. The advocated generalization is a probabilistic expression for the unitary group $\{e^{-itT_f}\}_{t\in\mathbb{R}}$ generated by a Berezin-Toeplitz operator T_f. More precisely, a Wiener-regularized path integral expresses the integral kernel of the time-evolution operator e^{-itT_f} as the ultra-diffusive limit of an expectation value over Brownian motion paths on the classical phase space. It is both comforting and convenient that the expression for $\{e^{-itT_f}\}_{t\in\mathbb{R}}$ is entirely geometric in nature and opens up a wealth of analytic tools from the extensively studied background of Brownian motion.

In contrast to the setting considered by Charles [Cha99], we include the case of unbounded Berezin-Toeplitz operators and non-compact manifolds in our results, subject to certain technical conditions. In addition, we show that the Wiener regularization may not only be realized with the Brownian motion governed by the original Kähler metric, but also by those obtained from a conformal rescaling, at the cost of adjusting the path measure with a suitable Feynman-Kac functional. A minor difference with the original intent of Daubechies and Klauder and its advocated generalizations [KO89, AKL93, Kla94, AK96] is that instead of unitary groups, we focus on the probabilistic representation of semigroups $\{e^{-tT_f}\}_{t\geq 0}$ that are generated by self-adjoint, semibounded Berezin-Toeplitz operators.

1.3 Structure and Contents

This dissertation contains three major parts. In the beginning, we study conditions for the self-adjointness and semiboundedness of Berezin-Toeplitz operators, which may then serve as generators of strongly continuous self-adjoint semigroups. The middle part establishes the stochastic formulation expressing these semigroups as Wiener-regularized path integrals. We finish the text by applying the probabilistic technique of path transformations in this setting.

The chapters of this dissertation have the following contents:

In Chapter 2, we review some elements of differential geometry that are needed in the coordinate-independent formulation of Berezin-Toeplitz quantization given in Chapter 3.

There, we show that a class of coherent states is essential to the understanding of this quantization scheme. After defining Berezin-Toeplitz operators in terms of semibounded quadratic forms, we give an abstract condition for their self-adjointness.

Chapter 4 establishes a relationship between Berezin-Toeplitz and Schrödinger operators, which makes standard techniques from the context of differential operators available to formulate more concrete conditions ensuring the self-adjointness of a Berezin-Toeplitz operator.

The main topic of Chapter 5 is the probabilistic representation of semigroups generated by self-adjoint, semibounded Berezin-Toeplitz operators. This result is called the Daubechies-Klauder formula. It is derived from a version of the Feynman-Kac formula for Schrödinger operators.

Chapter 6 explores a consequence of an invariance property of Brownian motion in the probabilistic representation of the preceding chapter. A change of variables in an expectation with respect to Brownian motion relates the expressions for the resolvents of different Berezin-Toeplitz operators.

Finally, we summarize the results in Chapter 7 and conclude with an outlook on further developments.

CHAPTER 2
BASIC DEFINITIONS AND CONCEPTS OF COMPLEX DIFFERENTIAL GEOMETRY

This chapter reviews some basic definitions and conventions that can be found in standard textbooks and monographs on differential geometry [KN63, LM89, Zha00]. The content prepares the discussion in the following chapters, with the main intent of fixing the differential geometric notation and terminology. Two passages at the end of this chapter deserve most of the attention. Both concern the concept of a connection in a holomorphic line bundle with a Hermitian structure. The first passage shows uniqueness of a connection that meets certain compatibility requirements. The second one regards the construction of a horizontal transport related to the connection.

2.1 Elements of Linear Algebra

The vector-space notation introduced here aims at one particular example: the tangent space of a point in a complex manifold.

Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, respectively. We will tacitly identify $(z_1, z_2) \in \mathbb{R}^2$ and $z = z_1 + iz_2 \in \mathbb{C}$, with $i := \sqrt{-1}$ denoting the imaginary unit. The number $\overline{z} := z_1 - iz_2$ is the complex conjugate of z, the real and imaginary parts of z are sometimes written as $\mathfrak{Re}\, z := (z + \overline{z})/2$ and $\mathfrak{Im}\, z := (z - \overline{z})/2i$.

2.1.1. Definition. Let $\mathbb{V}$ be a finite-dimensional vector space over $\mathbb{R}$ with an inner product $(\cdot, \cdot)$. An almost complex structure that is compatible with the inner product is a linear mapping J on $\mathbb{V}$ that squares to the negative identity, $J^2 = -\mathrm{id}_{\mathbb{V}}$, and preserves the inner product of all $X, Y \in \mathbb{V}$, $(JX, JY) = (X, Y)$.

2.1.2. Remark. If a vector space $\mathbb{V}$ over $\mathbb{R}$ has $d < \infty$ dimensions and is equipped with an inner product and a compatible almost complex structure J, then d is an even number and by a variation of the Gram-Schmidt procedure, one can choose an orthonormal basis of $\mathbb{V}$ in the form $\{E_1, JE_1,\, E_2, JE_2, \ldots,\, E_{d/2}, JE_{d/2}\}$. This is a simple consequence of the orthogonality relation $X \perp JX$ for all $X \in \mathbb{V}$.

With the help of the almost complex structure, $\mathbb{V}$ can be turned into a vector space over $\mathbb{C}$ by defining the scalar multiplication of $X \in \mathbb{V}$ by $c \in \mathbb{C}$ according to the rule $c\,X = \mathfrak{Re}\,c\,X + \mathfrak{Im}\,c\,JX$. Thus, after choosing an orthonormal basis in the above manner, the coordinate maps identify $\mathbb{V}$ with $\mathbb{C}^{d/2}$.

2.1.3. Definition. On the complexified vector space $\mathbb{V}^{\mathbb{C}} := \mathbb{V} \otimes_{\mathbb{R}} \mathbb{C}$, we extend the inner product to be conjugate linear in its first entry. The mapping $J \otimes \mathrm{id}_{\mathbb{C}}$, henceforth also denoted by J, can be diagonalized in $\mathbb{V}^{\mathbb{C}}$ and possesses eigenvalues $+i$ and $-i$ with equal degeneracy. We call $\mathbb{V}^{(1,0)} := \{X \in \mathbb{V}^{\mathbb{C}} : JX = iX\}$ the holomorphic polarization of $\mathbb{V}^{\mathbb{C}}$, and the orthogonal complement $\mathbb{V}^{(0,1)}$ containing vectors corresponding to the eigenvalue $-i$ the antiholomorphic polarization. The image of a vector $X \in \mathbb{V}^{\mathbb{C}}$ under the orthogonal projection onto the holomorphic polarization $\mathbb{V}^{(1,0)}$ is written as $X^{(1,0)}$, and a similar notation applies to the antiholomorphic case. Thus, X can be uniquely decomposed into its components $X = X^{(1,0)} + X^{(0,1)}$.

2.1.4. Remark. In terms of an orthonormal basis $\{E_1, JE_1, E_2, JE_2, \ldots,\, E_{d/2}, JE_{d/2}\}$ as in the preceding remark, the holomorphic polarization $\mathbb{V}^{(1,0)}$ is spanned by the orthonormal set $\{Z_k := \frac{1}{\sqrt{2}}(E_k - iJE_k),\, 1 \le k \le d/2\}$ and $\mathbb{V}^{(0,1)}$ analogously by $\{\overline{Z}_k := \frac{1}{\sqrt{2}}(E_k + iJE_k),$ $1 \le k \le d/2\}$. In fact, this notation suggests the definition of a conjugation operation $X \mapsto \overline{X}$ on $\mathbb{V}^{\mathbb{C}}$ which is the conjugate-linear mapping taking each Z_k to its counterpart $\overline{Z}_k$.

2.1.5. Definition. A symplectic form on a real vector space $\mathbb{V}$ is a non-degenerate antisymmetric bilinear form $\omega : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$. Non-degeneracy means in this context that if $X \in \mathbb{V}$ and $\omega(X,Y) = 0$ for all $Y \in \mathbb{V}$, then $X = 0$.

Remark. With an inner product $(\cdot,\cdot)$ and a compatible almost complex structure J, we can always associate a J-invariant symplectic form having the value $\omega(X,Y) := (X, JY) = (JX, J^2Y) = -\omega(X,Y)$ for $X, Y \in \mathbb{V}$.

2.2 From Differentiable Manifolds to Holomorphic Line Bundles

Whenever referring to $\mathbb{R}$ and $\mathbb{C}$ as topological spaces, the usual definition of open sets is implied. In this chapter, all functions are tacitly assumed to be measurable, each one in its appropriate sense. The characteristic function χ_A of a set A has the value one whenever the argument is contained in A and vanishes otherwise. A complex-valued function f defined on d-dimensional Euclidean space $\mathbb{R}^d$ is said to be smooth, abbreviated by $f \in C^\infty(\mathbb{R}^d)$, if it is arbitrarily often differentiable. If f is smooth and real-valued, we write $f \in C^\infty_{\mathbb{R}}(\mathbb{R}^d)$. Without any exception, **repeated indices are never summed over automatically**.

2.2.1. Definition. A topological manifold is a Hausdorff space that has a countable topological basis and is locally homeomorphic to $\mathbb{R}^d$. A coordinate chart on a topological manifold $\mathcal{M}$ is a pair (U, ϕ) consisting of an open set $U \subset \mathcal{M}$ and a homeomorphism $\phi : U \to V \subset \mathbb{R}^d$. An atlas is a family of charts $\{(U_j, \phi_j)\}_{j\in I}$ covering $\mathcal{M}$, $\bigcup_{j\in I} U_j = \mathcal{M}$.

2.2.2. Definition. A differentiable manifold is a topological manifold equipped with a C^∞-structure. This structure is specified by an atlas $\mathcal{A} = \{(U_j, \phi_j)\}_{j\in I}$ containing charts $\phi_j : U_j \to V_j \subset \mathbb{R}^d, d \in \mathbb{N}$, that are C^∞-compatible in the sense that for each pair $j, k \in I$, the composition $\phi_j \circ \phi_k^{-1}$ is either defined on the empty set or a diffeomorphism. The common exponent d is called the dimension of the manifold $\mathcal{M}$.

2.2.3. Convention. Without loss of generality, we will always assume that the index set I of an atlas is countable, and that for a given atlas, each point is contained in the domains of only finitely many charts, which are all contractible sets. Moreover, we exclusively consider pathwise connected manifolds.

2.2.4. Definition. Let $\mathcal{M}$ be a differentiable manifold with an atlas $\{(U_j, \phi_j)\}_{j\in I}$. A function $f : \mathcal{M} \to \mathbb{C}$ is called smooth if for each $j \in I$, the composition $f \circ \phi_j^{-1}$ is arbitrarily often differentiable on the image $\phi_j(U_j)$.

Given an additional differentiable manifold $\mathcal{M}'$ and a mapping $\Phi : \mathcal{M}' \to \mathcal{M}$, then Φ is said to be smooth if this applies to the composition $f \circ \Phi$ for every smooth function f on $\mathcal{M}$.

2.2.5. Definition. Let $\pi : \mathcal{V} \to \mathcal{M}$ be a smooth, surjective mapping between differentiable manifolds $\mathcal{V}$ and $\mathcal{M}$. We say $(\mathcal{V}, \pi)$ is a vector bundle over $\mathcal{M}$ with the real vector space $\mathbb{F}$ as a typical fiber if the following conditions are satisfied:

- The set $\mathcal{V}_x := \pi^{-1}(\{x\})$ is called the fiber associated with the base point $x \in \mathcal{M}$. Every fiber has a vector space structure isomorphic to $\mathbb{F}$, symbolically $\pi^{-1}(\{x\}) \cong \mathbb{F}$ for all $x \in \mathcal{M}$.
- Every point x in the base manifold $\mathcal{M}$ is contained in an open set U carrying a local trivialization $\xi : \pi^{-1}(U) \to V \times \mathbb{F}$. This means, ξ is a diffeomorphism, V is an open set in $\mathbb{R}^d$, and for all $y \in U$, the restriction $\xi|_{\pi^{-1}(\{y\})}$ assumes a constant value in the first component of $V \times \mathbb{F}$ and maps linearly onto the second. As usual, the exponent d is fixed and denotes the dimension of the manifold $\mathcal{M}$.

Unless there is danger of confusion, the symbol $\mathcal{V}$ refers henceforth not only to the so-called total space as a manifold, but to the vector bundle $(\mathcal{V}, \pi)$ as a whole.

Remark. Since we view π in a local trivialization $\xi : \pi^{-1}(U) \to V \times \mathbb{F}$ as the projection on the first component V, a local trivialization is, in short, a special chart of $\mathcal{V}$ that respects the fiber structure.

2.2.6. Definition. A section Σ in a vector bundle $\mathcal{V}$ is a mapping that assigns to each $x \in \mathcal{M}$ a vector in the associated fiber $\pi^{-1}(\{x\})$, so the composition $\pi \circ \Sigma$ is the identity map $\mathrm{id}_{\mathcal{M}}$ on $\mathcal{M}$. We denote the linear space of all such sections by $\Gamma_{\mathcal{V}}(\mathcal{M})$ and that of smooth sections by $C^\infty_{\mathcal{V}}(\mathcal{M})$. The space of compactly supported smooth sections is written as $C^\infty_{c\mathcal{V}}(\mathcal{M})$.

The typical example of a vector bundle is the tangent bundle of a differentiable manifold.

2.2.7. Definition. Let $\mathcal{M}$ be a differentiable manifold. A tangent vector X in the tangent space $T_x\mathcal{M}$ of a point $x \in \mathcal{M}$ is a first-order differential operator $X : C^\infty_{\mathbb{R}}(\mathcal{M}) \to \mathbb{R}$ without zero-order terms. Equivalently, it satisfies $(Xf^2)(x) = 2f(x)(Xf)(x)$ for all real-valued, smooth functions f on $\mathcal{M}$. If the manifold $\mathcal{M}$ has the dimension $d \in \mathbb{N}$, then $T_x\mathcal{M}$ is a vector space isomorphic to $\mathbb{R}^d$.

The tangent bundle $T\mathcal{M}$ over the manifold $\mathcal{M}$ is the vector bundle with fibers $T_x\mathcal{M}$. The space of smooth sections in $T\mathcal{M}$ is for convenience abbreviated as $\Upsilon_{\mathbb{R}}(\mathcal{M}) := C^\infty_{T\mathcal{M}}(\mathcal{M})$.

2.2.8. Definition. An l-form ω on a differentiable manifold $\mathcal{M}$ is given by a smooth family $\{\omega_x\}_{x\in\mathcal{M}}$ of linear maps $\omega_x : (T_x\mathcal{M})^{\otimes l} \to \mathbb{R}$ defined on the l-th tensor power of $T_x\mathcal{M}$, which are antisymmetric with respect to permutation of the factor spaces. The non-negative integer l is called the degree of ω. If $l = 0$ then ω is by convention a real-valued function.

2.2.9. Definition. The exterior product $\wedge$ maps a pair of a l_1-form ω_1 and a l_2-form ω_2 to the $(l_1 + l_2)$-form $\omega_1 \wedge \omega_2 = (-1)^{l_1 l_2}\omega_2 \wedge \omega_1$. The only property used in this work is the expression $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_l(X_1, X_2 \ldots X_l) = \det(\langle\alpha_j, X_k\rangle)$ involving the exterior product of $l \in \mathbb{N}$ one-forms $\alpha_1, \ldots \alpha_l$ applied to l tangent vectors $X_1, X_2, \ldots X_l$, given by the determinant of canonical pairings $\langle\alpha_j, X_k\rangle := \alpha_j(X_k)$, with $j, k \in \{1, \ldots l\}$.

2.2.10. Definition. The exterior derivative d on a differentiable manifold is a linear mapping from l to $(l+1)$-forms satisfying the following axioms, valid for all l_1-forms ω_1, l_2-forms ω_2, functions $f \in C^\infty_{\mathbb{R}}(\mathcal{M})$, and tangent vectors X:

- $df(X) = X(f)\,,$
- $d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^{l_1}\omega_1 \wedge d\omega_2\,,$
- $d(d\omega_1) = 0\,.$

At this point, we turn our attention to complex geometry. The manifolds and vector bundles we have considered so far will now be endowed with complex structures.

2.2.11. Convention. The complexified tangent bundle of the manifold $\mathcal{M}$ is denoted by $T^{\mathbb{C}}\mathcal{M}$. It is the vector bundle $T^{\mathbb{C}}\mathcal{M} \to \mathcal{M}$ with fibers $T^{\mathbb{C}}_x\mathcal{M} \cong \mathbb{R}^{2d}$ at each $x \in \mathcal{M}$. The smooth sections in $T^{\mathbb{C}}\mathcal{M}$ are denoted by $\Upsilon(\mathcal{M}) := C^\infty_{T^{\mathbb{C}}\mathcal{M}}(\mathcal{M})$.

From now on, differential forms are allowed to assume complex values. When they are applied to vectors in the complexified tangent space, they are by default complex linear in each argument. Similarly, the exterior product and derivative are henceforth defined for complex-valued forms.

2.2.12. Definition. A complex analytic atlas $\{(U_j, \phi_j)\}_{j\in I}$ is a family of charts $\phi_j : U_j \to V_j \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, $n \in \mathbb{N}$, which are biholomorphically related, so all compositions $\phi_j \circ \phi_k^{-1}$ are holomorphic on their respective domains.

Every choice of a complex analytic atlas fixes a so-called complex structure, the maximal family of compatible charts, that is, those biholomorphically related to the charts in the given atlas. A complex manifold is a differentiable manifold that is equipped with such a complex structure.

Remark. Sometimes we need to refer to $\mathcal{M}$ as a real manifold. In this case, the convention $d = 2n$ is always in force, where n denotes the complex dimension of $\mathcal{M}$ and d that of the underlying real manifold.

2.2.13. Examples. All of the following manifolds have the complex dimension $n \in \mathbb{N}$.

1. *Domains of* $\mathbb{C}^n$. Let $\mathcal{M}$ be a domain of $\mathbb{C}^n$, in other words, a connected open subset of $\mathbb{C}^n$. In this case, the identity map may serve as a global chart. The open unit ball $B(0,1) = \{z \in \mathbb{C}^n : \sum_{k=1}^n \left|z^{(k)}\right|\}$ is an example for such a manifold.

2. *Complex projective space.* Consider the quotient space $\mathbb{C}P^n$ obtained from $\mathbb{C}^{n+1} \setminus \{0\}$ by identifying two nonzero vectors w and w' whenever they are collinear, $w = cw'$ for some $c \in \mathbb{C}\setminus\{0\}$. The equivalence class of w will be written as $[w]$. The chart domains U_j, $j \in \{1, \ldots, n+1\}$ may be chosen as the sets $U_j := \{[w] : w^{(j)} \neq 0\}$, and the corresponding charts are defined as $\phi_j : U_j \to \mathbb{C}^n, [w] \mapsto \left(\frac{w^{(1)}}{w^{(j)}}, \frac{w^{(2)}}{w^{(j)}}, \ldots, \cancel{\frac{w^{(j)}}{w^{(j)}}} \ldots \frac{w^{(n+1)}}{w^{(j)}}\right)$, where the j-th component gets dropped. It is left to the reader to verify that the mappings $\phi_j \circ \phi_l^{-1}$ are holomorphic on $\phi_l(U_j \cap U_l)$ for $j, l \in \{1, \ldots n+1\}$.

3. *Complex tori.* Given a lattice $\mathbb{G} := \{\sum_{k=1}^{2n} l_k \epsilon_k : l \in \mathbb{Z}^{2n}\}$ with spacings $\epsilon_k \in \mathbb{C}^n$, $k \in \{1, \ldots 2n\}$, that are linearly independent over $\mathbb{R}$, we consider the quotient $\mathbb{C}^n/\mathbb{G}$. The underlying identification is understood as the equivalence relation $z \sim z'$ between z and z' in $\mathbb{C}^n$ whenever $z = z' + \sum_{k=1}^{2n} l_k \epsilon_k$ for some choice of $l \in \mathbb{Z}^{2n}$. The resulting

compact manifold is called a complex n-torus. A family of identity maps restricted to sets with diameter less than $\min_{1\le k\le 2n}\{|\epsilon_k|\}$ covering $\mathbb{C}^n$ may serve as a complex analytic atlas, because all these maps are invertible.

2.2.14. Definition. A function $f : \mathcal{M} \to \mathbb{C}$ is called holomorphic with respect to a given complex structure if for every compatible chart (U, ϕ), the composition $f \circ \phi^{-1} : \phi(U) \to \mathbb{C}$ is holomorphic in the usual sense.

Let Φ be a mapping from another complex manifold $\mathcal{M}'$ to $\mathcal{M}$. We call Φ holomorphic whenever this notion applies to the composition $\phi_j \circ \Phi$ with every chart $\phi_j : U_j \to V_j \subset \mathbb{C}^n$ of the complex analytic atlas on $\mathcal{M}$.

2.2.15. Definition. A complex vector bundle $\mathcal{V}$ over a differentiable manifold $\mathcal{M}$ is a vector bundle with the complex vector space $\mathbb{F} = \mathbb{C}^k$, $k \in \mathbb{N}$, as the typical fiber. In addition to the conditions in Definition 2.2.4, the restriction of a local trivialization $\xi : \pi^{-1}(U) \to V \times \mathbb{C}^k \subset \mathbb{R}^d \times \mathbb{C}^k$ to the fiber $\mathcal{V}_x$ associated with $x \in U$ is required to map complex linearly onto the second component of $V \times \mathbb{C}^k$. In the case $k = 1$, we refer to $\mathcal{V}$ as a complex line bundle.

In the forthcoming discussion, complex line bundles will gain particular importance. To keep track of the structure of a complex line bundle in calculations, we will use the concepts of transition functions and local reference sections. These concepts can of course be applied to arbitrary vector bundles, but for our purposes we do not need them in such generality.

2.2.16. Definition. Given an atlas of local trivializations $\{\xi_j\}_{j\in I}$ of a complex line bundle $\mathcal{L}$ with an underlying cover $\{U_j\}_{j\in I}$ of the base manifold $\mathcal{M}$, there is a corresponding set of transition functions $\{c_{jk} : U_j \cap U_k \to \mathbb{C}\}_{j,k\in I}$ characterized by the property $\xi_j = c_{jk} \circ \pi\ \xi_k$ whenever $U_j \cap U_k \neq \emptyset$.

2.2.17. Remarks. Via an atlas of local trivializations, a complex line bundle can be identified with a set of product manifolds $U_j \times \mathbb{C}$ that are glued together in a certain way. The sets U_j are patched up to form the base manifold $\mathcal{M} = \bigcup_{j\in I} U_j$ and the transition functions govern the twists in the fibers.

Sections in line bundles $\mathcal{L}$ may be visualized as functions on the base manifold $\mathcal{M}$ with the concept of local reference sections. The local reference sections $s_j : U_j \to \mathcal{L}$ adapted to an atlas of local trivializations $\{\xi_j\}_{j\in I}$ are characterized by the property $\xi_j \circ s_j : U_j \to V_j \times \{1\}$ for all $j \in I$. Each s_j is zero-free in its respective domain, and thus every section ψ can locally be written as a product $\psi|_{U_j} = \psi_j s_j$ (without summation), where ψ_j is a complex-valued function on U_j acting on the fibers by scalar multiplication. The equation $s_k = c_{jk} s_j$ on nonempty intersections $U_j \cap U_k$ explains the relation between reference sections and transition functions.

2.2.18. Definition. A holomorphic line bundle is a complex line bundle with additional complex structures on $\mathcal{L}$ and $\mathcal{M}$ such that π is a holomorphic mapping. In this case, the transition functions are holomorphic.

A section ψ in a holomorphic line bundle $\mathcal{L}$ over $\mathcal{M}$ is holomorphic if for any local trivialization ξ, the composition $\xi \circ \psi$ is holomorphic wherever defined. We will call the linear space of such sections $\Gamma^{hol}_{\mathcal{L}}(\mathcal{M})$. If ψ is a holomorphic section in a holomorphic line bundle with an atlas of local trivializations ξ_j and reference sections s_j as in the previous remark, then of course all $\psi_j : U_j \to \mathbb{C}$ are holomorphic.

If a real $2n$-dimensional manifold is equipped with a complex structure, the tangent bundle $T\mathcal{M}$ can be identified with a holomorphic vector bundle having the typical fiber $\mathbb{C}^n$. The essential ingredient in this identification is the canonical almost complex structure J on $T\mathcal{M}$ induced by the complex analytic atlas.

2.2.19. Definition. Every chart $\phi : U \to V \subset \mathbb{C}^n$ of a complex analytic atlas $\mathcal{A}$ gives rise to holomorphic coordinate functions $z^{(1)}, z^{(2)}, \ldots, z^{(n)}$ via the identification $\phi = \big(z^{(1)}, z^{(2)}, \ldots z^{(n)}\big)$. The canonical almost complex structure J induced by the atlas acts by $J\partial_k = i\partial_k$ and $J\overline{\partial}_k = -i\overline{\partial}_k$, where $\partial_k = \frac{1}{2}\big(\partial/\partial z_1^{(k)} - i\partial/\partial z_2^{(k)}\big)$, $\overline{\partial}_k = \frac{1}{2}\big(\partial/\partial z_1^{(k)} + i\partial/\partial z_2^{(k)}\big)$, with $z_1^{(k)}$ and $z_2^{(k)}$ denoting the real and imaginary parts of the component $z^{(k)}$, $k \in \{1, \ldots n\}$. Using the chain rule, one may verify that J is independent of the chosen chart because it is preserved under biholomorphic mappings.

Thus, the complexified tangent space at a point $x \in \mathcal{M}$ is split into a direct sum $T_x^{\mathbb{C}}\mathcal{M} = T_x^{(1,0)}\mathcal{M} \oplus T_x^{(0,1)}\mathcal{M}$. The almost complex structure J acts on each vector $X \in$

$T_x^{\mathbb{C}}\mathcal{M}$ by $JX = iX^{(1,0)} - iX^{(0,1)}$. If a vector field X has exclusively values in $T^{(1,0)}\mathcal{M}$, it is called holomorphic, in the analogous case of values in $T^{(0,1)}\mathcal{M}$ it is antiholomorphic.

In a similar vein, we say forms are of type (k,l), if they depend on k holomorphic and l antiholomorphic vector fields, with fixed non-negative integers k and l throughout $\mathcal{M}$. The exterior derivative of a form ω of type (k,l) is a sum of a $(k+1,l)$ and a $(k,l+1)$-form, denoted by $\partial\omega$ and $\overline{\partial}\omega$, respectively.

2.2.20. Definition. A Hermitian metric h on a complex vector bundle $\mathcal{V} \xrightarrow{\pi} \mathcal{M}$ is a family $\{h_x\}_{x\in\mathcal{M}}$ of positive definite sesquilinear forms h_x on the fibers $\mathcal{V}_x$. By convention, each $h_x : \mathcal{V}_x \times \mathcal{V}_x \to \mathbb{C}$ is conjugate linear in the first entry. A complex vector bundle equipped with such a Hermitian metric will be referred to as a Hermitian vector bundle.

2.2.21. Examples. In the following examples, each base manifold $\mathcal{M}$ carries two Hermitian holomorphic vector bundles: one is the holomorphic tangent space $T\mathcal{M}$ and the other is a holomorphic line bundle $\mathcal{L}$ over $\mathcal{M}$. We denote the associated Hermitian metrics by $h^{T\mathcal{M}}$ and h, respectively.

1. (a) *Euclidean space.* Let $\mathcal{M} = \mathbb{C}^n$ and $\mathcal{L} = \mathcal{M}\times\mathbb{C}$. The Hermitian metric on $T\mathcal{M}$ is given by $h^{T\mathcal{M}} := \frac{1}{2}\sum_{k=1}^n \big(dz^{(k)}\otimes d\overline{z}^{(k)} + d\overline{z}^{(k)}\otimes dz^{(k)}\big)$. The Hermitian metric on the fibers of $\mathcal{L}$ is defined over a base point $z\in\mathbb{C}^n$ by $h_z((z,u),(z,v)) = e^{-|z|^2}\overline{u}v$ with $u, v \in \mathbb{C}$ denoting the fiber component and $|z|^2 = \sum_{k=1}^n \overline{z}^{(k)} z^{(k)}$.

   (b) *Bergman space.* Let $\mathcal{M} = B(0,1)$, the unit ball in $\mathbb{C}^n$, and again choose a product bundle $\mathcal{L} = \mathcal{M}\times\mathbb{C}$. We define $h^{T\mathcal{M}} = \frac{1}{2}\sum_{k,l=1}^n \frac{(1-|z|^2)\delta_{kl}+\overline{z}^{(k)}z^{(l)}}{(1-|z|^2)^2}\big(dz^{(k)}\otimes d\overline{z}^{(l)} + d\overline{z}^{(k)}\otimes dz^{(l)}\big)$, and $h_z((z,u),(z,v)) = (1-|z|^2)^r\overline{u}v$, with some fixed $r>0$.

2. *Complex projective space.* Let $\mathcal{L}^{\times} = \mathbb{C}^{n+1}\setminus\{0\}$ and $\mathcal{M} = \mathbb{C}P^n$ as described in the previous Examples 2.2.13. After filling in the missing zero in each fiber, we obtain the so-called tautological line bundle $\mathcal{L}$. Let $U_1 := \{[w] : w\in\mathbb{C}^{n+1}, w^{(1)}\neq 0\}$ as before and write $\phi_1([w]) =: (z^{(1)}, z^{(2)}, \dots z^{(n)})$. The local expression $h^{T\mathcal{M}} = \frac{1}{2}\sum_{k,l=1}^n \frac{(1+|z|^2)\delta_{kl}-\overline{z}^{(k)}z^{(l)}}{(1+|z|^2)^2}\big(dz^{(k)}\otimes d\overline{z}^{(l)} + d\overline{z}^{(k)}\otimes dz^{(l)}\big)$ defines the so-called Fubini-Study metric on $U_1\subset\mathbb{C}P^n$. It is left to the reader to verify that this definition is consistent when U_1 is replaced by any U_j, $j\in\{2,\dots n+1\}$. The Hermitian metric

h is obtained from the inner product on $\mathbb{C}^{n+1}$. In terms of the local coordinate z, $h_z(w, w') = (1 + |z|^2)\overline{w}_1 \cdot w'_1$, where $\phi_1([w]) = \phi_1([w']) = z$.

One may verify that this bundle does not admit any global holomorphic sections, whereas its dual does. For more details, see [Wel80].

3. *The Riemann sphere.* Consider the manifold $\mathcal{M} = \mathbb{C}P^1$ equipped with a tensor power of the dual bundle corresponding to the preceding example. To give a more explicit description, we identify $\mathcal{M}$ with the compactified complex plane $\mathcal{M} = \mathbb{C} \cup \{\infty\}$ that is equipped with two charts: ϕ_1 is the identity on $U_1 := \mathbb{C}$ and ϕ_2 is the inverse function $z \mapsto 1/z$ defined on $U_2 := \{z \neq 0\} \cup \{\infty\}$ with the convention $1/\infty = 0$. The holomorphic bundle $\mathcal{L}$ is specified by the relation between the reference sections $s_1(z) = z^r s_2(z)$, with $r \in \mathbb{N}$. The Hermitian metric is chosen in accordance with the preceding example, so $h^{T\mathcal{M}} = \frac{1}{2}(1 + |z|^2)^{-2}(dz \otimes d\overline{z} + d\overline{z} \otimes dz)$ on U_1. The fiber metric h is in the local trivialization corresponding to s_1 given as $h_z((z, u), (z, v)) = (1 + |z|^2)^{-r}\overline{u}v$, where the negative integer $-r$ in the exponent shows that this bundle can be thought of as the r-th tensor power of the dual to the tautological line bundle for $n = 1$.

All of these examples share one property that will become important later: The imaginary part of the Hermitian metric on the complexified tangent space of each $\mathcal{M}$ is a closed two-form. Thus, the base manifolds are Kähler.

2.2.22. Definition. A connection ∇ on a complex line bundle $\mathcal{L}$ is a mapping $X \mapsto \nabla_X$ such that for each smooth vector field $X \in \Upsilon(\mathcal{M})$, ∇_X is an endomorphism on the space of smooth sections $C^\infty_{\mathcal{L}}(\mathcal{M})$ satisfying the following conditions for all smooth sections $\psi, \sigma \in C^\infty_{\mathcal{L}}(\mathcal{M})$, functions $f \in C^\infty(\mathcal{M})$, and vector fields $X, Y \in \Upsilon(\mathcal{M})$:

- $\nabla_X(\psi + \sigma) = \nabla_X\psi + \nabla_X\sigma\,,$
- $\nabla_{fX+Y} = f\nabla_X + \nabla_Y\,,$
- $\nabla_X(f\psi) = X(f)\psi + f\nabla_X\psi\,.$

A connection is compatible with a Hermitian metric h on a complex line bundle $\mathcal{L}$ if

$$X\, h(\psi, \sigma) = h(\nabla_{\overline{X}}\psi, \sigma) + h(\psi, \nabla_X \sigma) \tag{2.1}$$

for every $X \in \Upsilon(\mathcal{M})$ and $\psi, \sigma \in C^\infty_{\mathcal{L}}(\mathcal{M})$.

2.2.23. Definition. The group of nonzero complex numbers $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ acts by scalar multiplication on each fiber of $\mathcal{L}$. To make this action transitive, we remove the image of $\mathcal{M}$ under the zero-section o from $\mathcal{L}$ and write $\mathcal{L}^\times := \mathcal{L} \setminus o(\mathcal{M})$.

2.2.24. Definition. The connection one-form α on $\mathcal{L}^\times$ belonging to a connection ∇ is characterized by the following properties:

- The pull-back formula

$$\nabla_X \psi = -i(\psi^* \alpha)(X)\psi \tag{2.2}$$

  holds for all $\psi \in C^\infty_{\mathcal{L}}(\mathcal{M})$ and $X \in \Upsilon(\mathcal{M})$.

- The remaining component of α is determined by the equation

$$\alpha(X_\perp) = i\frac{X_\perp(\vartheta)}{\vartheta} \tag{2.3}$$

  for any vertical vector $X_\perp$, meaning $\pi_* X_\perp = 0$, and any function ϑ that is complex linear in each fiber $\mathcal{L}_x$ and non-vanishing on $\mathcal{L}^\times$.

The combination of both properties shows that α is invariant under the fiber-preserving group action of all nonzero complex numbers $c \in \mathbb{C}^\times$ on the bundle, $(c\,\mathrm{id}_{\mathcal{L}})^*\alpha = \alpha$.

2.2.25. Remark. Given an atlas of local trivializations ξ_j with underlying contractible open sets U_j covering $\mathcal{M}$ and local reference sections s_j, we can define one-forms α_j on each U_j by $\alpha_j := s_j^* \alpha$. With $\psi|_{U_j} = \psi_j s_j$ we then have the local expression

$$\nabla_X \psi = \nabla_X(\psi_j s_j) = (X(\psi_j) - i\alpha_j(X)\psi_j)s_j \tag{2.4}$$

on each set U_j.

2.2.26. Proposition. Given a holomorphic line bundle $\mathcal{L}$ with a smooth, nowhere degenerate Hermitian metric h, there is a unique connection ∇ compatible with h that renders every local, holomorphic section $\psi = \psi_j s_j \in \Gamma^{hol}_{\mathcal{L}}(U_j)$, $j \in I$, covariantly constant with respect to any local antiholomorphic vector field $X \in \Upsilon^{(0,1)}(U_j)$, $\nabla_X \psi = 0$. The smoothness of h is understood in the sense of the function $\tilde{h} : \mathcal{L} \to \mathbb{R}, u \mapsto h_{\pi(u)}(u, u)$ being smooth.

Proof. It is straightforward to check that the expression

$$\alpha := i \frac{\partial \tilde{h}}{\tilde{h}} \tag{2.5}$$

has all the properties required of a connection one-form, and therefore defines a connection ∇. In addition, inserting the definition (2.5) in equation (2.2) shows that ∇ is compatible with the holomorphic and Hermitian structures.

The uniqueness of this connection is guaranteed by the following argument. Suppose α is any connection one-form with the desired compatibility properties. Considering that the vertical component of α is specified by equation (2.3), it is enough to show that the local expression α_j is uniquely determined.

By compatibility with the complex structure, α_j must vanish on $\Upsilon^{(0,1)}(U_j)$. In addition, since any $X \in \Upsilon_{\mathbb{R}}(U_j)$ appearing in the differential equation

$$X\, h(s_j, s_j) = -i h(s_j, s_j)(\alpha_j(X) - \overline{\alpha}_j(X)) \tag{2.6}$$

can be uniquely decomposed into holomorphic and antiholomorphic parts, the identity

$$\alpha_j = i \frac{\partial h(s_j, s_j)}{h(s_j, s_j)} \tag{2.7}$$

follows from considering each part separately in equation (2.6). □

Consequence. In a local trivialization of a holomorphic line bundle, the covariant derivative of a section $\psi|_{U_j} = s_j \psi_j$ with respect to an antiholomorphic vector field $X \in \Upsilon^{(0,1)}(U_j)$

appears as

$$\nabla_X(\psi_j s_j) = X(\psi_j)s_j\,,$$

because the term $\alpha_j(X) = 0$ does not contribute.

2.2.27. Definition. Given a connection ∇ on a complex line bundle $\mathcal{L}$, the horizontal transport $H_\zeta\Psi$ of a vector $\Psi \in \mathcal{L}_{\zeta(0)}$ along a smooth curve $\zeta : [0,t] \to \mathcal{M}$ is the curve $H_\zeta\Psi$ in $\mathcal{L}$ satisfying the following conditions:

- $H_\zeta\Psi(0) = \Psi$, $\pi \circ H_\zeta\Psi = \zeta$.
- $H_\zeta\Psi$ is covariantly constant along ζ. To be more specific, for all $t_0 \in (0,t)$ and all smooth sections ψ that satisfy $\psi \circ \zeta = H_\zeta\Psi$ in some open interval around t_0, we have $\nabla_{\dot\zeta(t_0)}\psi(\zeta(t_0)) = 0$. Hereby, $\dot\zeta(t_0)$ is the velocity vector defined at $\zeta(t_0)$ by $\dot\zeta : f \mapsto \frac{d}{dt} f \circ \zeta$ for $f \in C^\infty(\mathcal{M})$.

From now on, we use the abbreviations $H_{\zeta,t}\Psi := H_\zeta\Psi(t)$ and, if the choice of ζ is clear from the context, $\widehat{\Psi} := H_\zeta\Psi$.

Consequence. If $\mathcal{L}$ is a holomorphic line bundle with a smooth Hermitian metric h and ∇ is the unique connection constructed in Proposition 2.2.26, then the compatibility requirement (2.1) implies that the horizontal transport of $\Psi \in \mathcal{L}_{\zeta(0)}$ along ζ preserves the inner product, $h_{\zeta(t)}(\widehat{\Psi}(t), \widehat{\Psi}(t)) = h_{\zeta(0)}(\Psi, \Psi)$.

2.2.28. Lemma. Let $\mathcal{L}$ be a complex line bundle with a connection ∇. When $\zeta : [0,t] \to U_j$ stays in the domain U_j of a chart, the horizontal transport of $\Psi = \widehat{\Psi}(0) \in \mathcal{L}_{\zeta(0)}$ is locally given by $\widehat{\Psi} = \widehat{\Psi}_j\, s_j \circ \zeta$ in terms of the reference section s_j and the values

$$\widehat{\Psi}_j(t) = e^{i\int_\zeta \alpha_j}\,\widehat{\Psi}_j(0)\,, \tag{2.8}$$

derived from the local representation α_j of the connection one-form.

Proof. Given $t_0 \in (0,t)$ and a section $\psi = \psi_j s_j$ that is defined near a piece of the curve around $\zeta(t_0)$, the equation $\nabla_{\dot\zeta}\psi(\zeta(t)) = 0$ is locally expressed as the differential equation

$\frac{d}{dt}\psi_j(\zeta(t)) = i\alpha_j(\dot\zeta(t))\psi_j(\zeta(t))$ with the solution $\psi_j(\zeta(r)) = e^{i\int_{\zeta|_{[t_0,r]}}\alpha_j}\psi_j(\zeta(t_0))$ for r sufficiently close to t_0. Thus, $\widehat\Psi_j = \psi_j\circ\zeta$ locally lifts the curve ζ, and $\widehat\Psi = \psi\circ\zeta$ is the horizontal transport specified by (2.8) in a neighborhood of t_0. The general statement follows from repeating this procedure for a dense set of $t_0 \in (0,t)$ and appropriate sections ψ. □

2.2.29. Proposition. Consider a smooth section $\psi \in C^\infty_{\mathcal{L}}(\mathcal{M})$ in a complex line bundle $\mathcal{L}$ with a connection ∇. For any smooth curve $\zeta : [0,t] \to \mathcal{M}$, the horizontal transport and the connection are related by the integral equation

$$H^{-1}_{\zeta,t}\psi(\zeta(t)) = \psi(\zeta(0)) + \int_0^t H^{-1}_{\zeta,r}\nabla_{\dot\zeta(r)}\psi(\zeta(r))\,dr\,, \tag{2.9}$$

where the reverse horizontal transport $H^{-1}_{\zeta,r}$ is understood as the map $H^{-1}_{\zeta,r} : \mathcal{L}_{\zeta(r)} \to \mathcal{L}_{\zeta(0)}$ satisfying $H^{-1}_{\zeta,r}\circ H_{\zeta,r}\Psi = \Psi$ for all $\Psi \in \mathcal{L}_{\zeta(0)}$ and $r \in [0,t]$.

Proof. The equation

$$\nabla_{\dot\zeta(r)}\psi(\zeta(r)) = \lim_{s\searrow r}\frac{1}{s-r}\left(H^{-1}_{\zeta|_{[r,s]},s}\psi(\zeta(s)) - \psi(\zeta(r))\right) \tag{2.10}$$

can be verified for any $r \in (0,t)$ in the local formulation of the preceding lemma. Using the pre-limit expression in (2.10) as in the construction of the Riemann integral, we obtain (2.9) as an analogue of the Fundamental Theorem of Calculus. □

2.2.30. Definition. Let $\mathcal{M}$ be the base manifold of a bundle with total space $\mathcal{L}$. The horizontal lifting operator L in $\mathcal{L}$ associated with the connection ∇ is a family $\{L_y\}_{y\in\mathcal{L}}$ of mappings $L_y : T_{\pi(y)}\mathcal{M} \to T_y\mathcal{L}$ having the push-forward with the projection π as a left inverse, $\pi_* L_y X = X_{\pi(y)}$, and mapping every vector field $X \in \Upsilon(\mathcal{M})$ into the kernel of the connection one-form, $\langle\alpha, L_y X\rangle = 0$.

2.2.31. Remarks. The operator L serves to convert vector fields X on $\mathcal{M}$ into horizontal vector fields LX on $\mathcal{L}$. On the other hand, L can be used to pull back one-forms β on $\mathcal{L}$ by its adjoint, $L^*_y\beta(X) = \beta(L_y X)$. If L is clear from the context, we will write $\widehat X_y$ instead of $L_y X$ and $\check\beta_y(X)$ instead of $L^*_y\beta(X)$. Moreover, if β is invariant under the group action of $\mathbb{C}^\times$ on the fibers, then $\check{\beta}_y(X)$ is independent of the choice of $y \in \mathcal{L}_x$ and the subscript of $\check{\beta}$ may be dropped.

The construction of L is simply obtained from the lifting of local integral curves of X. To this end, we define for all functions $f \in C^\infty(\mathcal{M}), q \in C^\infty(\mathcal{L})$ and vector fields $Xf(\pi(y)) = \frac{d}{dt} f \circ \zeta(t_0)$ the lifted vectors by $L_y X : q \mapsto \frac{d}{dt} q \circ \hat{\zeta}(t_0)$ with the horizontal transport $\hat{\zeta}$ passing through y at time t_0. From the local representation of the preceding lemma it follows that $\alpha(L_y X) = 0$. The lifting operator L is unique because $\pi_* LX = X$ determines LX up to a vertical component, which must vanish by $\alpha(L_y X) = 0$.

CHAPTER 3
BEREZIN-TOEPLITZ QUANTIZATION FROM A COHERENT-STATE PERSPECTIVE

The focus of this chaper is the construction of operators according to a quantization scheme in the spirit of Berezin [Ber72, Ber74]. In a geometric formulation of this scheme [CGR90, CGR93, BMS94, CGR94, CGR95], Schrödinger's concept of wavefunctions on configuration space gets replaced by holomorphic sections in a holomorphic line bundle $\mathcal{L}$ with a connection ∇ over the classical phase space $\mathcal{M}$. The correspondence between the classical phase-space structure and the line bundle is implicit in the fundamental assumption that the symplectic form on $\mathcal{M}$ can be reconstructed as a multiple of the curvature associated with the connection. In analogy with Schrödinger's interpretation of probability amplitudes, the probability that a measurement will find a quantum system described by such a section in a given subset of phase space emerges according to the following procedure: First, the length of a section at any base point must be defined. Up to an overall constant, this is determined by asking the horizontal transport determined by the connection to be length-preserving. Integrating the square of this length against Liouville's volume form over the phase-space subset in question then gives the desired probability. Hereby, the constant is chosen in order to normalize the resulting probability measure.

3.1 From Continuous Representations to Berezin-Toeplitz Quantization

Klauder's concept of a continuous representation [Kla63a, Kla63b, Kla64, MK64, KM65, KMC65] postulates the existence of a family of orthogonal projectors $\{\Pi_x\}_{x\in\mathcal{M}}$ onto one-dimensional subspaces of a separable Hilbert space $\mathcal{H}$, indexed by points in a topological manifold such that $x \mapsto \Pi_x$ is weakly continuous. If there is a measure $\mathfrak{m}$ on $\mathcal{M}$ such that the integral $\int_{\mathcal{M}} \Pi_x d\mathfrak{m}(x) = \mathrm{id}_{\mathcal{H}}$ provides a weakly convergent resolution of the identity mapping $\mathrm{id}_{\mathcal{H}}$, then we call each one-dimensional subspace $e(x) := \Pi_x\mathcal{H}$ a coherent state. Thus, one can think of the manifold $\mathcal{M}$ as being embedded in the projective Hilbert space $P\mathcal{H}$, the set of all one-dimensional subspaces of $\mathcal{H}$. By definition, the image of the embedding constitutes the family of coherent states. The identification of collinear vectors in $\mathcal{H}$ to describe a (pure) quantum state induces additional structures on $\mathcal{M}$. The details are explained in the following exposition.

Since $P\mathcal{H}$ is the base manifold of a bundle $P : \mathcal{H}\backslash\{0\} \to P\mathcal{H}$, where the projection P maps any nonzero vector in $\mathcal{H}$ to the one-dimensional subspace it generates, the embedding of $\mathcal{M}$ pulls back the fibers $\pi^{-1}(\{x\}) := P^{-1}(\{e(x)\}), x \in \mathcal{M}$. To make $\mathcal{M}$ the base manifold of a complex line bundle, the missing zero vector must be inserted in every fiber and thus a so-called tautological bundle is created with total space $\mathcal{L}$ and projection π. If we suppose that $\mathcal{H}$ is the closure of the linear hull $\mathrm{Lin}(\mathcal{L})$, then the linear functional $\vartheta_v : \psi \mapsto (v, \psi)$ restricted to $\psi \in \mathcal{L}$ provides a realization of $v \in \mathcal{H}$ as a function on $\mathcal{L}$ that is complex linear in the fibers.

If $\mathcal{M}$ is a differentiable manifold and the mapping $x \mapsto \Pi_x$ is in some sense smooth, then as subsets of $\mathcal{H}$, the fibers in the total space $\mathcal{L}$ inherit additional features. For example, the scalar multiplication $\psi \mapsto c\psi$ of vectors $\psi \in \mathcal{H}$ provides a natural fiber-preserving group action of nonzero complex numbers $c \in \mathbb{C}^{\times}$. The scalar product $(\cdot, \cdot)$ serves simultaneously as a Hermitian metric on both the total space $\mathcal{L}$ and the tangent space $T\mathcal{L}$. By definition, these metrics are invariant under all endomorphisms mapping $\mathcal{L}$ to $\mathcal{L}$ that are restrictions of unitary transformations on $\mathcal{H}$, such as the scalar multiplication of all vectors by a unimodular number $c, |c| = 1$.

The notion of horizontal transport passes from $\mathcal{H}$ to $\mathcal{L}$, which takes a smooth curve $\zeta : \mathbb{R} \to \mathcal{M}$ together with a starting point $\hat{\zeta}(0)$ in $\pi^{-1}(\zeta(0))$ and produces the lifted curve $\hat{\zeta}$ in $\mathcal{L}$ by moving in an infinitesimal time step dt from $\hat{\zeta}(t), t \in \mathbb{R}$, to the orthogonal projection of $\hat{\zeta}(t)$ onto the space $e(\zeta(t + dt))$. In fact, this way the norm of a horizontally transported vector in the fiber is left invariant while its base point moves along the curve in $\mathcal{M}$. In other words, the connection on the bundle corresponding to the transport is compatible with the Hermitian structure.

Berezin-Toeplitz quantization realizes a class of continuous representations in a setting that is familiar in algebraic geometry [GH78]: If $\mathcal{L}$ is a holomorphic line bundle over a Kähler manifold, then the curvature of the line bundle is a closed two-form [Zha00]. This two-form is up to an imaginary factor assumed to be equal to the symplectic form on $\mathcal{M}$. The Hilbert space chosen by Berezin-Toeplitz quantization is the space of holomorphic sections that are square-integrable with respect to Liouville's measure, the maximal exterior power of the curvature form.

These requirements are needed to show a correspondence principle for compact $\mathcal{M}$ [BMS94, Sch98]. Unfortunately, they also restrict the universality of Berezin-Toeplitz quantization. Not all symplectic manifolds can be equipped with a compatible complex structure, and even less may be obtained as the base manifold of a holomorphic line bundle such that its curvature is a constant multiple of the original symplectic form [Sch98].

Because of the demonstrated invariance properties of the Hermitian metric on $T\mathcal{L}$, the tangent bundle $T\mathcal{M}$ inherits a Fubini-Study type metric [Arn89, Appendix 3]. It is straightforward to check that the imaginary, skew-symmetric part of the Fubini-Study metric is closed and therefore constitutes yet another way to derive a symplectic form from the embedding, which makes $\mathcal{M}$ a Kähler manifold. It turns out, however, that this symplectic form may or may not coincide up to a constant factor with the curvature of the line bundle [CGR93].

In order to provide a resolution of the identity $\mathrm{id}_{\mathcal{H}}$ according to $\int_{\mathcal{M}} \Pi_x d\mathfrak{m}(x) = \mathrm{id}_{\mathcal{H}}$, the measure $\mathfrak{m}$ is chosen as a locally rescaled version of the Liouville form, see [CGR93]. More generally, Berezin-Toeplitz quantization maps the classical observable represented by a bounded real-valued function $f : \mathcal{M} \to \mathbb{R}$ to the self-adjoint operator $T_f := \int_{\mathcal{M}} f(x)\Pi_x d\mathfrak{m}(x)$. In both cases, the integral converges in the strong sense. The quantization of dynamics is then realized with the unitary group $\{e^{-itT_f}\}_{t\in\mathbb{R}}$ that results from choosing f as the generator of classical time evolution.

3.2 Hilbert Spaces of Square-Integrable, Holomorphic Sections

3.2.1. Definition. Let us assume that the complex line bundle $\mathcal{L} \to \mathcal{M}$ is equipped with a Hermitian metric h. With the help of a measure μ on $\mathcal{M}$ we can then define an inner product

$$(\psi, \phi) := \int_{\mathcal{M}} h(\psi, \phi)\, d\mu \tag{3.1}$$

for sufficiently regular sections ψ and ϕ in $\Gamma_{\mathcal{L}}(\mathcal{M})$, where $h(\psi,\phi)$ is interpreted as the function $x \mapsto h_x(\psi(x),\phi(x))$.

3.2.2. Remark. In the definition of the inner product, h and μ can be combined to a Hermitian-metric valued measure, hereafter denoted by $h\mu$. Indeed, this is a more appropriate way to view the definition, since the redundancy of rescaling h while changing μ to compensate accordingly is manifest in the notation. We will revisit these rescaling operations at the end of this chapter. For now, we simply consider h and μ as fixed.

3.2.3. Definition. The linear space of square-integrable sections on a complex line bundle $\mathcal{L}$ over a base manifold $\mathcal{M}$ is denoted by

$$L^2(h\mu) := \left\{\psi \in \Gamma_{\mathcal{L}}(\mathcal{M}) : \int_{\mathcal{M}} h(\psi,\psi)\, d\mu < \infty\right\}. \tag{3.2}$$

When $\mathcal{L}$ is a holomorphic line bundle, we define the weighted Bergman space $L^2_{hol}(h\mu)$ as the space of all holomorphic sections in $L^2(h\mu)$.

Remark. It may happen that $L^2_{hol}(h\mu)$ only contains the zero section. Therefore, results about the dimensionality of this space are of a fundamental interest. For the case of compact $\mathcal{M}$, see [BMS94].

3.2.4. Theorem. Equipped with the previously defined inner product, the space $L^2(h\mu)$ containing all square-integrable sections becomes a Hilbert space. If $\mathcal{L}$ is a holomorphic line bundle and μ, interpreted as a volume form, and h are everywhere non-degenerate and smooth, then the weighted Bergman space $L^2_{hol}(h\mu)$ is a Hilbert-subspace of $L^2(h\mu)$.

Proof. To begin with, we choose an atlas of local trivializations $\{\xi_j\}_{j\in I}$ and the corresponding reference sections s_j. Thus, we can identify each section ψ with a set of functions $\{\psi_j\}_{j\in I}$ satisfying $\psi|_{U_j} = \psi_j s_j$.

With the help of standard arguments from measure theory we may now deduce that $L^2(h\mu)$ is complete, which means the space of equivalence classes of square-integrable sections that coincide up to sets of measure zero.

To see this, we consider that, using the local reference sections, a Cauchy sequence of sections $\psi^{(l)} \in L^2(h\mu)$ gets mapped to a Cauchy sequence of functions $\psi_j^{(l)} : U_j \to \mathbb{C}$ in $L^2(U_j, \mu_j)$, with the measure given by $\mu_j = h(s_j, s_j)\mu$. The non-degeneracy of μ and h ensure that each $L^2(U_j, \mu_j)$ is complete, and the only thing left to verify is that, passing to subsequences if necessary, the individual limits in $L^2(U_j, \mu_j)$ give rise to an almost everywhere well-defined global section that is square-integrable. Finally, we note that this section is almost everywhere independent of the particular choice of local trivializations.

To prove that $L^2_{hol}(h\mu)$ is a Hilbert space, it is enough to show that any given Cauchy sequence converges pointwise to a holomorphic section, which is a holomorphic representative of the limit in the L^2-sense.

To this end, we pick a local trivialization $\xi_j : \pi^{-1}(U_j) \to V_j \times \mathbb{C}$ and a local reference section $s_j : U_j \to \mathcal{L}$. With the help of the product decomposition $\psi|_{U_j} = s_j\psi_j$, the Cauchy sequence $\{\psi^{(l)}\}_{n\in\mathbb{N}}$ is represented by a sequence of holomorphic functions $\psi_j^{(l)} : U_j \to \mathbb{C}$. After introducing holomorphic coordinate functions $z^{(1)}, z^{(2)}, \ldots, z^{(n)}$ in U_j, the measure μ_j can be decomposed into the components $d\mu_j = \mu_j(z)d^{2n}z := \frac{\mu_j(z)}{(2i)^n n!}dz^{(1)} \wedge d\overline{z}^{(1)} \wedge \cdots \wedge dz^{(n)} \wedge d\overline{z}^{(n)}$. Suppose we have chosen local trivializations ξ_j with underlying charts $\phi_j : U_j \to V_j$ such that each V_j is a ball of radius r_j centered at the origin. Then $\int_{U_j} h(\psi^{(l)}, \psi^{(l)})d\mu = \int_{V_j} |\psi_j^{(l)}|^2\mu_j(z)d^{2n}z$. The non-degeneracy and smoothness of h and μ imply that for each U_j, there is a strictly positive lower bound $0 < \epsilon_j < \mu_j(z)$. By the inequality

$$\epsilon_j \int_{V_j} \left|\psi_j^{(l)}\right|^2 d^{2n}z \leq (\psi^{(l)}, \psi^{(l)})$$

we deduce that $\left\{\psi_j^{(l)}\right\}_{l\in\mathbb{N}}$ is a Cauchy sequence of holomorphic functions in the traditional Bergman space $L^2_{hol}(V_j, \epsilon_j d^{2n}z)$. According to Appendix A, the sequence converges pointwise to a holomorphic function. The limits obtained on each V_j can then be recombined with the help of the reference sections s_j to give a global, holomorphic section. This limit section is the holomorphic representative that coincides almost everywhere with the limit of the Cauchy sequence $\{\psi^{(l)}\}_{l\in\mathbb{N}}$ taken in $L^2(h\mu)$. □

3.2.5. Lemma. Given a vector u in a fiber above $x := \pi(u)$, the point evaluation

$$\begin{aligned} \vartheta_u : L^2_{hol}(h\mu) &\longrightarrow \mathbb{C} \\ \psi &\longmapsto h_x(u, \psi(x)) \end{aligned} \tag{3.3}$$

defines a bounded linear functional, and by the Riesz representation theorem this evaluation can be realized as an inner product $\tilde{\psi}(u) := (e_u, \psi) = \vartheta_u(\psi)$ with a section $e_u \in L^2_{hol}(h\mu)$. Two such sections form a kernel function $k(u, v) := (e_u, e_v)$ that is defined on $\mathcal{L} \times \mathcal{L}$ and sesquilinear in the fibers.

Proof. The detail that mostly deserves explanation is the boundedness of ϑ_u. To verify this, we choose a local trivialization ξ around the fiber generated by u, mapping $\pi^{-1}(U) \subset \mathcal{L}$ to $V \times \mathbb{C}$ with a ball $V \subset \mathbb{C}^n$ having the first component of $\xi(u)$ as the center.

Given a convergent sequence of sections $\{\psi^{(l)}\}_{l \in \mathbb{N}}$, we use as in Appendix A the mean value property of the associated holomorphic functions $\psi_j^{(l)}$ on V to bound the value of $\vartheta_u(\psi^{(l)})$ by a constant times the L^2-norm of $\psi^{(l)}$. Since the sequence has the Cauchy property, $\vartheta_u(\psi^{(l)})$ is also Cauchy, and therefore convergent.

The sesquilinearity of k results from the scaling property $e_{cu} = \overline{c}\, e_u$ for any $c \in \mathbb{C}$ and $u \in \mathcal{L}$. □

3.2.6. Definition. The dual bundle $\mathcal{L}^*$ is the bundle associating with each $x \in \mathcal{M}$ the space of linear forms on $\pi^{-1}(x)$. With the help of the Hermitian structure, we can identify sections ψ in $\mathcal{L}$ with their dual ψ^* in $\mathcal{L}^*$ by $\psi^*(u) = \vartheta_{\psi(x)}(u) = h_x(\psi(x), u)$ for $u \in \pi^{-1}(x), x \in \mathcal{M}$.

A Schwartz kernel in a complex line bundle $\mathcal{L}$ is a family of linear mappings $\{\mathcal{S}(x, y) : \mathcal{L}_y \to \mathcal{L}_x\}_{x,y \in \mathcal{M}}$, that is, $\mathcal{S}(x, y)$ is linear in vectors with base point y and has as its values vectors at x. If $\mathcal{S}(x, y)$ is jointly continuous in x and y, then it can be interpreted as continuous section in the bundle $\mathcal{L} \otimes \mathcal{L}^* \to \mathcal{M} \otimes \mathcal{M}$.

3.2.7. Proposition. The Schwartz kernel K given in the terminology of the preceding lemma by $K(x, y)v = e_v(x)$ for $v \in \pi^{-1}(y)$ is jointly continuous in x and y. We will call K

the reproducing kernel of $L^2_{hol}(h\mu)$ because of the identity

$$\psi(x) = \int_{\mathcal{M}} K(x,y)\psi(y)d\mu(y)\,. \tag{3.4}$$

Proof. The joint continuity follows from the continuity of e_v in v and the uniform convergence of Cauchy sequences in $L^2_{hol}(h\mu)$. These properties may be obtained using the definition of e_v via (3.3) and the argument in Appendix A.

To derive (3.4), we consider in a first step the adjoint map $(K(x,y))^* : \pi^{-1}(x) \to \pi^{-1}(y)$, in the usual way defined by $u \mapsto h_x(K(x,y)v,u)v$, independent of the choice of a normalized vector $v \in \pi^{-1}(y), \|v\| = 1$. We claim that $(K(x,y))^* = K(y,x)$, which means for all u,v in fibers above x and y, respectively, the equation $h_x(u,K(x,y)v) = h_y(K(y,x)u,v)$ holds. To simplify the following calculation, we assume that u and v are normalized; the general case follows by rescaling.

$$h_x(u,K(x,y)v) = h_x(u,e_v(x)) = h_x(u,\tilde{e}_v(u)u) \tag{3.5}$$

$$= \tilde{e}_v(u) = \overline{\tilde{e}_u(v)} \tag{3.6}$$

$$= h_y(\tilde{e}_u(v)v,v) = h_y(e_u(y),v) \tag{3.7}$$

$$= h_y(K(y,x)u,v) \tag{3.8}$$

The second step for the derivation of (3.4) uses again a normalized vector u above x,

$$\psi(x) = \tilde{\psi}(u)u = (e_u,\psi)\,u \tag{3.9}$$

$$= \int_{\mathcal{M}} h_y(e_u(y),\psi(y))\,u\,d\mu(y) \tag{3.10}$$

$$= \int_{\mathcal{M}} h_y(K(y,x)u,\psi(y))\,u\,d\mu(y) \tag{3.11}$$

$$= \int_{\mathcal{M}} h_y(u,K(x,y)\psi(y))\,u\,d\mu(y) \tag{3.12}$$

$$= \int_{\mathcal{M}} K(x,y)\psi(y)\,d\mu(y)\,. \tag{3.13}$$

□

Comment. One of the goals in this work is to find a formula for this kernel. In principle, one could follow a Gram-Schmidt orthogonalization procedure, construct an orthonormal basis of sections $\{\eta_l\}_{l\in\mathbb{N}}$ and then express the reproducing kernel as a series $K(x,y) = \sum_l \eta_l(x) h_y(\eta_l(y), \cdot)$ that terminates after finite terms or converges uniformly on compact sets in $\mathcal{M} \times \mathcal{M}$. However, this procedure is too abstract to show how the geometry of $\mathcal{L}$ shapes the kernel. We will therefore present an alternative strategy, expressing K in a probabilistic way.

3.2.8. Consequence. If $h\mu$ is smooth and nowhere degenerate, then any bounded operator B on $L^2_{hol}(h\mu)$ possesses an integral kernel $B(x,y)$ that is characterized by the equation $h_x(u, B(x,y)v) = (K(\cdot,x)u, BK(\cdot,y)v)$, and the image of $\psi \in L^2_{hol}(h\mu)$ is expressed as

$$B\psi(x) = \int_{\mathcal{M}} B(x,y)\psi(y)\, d\mu(y)\,. \tag{3.14}$$

Proof. That $B(x,y)$ is indeed an integral kernel results from the reproducing property (3.9) and Fubini's theorem. The sesqui-analyticity of $B(x,y)$ follows because the mapping $v \mapsto K(\cdot, \pi(v))v = e_v$ into $L^2_{hol}(h\mu)$ is antiholomorphic. □

3.2.9. Remark. Since the right-hand side of equation (3.14) is defined even for $\psi \in L^2(h\mu)$, any bounded operator extends naturally via its integral kernel to all of $L^2(h\mu)$. From this point of view, $K(x,y)$ is the integral kernel of an orthogonal projection operator, henceforth also called K, that maps $L^2(h\mu)$ onto $L^2_{hol}(h\mu)$.

3.3 Berezin-Toeplitz Operators Defined via Quadratic Forms

In the remaining text, we assume that h and μ are smooth and non-degenerate to ensure that $L^2_{hol}(h\mu)$ is complete.

3.3.1. Definition. Given the Hilbert space $L^2_{hol}(h\mu)$ and a real-valued function $f : \mathcal{M} \to \mathbb{R}$, we consider the sesquilinear form

$$\mathcal{T}_f : \mathcal{Q}(\mathcal{T}_f) \times \mathcal{Q}(\mathcal{T}_f) \longrightarrow \mathbb{C} \tag{3.15}$$

$$(\psi, \phi) \longmapsto \int_{\mathcal{M}} f(x) h_x(\psi(x), \phi(x)) d\mu(x) \tag{3.16}$$

with form domain

$$\mathcal{Q}(\mathcal{T}_f) := \left\{ \psi \in L^2_{hol}(h\mu) : \int_{\mathcal{M}} |f(x)|\, h_x(\psi(x), \psi(x)) d\mu < \infty \right\}. \tag{3.17}$$

When referring to $\mathcal{T}_f$ as a quadratic form, it is really the function $\psi \mapsto \mathcal{T}_f(\psi, \psi)$ that is meant.

3.3.2. Definition. Given a real-valued, bounded function $f : \mathcal{M} \to \mathbb{R}$, the form $\mathcal{T}_f$ specified in the preceding definition is bounded and symmetric. Therefore, it is associated with a self-adjoint operator T_f satisfying $(\psi, T_f \psi) = \mathcal{T}_f(\psi, \psi)$ for all $\psi \in L^2_{hol}(h\mu)$. In the context of weighted Bergman spaces, we call T_f a self-adjoint Berezin-Toeplitz operator and the function f its symbol.

Remarks. The original definition according to Berezin [Ber72, Ber74] and its geometric interpretation by Rawnsley and others [CGR90, CGR93, BMS94, CGR94, CGR95] do not refer to sesquilinear forms. Indeed, for bounded symbols the approach chosen here offers no new insights.

However, the use of sesquilinear forms is convenient for the construction of semibounded Berezin-Toeplitz operators described in the remaining part of this chapter. The implicit goal is to find a large class of possibly unbounded symbols f that lead to closed, semibounded quadratic forms $\mathcal{T}_f$ and thus yield unique self-adjoint Berezin-Toeplitz operators T_f via the Friedrichs construction characterized by equation (3.20). In fact, this goal leads the discussion from abstract conditions ensuring the semiboundedness of T_f to a more concrete class of admissible symbols presented in the next chapter.

3.3.3. Lemma. The sesquilinear form belonging to a non-negative function $f \geq 0$ is closed.

Proof. We need to show that $\mathcal{Q}(\mathcal{T}_f)$, equipped with the form-norm $\|\bullet\|_{\mathcal{T}_f}$ defined by

$$\|\psi\|_{\mathcal{T}_f} := (\mathcal{T}_f(\psi, \psi) + \|\psi\|^2)^{1/2} \quad \text{for} \quad \psi \in \mathcal{Q}(\mathcal{T}_f), \tag{3.18}$$

is complete.

Suppose $(\psi_l)_{l\in\mathbb{N}}$ is a Cauchy sequence with respect to the form-norm. Due to the estimate $\|\psi\| \le \|\psi\|_{\mathcal{T}_f}$ the sequence is convergent in $L^2_{hol}(h\mu)$, $\psi_l \to \psi$. Using pointwise convergence and Fatou's lemma, we obtain $\|\psi - \psi_l\|_{\mathcal{T}_f} \le \liminf_{k\to\infty} \|\psi_k - \psi_l\|_{\mathcal{T}_f}$ and therefore the sequence $(\psi_l)_{l\in\mathbb{N}}$ converges with respect to the form-norm. □

3.3.4. Fact. If the form $\mathcal{T}_{f^+}$ belonging to the positive part $f^+ : x \mapsto \max\{f(x), 0\}$ of a function $f : \mathcal{M} \to \mathbb{R}$ is densely defined and the negative part $f^- : x \mapsto \max\{-f(x), 0\}$ can be incorporated in $\mathcal{T}_f$ as a form-bounded perturbation, meaning

$$\mathcal{T}_{f^-}(\psi, \psi) \le c_1\, \mathcal{T}_{f^+}(\psi, \psi) + c_2\|\psi\|^2 \tag{3.19}$$

with a relative form bound $c_1 < 1$ and a constant $c_2 \ge 0$, then $\mathcal{T}_f$ is closed on $\mathcal{Q}(\mathcal{T}_f) = \mathcal{Q}(\mathcal{T}_{f^+})$ and has a lower bound $c \in \mathbb{R}$, such that $\mathcal{T}_f(\psi, \psi) \ge c\|\psi\|^2$.

Proof. This is the so-called KLMN theorem, see [Sim71] or [RS75, Theorem X.17]. It goes back to works of Kato [Kat55], Lax and Milgram [LM54], Lions [Lio61] and Nelson [Nel64b]. □

3.3.5. Fact. If the form $\mathcal{T}_f$ is closed and has the greatest lower bound $c \in \mathbb{R}$, then it belongs to a unique self-adjoint operator T_f that is characterized in terms of the square-root $\sqrt{T_f - c}$ satisfying

$$(\sqrt{T_f - c}\,\phi, \sqrt{T_f - c}\,\psi) + c(\phi, \psi) = \mathcal{T}_f(\phi, \psi) \quad \text{for all} \quad \phi, \psi \in \mathcal{Q}(\mathcal{T}_f)\,. \tag{3.20}$$

whenever $\psi, \phi \in \mathcal{D}(\sqrt{T_f - c}) = \mathcal{Q}(\mathcal{T}_f)$.

Proof. Again, we refer to the literature [RS80, Theorem VIII.15] or [Wei80, Theorem 5.36] for the proof of this result which we call the Friedrichs construction. □

3.3.6. Remarks. As a special case of Consequence 3.2.8, when f is a bounded function, T_f has an integral kernel $T_f(x, y)$ characterized by $h_x(u, T_f(x, y)v) = (K(\cdot, x)u, fK(\cdot, y)v)$, where $u, v \in \mathcal{L}$ have base points x and y, and the scalar product is taken in $L^2(h\mu)$.

For $\psi \in \mathcal{D}_{min}(T_f) := \{\psi \in L^2_{hol}(h\mu), f\psi \in L^2(h\mu)\}$, the identity $T_f\psi = K(f\psi)$ relates T_f to the traditional way of defining a Berezin-Toeplitz operator as a composition of a multiplication operator with the orthogonal projection K. However, it may happen that $\mathcal{D}_{min}(T_f)$ does not include all of $L^2_{hol}(h\mu)$, although the operator T_f is bounded.

A disadvantage of defining T_f by a semibounded form is that in general, nothing is known about its domain. The situation is different, if a domain of essential self-adjointness can be identified for T_f. Such situations have been investigated in detail [JS94, Cic96] for the case of the so-called Fock-Bargmann space.

The definition of Berezin-Toeplitz operators clearly does not rely on the validity of a correspondence principle and we will also not need to refer to it hereafter. However, because of its physical significance, we briefly mention some aspects concerning classical asymptotics in the context of Berezin-Toeplitz quantization on Kähler manifolds.

3.3.7. Definition. The curvature R of a complex line bundle $\mathcal{L}$ with a connection ∇ is given by the expression

$$R_{X,Y} := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{XY-YX} \tag{3.21}$$

for any two vector fields $X, Y \in \Upsilon(\mathcal{M})$. Contrary to its appearance, $R_{X,Y}$ is a zeroth-order differential operator that acts by scalar multiplication on the fibers. In other words, R can be interpreted as a two-form on $\mathcal{M}$ with values in the linear endomorphisms on the fibers of $\mathcal{L}$.

3.3.8. Definition. Let $\mathcal{M}$ be a complex manifold that is equipped with a Hermitian metric $h^{T\mathcal{M}}$ on the holomorphic tangent bundle $T\mathcal{M}$. Obviously, $g := \mathfrak{Re}\, h^{T\mathcal{M}}$ is a Riemannian metric and $\omega : (X,Y) \mapsto \frac{1}{2}g(X, JY)$ is antisymmetric in $X, Y \in T\mathcal{M}$. The latter is also called the Kähler form of $h^{T\mathcal{M}}$. If ω is closed, $d\omega = 0$, we call $\mathcal{M}$ a Kähler manifold.

3.3.9. Definition. A prequantum line bundle $\mathcal{L}$ over a Kähler manifold $\mathcal{M}$ with a is a holomorphic line bundle with a connection and a smooth non-degenerate Hermitian metric $h^{T\mathcal{M}}$ on $T\mathcal{M}$ such that the curvature R of the bundle and the Kähler form ω of the metric are in the relation

$$R_{X,Y} = \frac{i}{\hbar}\omega(X,Y) \tag{3.22}$$

with a fixed value of $\hbar > 0$ for all $X, Y \in T\mathcal{M}$.

3.3.10. Remarks. If the prequantum line bundle $\mathcal{L}$ has a non-trivial topology, a well-known integrality condition for Planck's constant $2\pi\hbar$ emerges [Woo92, Chapter 8].

If μ is chosen as the Liouville measure $\frac{1}{\hbar^n n!}\omega^{\wedge n} := (-1)^{n(n-1)/2}\frac{1}{\hbar^n n!}\omega\wedge\omega\wedge\cdots\wedge\omega$, then μ is invariant under canonical flows on $\mathcal{M}$ and one obtains a unitary group representation by lifting these flows to a horizontal transport of sections in $\mathcal{L}$. This idea is the starting point of geometric quantization [Tuy87, Woo92].

In the special setting of prequantum line bundles on homogeneous or compact Kähler manifolds, the Berezin-Toeplitz operators defined on the Hilbert space $L^2_{hol}(h\mu)$ are known to observe a correspondence principle, see [Ber74, Per86] or [BMS94, Sch98]. Moreover, in the compact case the same kind of classical asymptotics can be proved for more general almost-complex manifolds [BU96].

3.3.11. Remarks. Let $\mathcal{L}$ be a holomorphic line bundle with a Hermitian form h and suppose its base manifold $\mathcal{M}$ is Kähler. If initially the curvature R of $\mathcal{L}$ and the Kähler form ω of $\mathcal{M}$ are conformally related, meaning

$$R_{X,Y} = \frac{i}{\hbar}e^{2\nu/n}\omega(X,Y) \tag{3.23}$$

with a smooth logarithmic scaling function $\nu : \mathcal{M} \to \mathbb{R}$, then one may obtain the prequantum condition (3.22) by rescaling h and μ appropriately without changing the Hermitian-form valued measure $h\mu$.

Indeed, the desired relation (3.22) is obtained from equation (3.23) using

$$h' := e^{-2\nu}h \text{ and } \omega' := e^{2\nu/n}\omega\,, \tag{3.24}$$

which preserves $h\mu = \frac{1}{\hbar^n n!} h' \omega'^{\wedge n}$. To verify (3.22), we note that by the compatibility requirement (2.1), the change induces a new connection $\nabla_X \mapsto \nabla'_X := e^{\nu} \nabla_X e^{-\nu}$, but the curvature remains the same.

Consequently, relaxing the prequantum condition (3.22) to (3.23) does not provide any more generality. Nevertheless, the flexibility to choose "unphysical" combinations of h and μ will be used in a result in Chapter 6.

CHAPTER 4
SELF-ADJOINT BEREZIN-TOEPLITZ OPERATORS AS MONOTONE LIMITS OF SEMIBOUNDED SCHRÖDINGER OPERATORS

The main motivation for this chapter is to introduce a Riemannian metric on $\mathcal{M}$ in order to relate Berezin-Toeplitz and Schrödinger operators. An important application concerns the self-adjointness of Berezin-Toeplitz operators. In this chapter and the following one we derive conditions that are more accessible than verifying the abstract form-boundedness of T_{f^-} with respect to T_{f^+} according to inequality (3.19).

At first, the Riemannian structure seems to be an auxiliary element that is not needed in the definition of Berezin-Toeplitz operators according to the prescription of the preceding chapter. Nevertheless, the introductory remarks given there point out that the presence of coherent states leads to a Fubini-Study type metric, which is Kähler. The real part of this metric can then be used to define a Riemannian structure.

In the following, we consider Hilbert spaces $L^2_{hol}(hm)$ of square-integrable holomorphic sections in a holomorphic Hermitian line bundle $\mathcal{L}$ that has a base manifold $\mathcal{M}$ with a Kähler metric. The use of the natural volume measure m associated with the real part of the Kähler metric as apposed to the Liouville measure μ indicates that we do not require the curvature of $\mathcal{L}$ to be in a prequantum relation (3.22) with the Kähler structure of $\mathcal{M}$.

4.1 Elementary Definitions of Riemannian Geometry

4.1.1. Definition. A metric $g = \{g_x\}_{x \in \mathcal{M}}$ on a differentiable manifold $\mathcal{M}$ is a smooth family of non-degenerate, positive bilinear forms g_x on $T_x\mathcal{M}$. Whenever g is present, we will say $\mathcal{M}$ is a Riemannian manifold. As usual, the metric is accompanied by several derived concepts.

- A canonical bijection between the space of tangent vectors and one-forms is given by $X \mapsto X^\flat$,

$$X^\flat(Y) = g(X,Y)\,, \tag{4.1}$$

and its inverse $\beta \mapsto \beta^\sharp$ satisfying

$$g(\beta^\sharp, Y) = \beta(Y) \tag{4.2}$$

for all $X, Y \in T_x\mathcal{M}$ and one-forms $\beta \in T^*\mathcal{M}$.

- The gradient of a smooth function f on $\mathcal{M}$ is the vector field grad $f := (df)^\sharp \in \Upsilon(\mathcal{M})$.

- With an orthonormal frame $\{E_k\}_{k=1}^d$, such that $g(E_k, E_l) = \delta_{kl}$ in each $T_x\mathcal{M}$, we can express a natural, coordinate-invariant volume element $dm := \sqrt{g(x)}d^dx := \frac{1}{n!}E_1^\flat \wedge E_2^\flat \cdots \wedge E_d^\flat$, where $g(x)$ denotes the determinant of the matrix $g_x(\partial/\partial x^{(k)}, \partial/\partial x^{(l)})$ with the coordinate vector fields corresponding to $x^{(k)}$ and $x^{(l)}$ for $k, l \in \{1, \ldots d\}$.

  In addition, we can perform a trace operation $\mathrm{Tr}^{T\mathcal{M}} b := \sum_{k=1}^d b(E_k, E_k)$ on any bilinear form $b : T\mathcal{M} \otimes T\mathcal{M} \to \mathbb{C}$.

- The divergence of a vector field $Y \in \Upsilon_\mathbb{R}(\mathcal{M})$ is given as the trace $\mathrm{div}Y := \mathrm{Tr}^{T\mathcal{M}} b_Y$ of the bilinear form $b_Y : (X, Z) \mapsto (X, \mathrm{Cov}_Z Y)$.

- One of the fundamentals of Riemannian geometry is the existence of the unique affine Levi-Civita connection $\mathrm{Cov} : \Upsilon_\mathbb{R}(\mathcal{M}) \times \Upsilon_\mathbb{R}(\mathcal{M}) \to \Upsilon_\mathbb{R}(\mathcal{M})$. By definition, it is compatible with the metric in the sense that for $X, Y, Z \in \Upsilon_\mathbb{R}(\mathcal{M})$

$$Xg(Y,Z) = g(\mathrm{Cov}_X Y, Z) + g(Y, \mathrm{Cov}_X Z)\,, \tag{4.3}$$

and it has vanishing torsion,

$$\mathrm{Cov}_X Y - \mathrm{Cov}_Y X - XY + YX = 0\,. \tag{4.4}$$

4.1.2. Convention. We will not distinguish between a metric g on the tangent bundle $T\mathcal{M}$ and its sesquilinear extension to the complexified tangent bundle $T^{\mathbb{C}}\mathcal{M}$, as usual conjugate linear in the first argument. Similarly, the connection Cov is made complex linear on $T^{\mathbb{C}}\mathcal{M} \times T^{\mathbb{C}}\mathcal{M}$, and the divergence is thus defined as $\mathrm{Tr}^{T\mathcal{M}} b_Y$ for all $Y \in \Upsilon(\mathcal{M})$.

4.2 Bochner's Laplacian and its Relation to the Holomorphic Laplacian

Several Laplacians will be introduced in this section, each one is characterized by an associated positive definite quadratic form. Later, Schrödinger operators will arise from perturbations of these forms.

By default, $\mathcal{M}$ is always a d-dimensional Riemannian manifold, and whenever it appears in conjunction with the $\mathcal{L}$ holomorphic line bundle $\mathcal{L}$, it is tacitly understood to be the base manifold. The Hermitian metric h on $\mathcal{L}$ and the natural volume measure m on $\mathcal{M}$ are as forms assumed to be smooth and non-degenerate.

4.2.1. Definition. The Friedrichs construction corresponding to the closure of the quadratic form

$$\mathcal{E}(f, f) := \int_{\mathcal{M}} g(\operatorname{grad} f, \operatorname{grad} f) dm \tag{4.5}$$

with initial form domain $C_c^{\infty}(\mathcal{M})$ is called the negative Dirichlet Laplacian $-\Delta$ on $L^2(m)$.

4.2.2. Definition. The negative Bochner Laplacian $-\Delta^{\mathcal{L}}$ on $L^2(hm)$ is the Friedrichs construction corresponding to

$$\mathcal{E}^{\mathcal{L}}(\psi, \psi) := \int_{\mathcal{M}} \mathrm{Tr}^{T\mathcal{M}}\, h(\nabla\psi, \nabla\psi) dm \tag{4.6}$$

defined on $C_{c\mathcal{L}}^{\infty}(\mathcal{M})$, the space of smooth sections with compact support. Hereby, the trace operation is defined as before by choosing an orthonormal basis $\{E_k\}_{k=1}^{d}$ in each $T_x\mathcal{M}$ such that $\mathrm{Tr}^{T\mathcal{M}}\, h(\nabla\psi, \nabla\psi) = \sum_{k=1}^{d} h(\nabla_{E_k}\psi, \nabla_{E_k}\psi)$.

4.2.3. Proposition. On twice differentiable, compactly supported sections ψ, $\Delta^{\mathcal{L}}$ is expressed with the use of the orthonormal frame $\{E_k\}_{k=1}^d$ as the differential operator

$$\Delta^{\mathcal{L}}\psi = \sum_{k=1}^{d} \nabla_{E_k}\nabla_{E_k}\psi - \nabla_{\mathrm{Cov}_{E_k}E_k}\psi\,. \tag{4.7}$$

Proof. In the following, we use the abbreviations $\nabla_k := \nabla_{E_k}$ and $\mathrm{Cov}_k := \mathrm{Cov}_{E_k}$. We employ the compatibility requirement for the connection and write

$$h(\nabla_k\psi, \nabla_k\psi) = E_k h(\psi, \nabla_k\psi) - h(\psi, \nabla_k\nabla_k\psi)\,. \tag{4.8}$$

Under the integral, we can convert the first term on the right-hand side

$$\int_{\mathcal{M}} E_k h(\psi, \nabla_k\psi) dm = \int_{\mathcal{M}} (-\operatorname{div} E_k) h(\psi, \nabla_k\psi) dm\,. \tag{4.9}$$

Expressing the divergence in terms of a trace with respect to the orthonormal frame provided by the vectors E_k yields

$$(-\operatorname{div} E_k)h(\psi, \nabla_k\psi) = -\sum_{l=1}^{d}(E_l, \mathrm{Cov}_l E_k)h(\psi, \nabla_k\psi) = \sum_{l=1}^{d}(\mathrm{Cov}_l E_l, E_k)h(\psi, \nabla_k\psi)\,. \tag{4.10}$$

Now we use the completeness relation $\sum_k(\cdot, E_k)E_k = \mathrm{id}_{T\mathcal{M}}$ on the real vector space $T\mathcal{M}$ together with the linearity of the connection ∇ in the subscripted argument to obtain

$$\sum_{k,l=1}^{d}(\mathrm{Cov}_l E_l, E_k)\nabla_k = \sum_{l=1}^{d}\nabla_{\mathrm{Cov}_l E_l}\,. \tag{4.11}$$

Renaming the summation index and putting the previous equations together gives the desired expression (4.7). □

We recall that by assumption all manifolds we consider are pathwise connected.

4.2.4. Fact. Every complete Riemannian manifold $\mathcal{M}$ admits a localizing sequence of smooth cut-off functions with a uniformly attenuated gradient bound. This means, there is an increasing sequence $\{\eta_l\}_{l\in\mathbb{N}}$ of smooth functions η_l pointwise converging to unity, $\eta_l(x) \nearrow 1$ for all $x \in \mathcal{M}$, each η_l has compact support, and the uniform gradient bound $g(\operatorname{grad}\eta_l, \operatorname{grad}\eta_l) \leq C_l$ holds for some sequence $\{C_l\}_{l\in\mathbb{N}}$ of positive numbers $C_l \geq 0$ converging to zero.

Proof. The construction uses a result by Greene and Wu [GW79, Corollary to Proposition 2.1], by which one may approximate the distance from a fixed point $y \in \mathcal{M}$ with a smooth function. To be precise, one obtains a smooth function $\upsilon : \mathcal{M} \to \mathbb{R}$ such that $\|\operatorname{grad}\upsilon\| < 1$ and $|\upsilon(x) - \operatorname{dist}(x, y)| < 1$ for all $x \in \mathcal{M}$.

For the construction of the cut-off functions, we pick a real-valued smooth function $\eta : \mathbb{R} \to [0, 1]$ that is bounded above and below by characteristic functions $\chi_{[-1,1]} \leq \eta \leq \chi_{[-2,2]}$, ensuring compact support in the interval $[-2, 2]$. The composition $\eta_l(x) := \eta(\frac{1}{2^l}\upsilon(x))$ then defines an increasing sequence of smooth functions $\eta_l \nearrow 1$ with the gradient bound

$$\operatorname{grad}\eta_l = \frac{1}{2^l}\eta'(\frac{1}{2^l}\upsilon(x))\operatorname{grad}\upsilon(x) \leq \frac{1}{2^l}\sup_{r\in\mathbb{R}}\left|\eta'(r)\right| . \tag{4.12}$$

In addition, due to the completeness of the manifold, the support of each η_l is compact since it is contained in the closed set $\upsilon^{-1}([-2^l, 2^l])$. □

4.2.5. Theorem. If the Riemannian manifold $\mathcal{M}$ is complete, then $-\Delta$ is essentially self-adjoint on $C_c^\infty(\mathcal{M})$. The same holds for $-\Delta^{\mathcal{L}}$ with $C_{c\mathcal{L}}^\infty(\mathcal{M})$ as a domain of essential self-adjointness.

Proof. It is sufficient to show this for $\Delta^{\mathcal{L}}$, since Δ can be considered as the Bochner Laplacian on the trivial bundle $\mathcal{M} \times \mathbb{C}$ with the obvious Hermitian structure. We adapt Davies' treatment of the Dirichlet Laplacian [Dav89, Theorem 5.2.3] using the localizing sequence of cut-off functions described in the preceding construction.

The essential self-adjointness of $-\Delta^{\mathcal{L}}$ is by its positivity equivalent [RS75, Theorem X.26] to having only the zero vector in the orthogonal complement of $(-\Delta^{\mathcal{L}} + 1)\, C_{c\mathcal{L}}^\infty(\mathcal{M})$.

Suppose there is a nonzero vector $u \perp (-\Delta^{\mathcal{L}} + 1)\, C_{c\mathcal{L}}^\infty(\mathcal{M})$, in other words the equation $\Delta^{\mathcal{L}} u = u$ has a weak solution $u \in L^2(hm)$. Using the localizing sequence $\{\eta_l\}_{l\in\mathbb{N}}$ described above, we may estimate

$$
\begin{aligned}
0 \geq -\|\eta_l u\|_2^2 &= \mathcal{E}^{\mathcal{L}}(\eta_l^2 u, u) && (4.13)\\
&= \int_{\mathcal{M}} \sum_{k=1}^{d} h(\nabla_k \eta_l^2 u, \nabla_k u) dm && (4.14)\\
&= \int_{\mathcal{M}} \sum_{k=1}^{d} 2\eta_l E_k(\eta_l) h(u, \nabla_k u) dm \\
&\quad + \int_{\mathcal{M}} \sum_{k=1}^{d} \eta_l^2 h(\nabla_k u, \nabla_k u) dm\,. && (4.15)
\end{aligned}
$$

The last term is positive and we conclude that it must be bounded by

$$
\begin{aligned}
\int_{\mathcal{M}} \sum_{k=1}^{d} \eta_l^2 h(\nabla_k u, \nabla_k u) dm &\leq 2 \int_{\mathcal{M}} \sum_{k=1}^{d} \eta_l \left|E_k(\eta_l)\right| \left|h(u, \nabla_k u)\right| dm && (4.16)\\
&\leq 2 \int_{\mathcal{M}} \sum_{k=1}^{d} \eta_l \left|E_k(\eta_l)\right| \sqrt{h(u,u) h(\nabla_k u, \nabla_k u)}\, dm && (4.17)\\
&\leq 2 \int_{\mathcal{M}} \eta_l \|\operatorname{grad} \eta_l\| \sqrt{h(u,u) \textstyle\sum_k h(\nabla_k u, \nabla_k u)}\, dm && (4.18)
\end{aligned}
$$

where the Cauchy-Schwarz inequality has been used repeatedly. With the abbreviation $c_l := \eta_l \sqrt{\sum_k h(\nabla_k u, \nabla_k u)}$, we obtain

$$
\int_{\mathcal{M}} c_l^2 dm \leq 2 \int_{\mathcal{M}} c_l \|\operatorname{grad} \eta_l\| \sqrt{h(u,u)}\,, \tag{4.19}
$$

and after using the Cauchy-Schwarz inequality again,

$$
\|c_l\|_2^2 \leq 2 \|c_l\|_2 \|\operatorname{grad} \eta_l\|_\infty \|u\|_2\,. \tag{4.20}
$$

To avoid confusion, $\|\cdot\|_2$ denotes the L^2-norm and the term $\|\operatorname{grad} \eta_l\|_\infty$ the essential supremum of the Riemannian length of $\operatorname{grad} \eta_l(x)$ over $x \in \mathcal{M}$. This last inequality involves finite quantities on both sides, because u is a smooth function by an argument related to Sobolev norms as in Appendix B. The properties of the localizing sequence $\{\eta_l\}$ imply that the right-hand side approaches zero in the limit $l \to \infty$. Therefore, by Fatou's lemma $\mathcal{E}^{\mathcal{L}}(u,u) = 0$ or $\nabla^{\mathcal{L}} u = 0$, which is in contradiction to the assumption $\Delta^{\mathcal{L}} u = u \neq 0$. □

Now we will investigate the interplay between Riemannian and complex structures on $\mathcal{M}$.

4.2.6. Definition. Let $\mathcal{L}$ be a holomorphic line bundle with a Riemannian base manifold $\mathcal{M}$. The Hermitian metric on $\mathcal{L}$ is denoted by h, the Riemannian metric on $T\mathcal{M}$ by g, and the natural volume element on $\mathcal{M}$ by dm.

Suppose we pick a local antiholomorphic section in the orthonormal frame bundle of $T^{(0,1)}\mathcal{M}$, which means in a sufficiently small open set U, we have vector fields $Z_1, Z_2, \ldots Z_{d/2} \in \Upsilon^{(0,1)}(\mathcal{M})$ that are orthonormal, $g(Z_k, Z_l) = \delta_{kl}$. For a section ψ, the value of $\operatorname{Tr}^{(0,1)} h(\nabla\psi, \nabla\psi) := \sum_k h(\nabla_{Z_k}\psi, \nabla_{Z_k}\psi)$ depends on the metric and the connection ∇, not on the particular choice of orthonormal holomorphic vector fields. Therefore, we may define the negative holomorphic Laplacian $-\Delta^{(0,\bullet)}$, in a manner analogous to the previous definitions as the operator corresponding to the closure of the quadratic form

$$\mathcal{E}^{(0,\bullet)}(\psi,\psi) := \int_{\mathcal{M}} \operatorname{Tr}^{(0,1)} h(\nabla\psi, \nabla\psi) dm \tag{4.21}$$

initially defined on sections ψ in the domain $C^{\infty}_{c\mathcal{L}}(\mathcal{M})$.

4.2.7. Remark. Let g be a Riemannian metric on a complex manifold $\mathcal{M}$ and Cov its Levi-Civita connection. It is straightforward to check in local coordinates [Zha00, Proposition 7.14] that g is the real part of a Kähler metric if and only if it is compatible with the almost complex structure J and if Cov preserves the splitting of $\Upsilon(\mathcal{M})$ into holomorphic and antiholomorphic parts, that is, $\mathrm{Cov}_X JY = J\mathrm{Cov}_X Y$ for all $X, Y \in \Upsilon(\mathcal{M})$.

4.2.8. Proposition. If g is the real part of a Kähler metric, then the holomorphic Laplacian $\Delta^{(0,\bullet)}$ applied to twice continuously differentiable sections ψ with compact support is expressed by the formula

$$\Delta^{(0,\bullet)}\psi = \sum_{k=1}^{d/2} \left(\nabla_{Z_k} \nabla_{\overline{Z}_k} - \nabla_{\mathrm{Cov}_{Z_k} \overline{Z}_k} \right) \psi \tag{4.22}$$

involving the set of local holomorphic frame vectors $\{Z_k\}_{k=1}^{d/2}$ specified in Definition 4.2.6.

Proof. To obtain the differential operator $-\Delta^{(0,\bullet)}$ representing the quadratic form $\mathcal{E}^{(0,\bullet)}$, we use a strategy similar to the derivation of the differential operator $-\Delta^{\mathcal{L}}$:

$$(\psi, \Delta^{(0,\bullet)}\psi) = \sum_{k=1}^{d/2} \int_{\mathcal{M}} h(\nabla_{\overline{Z}_k}\psi, \nabla_{\overline{Z}_k}\psi) dm \tag{4.23}$$

$$= \int_{\mathcal{M}} \sum_{k=1}^{d/2} \Big(\sum_{l=1}^{d/2} (\mathrm{Cov}_{E_l} E_l + \mathrm{Cov}_{JE_l} JE_l, Z_k) h(\psi, \nabla_{\overline{Z}_k}\psi) - h(\psi, \nabla_{Z_k}\nabla_{\overline{Z}_k}\psi) \Big) dm \tag{4.24}$$

$$= \int_{\mathcal{M}} \sum_{k=1}^{d/2} h\Big(\psi, (\nabla_{\mathrm{Cov}_{Z_k}\overline{Z}_k} - \nabla_{Z_k}\nabla_{\overline{Z}_k})\psi\Big) dm \tag{4.25}$$

The last step

$$\sum_{k,l=1}^{d/2} (\mathrm{Cov}_{E_l} E_l + \mathrm{Cov}_{JE_l} JE_l, Z_k)\overline{Z}_k = \sum_{k=1}^{d/2} \mathrm{Cov}_{Z_k} \overline{Z}_k\,, \tag{4.26}$$

uses $\sum_{k=1}^{d/2} ((\cdot, E_k)E_k + (\cdot, JE_k)JE_k) = \mathrm{id}_{T\mathcal{M}}$ and the compatibility between the almost complex structure and the covariant derivative in the equation $(\mathrm{Cov}_{E_l} E_l + \mathrm{Cov}_{JE_l} JE_l, E_k - iJE_k) = (\mathrm{Cov}_{E_l} JE_l - \mathrm{Cov}_{JE_l} E_l, JE_k + iE_k)$. □

4.2.9. Proposition. Given an orthonormal frame $\{Z_k\}_{k=1}^{d/2}$ of $T^{(1,0)}\mathcal{M}$ obtained from the real frame $\{E_k, JE_k\}_{k=1}^{d/2}$ of $T\mathcal{M}$ by $Z_k := \frac{1}{\sqrt{2}}(E_k - iJE_k)$, then a Weitzenböck-type formula relates the holomorphic and Bochner Laplacians

$$\Delta^{(0,\bullet)} = \frac{1}{2}\big(\Delta^{\mathcal{L}} - \rho\big) \tag{4.27}$$

by a zeroth-order term ρ obtained from contracting the curvature R of the line bundle with the frame vectors, $\rho(x)\psi(x) = \sum_{k=1}^{d/2} R_{\overline{Z}_k, Z_k}\psi(x)$.

Proof. By commuting and anticommuting the terms involving the frame vectors and their conjugates, we obtain the relevant parts that need to be added and subtracted in order to isolate the Bochner Laplacian:

$$\begin{aligned}
\Delta^{(0,\bullet)} &= \sum_{k=1}^{d/2}\Big(\nabla_{Z_k}\nabla_{\overline{Z}_k} - \nabla_{\mathrm{Cov}_{Z_k}\overline{Z}_k}\Big) && (4.28)\\
&= \frac{1}{2}\sum_{k=1}^{d/2}\Big(\nabla_{Z_k}\nabla_{\overline{Z}_k} + \nabla_{\overline{Z}_k}\nabla_{Z_k} - [\nabla_{\overline{Z}_k},\nabla_{Z_k}] \\
&\qquad - \nabla_{\mathrm{Cov}_{Z_k}\overline{Z}_k+\mathrm{Cov}_{\overline{Z}_k}Z_k} + \nabla_{\mathrm{Cov}_{\overline{Z}_k}Z_k-\mathrm{Cov}_{Z_k}\overline{Z}_k}\Big) && (4.29)\\
&= \frac{1}{2}\sum_{k=1}^{d/2}\Big(\nabla_{Z_k}\nabla_{\overline{Z}_k} + \nabla_{\overline{Z}_k}\nabla_{Z_k} - \nabla_{\mathrm{Cov}_{Z_k}\overline{Z}_k+\mathrm{Cov}_{\overline{Z}_k}Z_k} \\
&\qquad - [\nabla_{\overline{Z}_k},\nabla_{Z_k}] + \nabla_{[\overline{Z}_k,Z_k]}\Big) && (4.30)\\
&= \frac{1}{2}\Big(\Delta^{\mathcal{L}} - \sum_{k=1}^{d/2} R_{\overline{Z}_k,Z_k}\Big)\,. && (4.31)
\end{aligned}$$

Hereby, the torsion-free property of the Levi-Civita connection Cov has been used in order to create the commutator $[\overline{Z}_k, Z_k] = \overline{Z}_k Z_k - Z_k\overline{Z}_k = \mathrm{Cov}_{\overline{Z}_k} Z_k - \mathrm{Cov}_{Z_k}\overline{Z}_k$ and thus the term involving the curvature of the line bundle. $\square$

4.2.10. Remark. If the curvature of the bundle and the Kähler form $\omega = \frac{1}{2}g(\cdot, J\cdot)$ are in the prequantum relation (3.22), then ρ is a constant,

$$\rho = \sum_{k=1}^{d/2} R_{\overline{Z}_k,Z_k} = -i\sum_{k=1}^{d/2} R_{E_k,JE_k} = -\frac{1}{2\hbar}\sum_{k=1}^{d/2} g(E_k,E_k) = -\frac{d}{4\hbar}\,. \qquad (4.32)$$

4.3 Berezin-Toeplitz Operators as Limits of Schrödinger Operators

This section shows how a Berezin-Toeplitz operator can be extended to a family of Schrödinger operators and reconstructed as a monotone limit of this family. A major benefit is that the knowledge about Schrödinger operators can be used to find sufficient conditions for the semiboundedness of $\mathcal{T}_f$, thereby ensuring the self-adjointness of the associated Berezin-Toeplitz operator.

4.3.1. Convention. In the following, $\mathcal{L}$ is always a holomorphic Hermitian line bundle and g is assumed to be the real part of a Kähler metric on the base manifold $\mathcal{M}$.

4.3.2. Proposition. If the manifold $\mathcal{M}$ is complete, then the space $L^2_{hol}(hm)$ is in the domain of the form-closure of $\mathcal{E}^{(0,\bullet)}$ and can be identified as the null-space $\{\psi \in L^2(hm) : -\Delta^{(0,\bullet)}\psi = 0\}$ of $-\Delta^{(0,\bullet)}$.

Proof. Given $\psi \in L^2_{hol}(hm)$, we need to construct a Cauchy sequence $\{\psi_l\}_{l\in\mathbb{N}}$ in $C^\infty_{c\mathcal{L}}(\mathcal{M})$ which converges to ψ with respect to the form-norm, $\|\psi_l - \psi\|_{\mathcal{E}^{(0,\bullet)}} \to 0$. To this end, we use an increasing sequence of localizing cut-off functions $\eta_l : \mathcal{M} \to [0,1]$ with the uniform gradient bound $\sup_{x\in\mathcal{M}} \|\mathrm{grad}\,\eta_l(x)\| \leq \frac{C}{2^l}$ for some constant $C > 0$, as described in the preceding section. Then by monotone convergence $\|\eta_l\psi - \psi\| \to 0$, and the remaining term in the form-norm can be estimated by

$$\begin{aligned}
\mathcal{E}^{(0,\bullet)}(\eta_l\psi, \eta_l\psi) &= \int_{\mathcal{M}} \sum_k h(\nabla_{\overline{Z}_k}\eta_l\psi, \nabla_{\overline{Z}_k}\eta_l\psi)\, dm && (4.33)\\
&= \sum_k \int_{\mathcal{M}} \Big(|Z_k(\overline{\eta}_l)|^2\, h(\psi,\psi) + |\eta_l|^2\, h(\nabla_{\overline{Z}_k}\psi, \nabla_{\overline{Z}_k}\psi) \\
&\qquad + 2\,\mathfrak{Re}\left(Z_k(\overline{\eta}_l) h(\psi, \nabla_{\overline{Z}_k}\psi)\right)\Big)\, dm && (4.34)\\
&\leq \frac{C^2}{2^{2l}}\|\psi\|^2 + \int_{\mathcal{M}} |\eta_l|^2 \sum_k h(\nabla_{\overline{Z}_k}\psi, \nabla_{\overline{Z}_k}\psi)\, dm \\
&\quad +2\int_{\mathcal{M}} \sum_k \left| Z_k(\overline{\eta}_l) h(\psi, \nabla_{\overline{Z}_k}\psi)\right|\, dm\,. && (4.35)
\end{aligned}$$

Using the Cauchy-Schwarz inequality, we have

$$\mathcal{E}^{(0,\bullet)}(\eta_l\psi, \eta_l\psi) \leq \frac{C^2}{2^{2l}}\|\psi\|^2 + \mathcal{E}^{(0,\bullet)}(\psi,\psi) + \frac{2C}{2^l}\|\psi\|(\mathcal{E}^{(0,\bullet)}(\psi,\psi))^{1/2} \tag{4.36}$$

so by dominated convergence $\mathcal{E}^{(0,\bullet)}(\eta_l\psi - \psi, \eta_l\psi - \psi) \to 0$. Thus, both terms in the form-norm converge to zero. □

4.3.3. Definition. A semibounded Schrödinger operator $S^{\mathcal{L}}_{D,q}$ on $L^2(hm)$ is the self-adjoint operator associated with the form

$$\mathfrak{S}^{\mathcal{L}}_{D,q}(\psi,\psi) = D\mathcal{E}^{\mathcal{L}}(\psi,\psi) + (\psi, q\psi)\,, \tag{4.37}$$

where $D > 0$ is some coupling constant and the requirement

$$(\psi, q^- \psi) \leq c_1 \mathfrak{S}^{\mathcal{L}}_{D,q^+}(\psi,\psi) + c_2(\psi,\psi) \tag{4.38}$$

is satisfied with relative form bound $c_1 < 1$ and some constant $c_2 \geq 0$. Thus, the form domain of $\mathfrak{S}^{\mathcal{L}}_{D,q}$ is obtained from the closure of $C^\infty_{c\mathcal{L}}(\mathcal{M}) \cap \{\psi : (\psi, q^+\psi) < \infty\}$.

4.3.4. Remark. If in addition to the requirement (4.38), the curvature term ρ of Proposition 4.2.9 is also form-bounded perturbation of $S^{\mathcal{L}}_{q^+}$, then

$$\mathfrak{S}^{(0,\bullet)}_{D,q} : \psi \mapsto \mathfrak{S}^{\mathcal{L}}_{D,D\rho+q}(\psi,\psi) = D\mathcal{E}^{(0,\bullet)}(\psi,\psi) + (\psi, q\psi) \tag{4.39}$$

has the same form domain as $\mathfrak{S}^{\mathcal{L}}_{D,q}$ and also defines a generalized Schrödinger operator, hereafter referred to as $S^{(0,\bullet)}_{D,q}$.

4.3.5. Proposition. If the assumptions of Proposition 4.3.2 and Remark 4.3.4 are fulfilled, then $\mathfrak{S}^{(0,\bullet)}_{D,f}$ is semibounded and $\mathcal{T}_f$ on $L^2_{hol}(h\mu)$ is closed and semibounded.

Proof. First we note $\mathfrak{S}^{(0,\bullet)}_{D,f}(\psi,\psi) = \mathcal{T}_f(\psi,\psi)$ for any $D > 0$ and $\psi \in \mathcal{Q}(\mathcal{T}_f)$. Thus, we only need to show that the restriction of $\mathfrak{S}^{(0,\bullet)}_{D,f}$ to the closed subspace $L^2_{hol}(hm)$ is again a closed and semibounded form.

To show closedness, assume a sequence $(\psi_l)_{l\in\mathbb{N}}$ in $L^2_{hol}(hm)$ which is Cauchy with respect to the form-norm. Then by the closedness of $\mathfrak{S}^{(0,\bullet)}_{D,f}$ the sequence has a limit $\psi \in \mathcal{Q}(\mathfrak{S}^{(0,\bullet)}_{D,f})$. However, this limit is contained in $L^2_{hol}(hm)$, because the sequence $(\psi_l)_{l\in\mathbb{N}}$ also converges with respect to the usual norm on $L^2_{hol}(hm)$.

Semiboundedness follows from the inequality

$$\inf_{\substack{\psi\in L^2(hm),\\ \|\psi\|=1}} \mathfrak{S}^{(0,\bullet)}_{D,f}(\psi,\psi) \leq \inf_{\substack{\psi\in L^2_{hol}(hm),\\ \|\psi\|=1}} \mathfrak{S}^{(0,\bullet)}_{D,f}(\psi,\psi) \tag{4.40}$$

due to the set inclusion $L^2_{hol}(hm) \subset L^2(hm)$. □

4.3.6. Remarks. As stated, the above theorem does not imply that $\mathcal{T}_f$ is densely defined. Therefore, T_f might be self-adjoint only on a Hilbert-subspace of $L^2_{hol}(hm)$.

One motivation for choosing m different from the Liouville measure μ becomes apparent here. It may be beneficial to introduce additional form-bounded perturbations derived from a conformal rescaling of h and μ in order to enlarge the class of admissible symbols f.

4.3.7. Proposition. If the assumptions of the preceding proposition hold, then the semigroup generated by $S^{(0,\bullet)}_{D,f}$ converges in the limit $D \to \infty$ strongly to a Berezin-Toeplitz semigroup,

$$\lim_{D\to\infty} e^{-tS^{(0,\bullet)}_{D,f}}\psi = e^{-tT_f}K_f\psi, \tag{4.41}$$

where $t > 0$, $\psi \in L^2(hm)$, and the orthogonal projector $K_f = K^*_f K_f$ maps onto the closure of $\mathcal{Q}(\mathcal{T}_f)$ in $L^2_{hol}(hm)$.

Proof. The limit $D \to \infty$ of $\mathfrak{S}^{(0,\bullet)}_{D,f}$ yields a non-densely defined form

$$\mathfrak{S}^{(0,\bullet)}_{\infty,f} : \psi \mapsto \lim_{D\to\infty} \mathfrak{S}^{(0,\bullet)}_{D,f}(\psi,\psi) \tag{4.42}$$

which is by inspection identical with $\mathcal{T}_f$.

The monotone convergence implies then that $\mathcal{T}_f$ is closed [Sim78] and the semiboundedness follows from that of $\mathfrak{S}^{(0,\bullet)}_{D,f} \le \mathcal{T}_f$ for some $D > 0$. These properties imply that the Berezin-Toeplitz operator T_f associated with $\mathcal{T}_f$ is self-adjoint on the closure $\overline{\mathcal{Q}(\mathcal{T}_f)} \subset L^2_{hol}(hm)$.

By the monotone convergence of forms the self-adjoint operators $S^{(0,\bullet)}_{D,f}$ converge in the strong resolvent sense [Sim78], which in turn implies strong convergence of the semigroups they generate [RS80, Theorem S.14]. □

CHAPTER 5
PROBABILISTIC REPRESENTATION OF BEREZIN-TOEPLITZ SEMIGROUPS

This chapter introduces a new element into the discussion of Berezin-Toeplitz operators, the concept of Brownian motion on the base manifold of the holomorphic line bundle $\mathcal{L}$. In terms of this stochastic process, one may characterize the Kato class of functions on $\mathcal{M}$. It turns out that Kato decomposable functions f lead to semibounded, self-adjoint Berezin-Toeplitz operators T_f on $L^2_{hol}(hm)$, where m is the Riemannian volume measure on $\mathcal{M}$. The final result in this chapter is a probabilistic expression for Berezin-Toeplitz semigroups, referred to as the Daubechies-Klauder formula.

5.1 A Starter in Stochastic Analysis

Henceforth, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, and $\mathbb{T} := \mathbb{R}^+ := \{t \geq 0\}$ denotes the time domain for stochastic processes. The qualifier almost surely (a.s.) means on a measurable subset $A \in \mathcal{F}$ with probability one, $\mathbb{P}(A) = 1$. Integration with respect to the probability measure $\mathbb{P}$ is customarily expressed in the expectation value $\mathbb{E}[\bullet] := \int (\bullet) d\mathbb{P}$.

5.1.1. Definition. Let $(E, \mathcal{B})$ be a metrizable topological space with the Borel σ-algebra $\mathcal{B}$ generated by all open sets. If E happens to be the d-dimensional Euclidean space $\mathbb{R}^d$, we will always interpret $\mathcal{B}$ as the Borel σ-algebra belonging to the usual metric and topology. A stochastic process X with values in E is a family $\{\mathsf{X}_t\}_{t\in\mathbb{T}}$ of measurable mappings $\mathsf{X}_t : \Omega \to E$.

5.1.2. Definition. A filtration of the σ-algebra $\mathcal{F}$ is a nondecreasing family $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ of sub-σ-algebras, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $s, t \in \mathbb{T}$ with $s \leq t$, such that $\bigcup_{t\geq 0} \mathcal{F}_t = \mathcal{F}$.

The filtration is called right-continuous if $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{n\in\mathbb{N}} \mathcal{F}_{t+1/n}$ for all $t \in \mathbb{T}$. A standard filtration is by definition right-continuous and $\mathcal{F}_0$ already contains all sets of zero probability. A stochastic process X is called adapted to a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ if each X_t is $\mathcal{F}_t$-measurable.

5.1.3. Definition. A stochastic process X is called (right-)continuous if its realizations $t \mapsto \mathsf{X}_t(\omega)$ are a.s. (right-)continuous. In this case, we refer to the realizations as (right-)continuous paths. The predictable σ-algebra $\mathcal{P}$ is the smallest σ-algebra of sets making all continuous, $\{\mathcal{F}_{t+}\}$-adapted processes X measurable when interpreted as mappings $\mathsf{X} : \mathbb{T} \times \Omega \to E$. The $\mathcal{P}$-measurable processes are called predictable.

Remark. All predictable processes are $\{\mathcal{F}_{t+}\}$-adapted [RY94, Exercise I.4.20].

5.1.4. Definition. A stopping time τ is a random variable taking values in $\mathbb{T} \cup \{\infty\}$ such that for all $t \in \mathbb{T}$, the event $\{\tau \le t\}$ belongs to $\mathcal{F}_t$.

5.1.5. Example. Let X be a right-continuous E-valued process, adapted to a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$. If the set Λ is closed in E, then the hitting time

$$\tau_\Lambda := \inf\{t \in \mathbb{T} : \mathsf{X}_t \in \Lambda\} \qquad\qquad (\inf \emptyset := \infty) \tag{5.1}$$

is a stopping time [KS91, Section 1.2]. If, in addition, the filtration is right-continuous, then Λ can be open or closed and τ_Λ is a stopping time [RY94, Proposition I.4.6].

5.1.6. Definition. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. The set of $\mathcal{G}$-measurable, p-integrable random variables is denoted by $L^p(\mathbb{P}, \mathcal{G})$.

The mapping which takes a random variable $\mathsf{X} \in L^1(\mathbb{P}, \mathcal{F})$ to the conditional expectation $\mathbb{E}[\mathsf{X}|\mathcal{G}] \in L^1(\mathbb{P}, \mathcal{G})$ is the unique extension of the orthogonal projection from $L^2(\mathbb{P}, \mathcal{F})$ onto $L^2(\mathbb{P}, \mathcal{G})$.

5.1.7. Definition. An $\{\mathcal{F}_t\}$-adapted process $\{\mathsf{M}_t\}_{t\in\mathbb{T}}$ with values in $\mathbb{R}^d$ is called a martingale if for all $t \in \mathbb{T}$ the following conditions hold:

- Each coordinate process $\mathsf{M}_t^{(k)}, k \in \{1, \dots d\}$ of M_t is integrable, $\mathsf{M}_t^{(k)} \in L^1(\mathbb{P}, \mathcal{F}_t)$.
- For all $0 \le s \le t : \mathbb{E}[\mathsf{M}_t|\mathcal{F}_s] = \mathsf{M}_s$ with probability one.

Remark. If M is a martingale and τ an almost-surely finite stopping time, then the stopped process M^τ given by $\{\mathsf{M}_t^\tau := \mathsf{M}_{\min(t,\tau)}\}_{t\in\mathbb{T}}$ is a martingale [RY94, Corollary II.3.6].

5.2 Stochastic Integrals and Stochastic Differential Equations

From now on, we assume that $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ is a standard filtration, so we do not need to distinguish between $\mathcal{F}_{t+}$ and $\mathcal{F}_t$ any more, and thus all continuous, adapted processes are predictable.

5.2.1. Definition. Let $\mathbb{M}_2$ be the vector space of all continuous, adapted martingales M with values in $\mathbb{R}$ such that $\mathsf{M}_0 = 0$ and for all $t \geq 0$, the random variable $\mathsf{M}_t \in L^2(\mathbb{P}, \mathcal{F}_t)$. The quadratic variation $[\mathsf{M}]$ of $\mathsf{M} \in \mathbb{M}_2$ is the unique [RY94, Theorem IV.1.3] nondecreasing continuous process with initial value $[\mathsf{M}]_0 = 0$, such that the compensated process $\mathsf{M}^2 - [\mathsf{M}]$ is a martingale. For $\mathsf{M}, \mathsf{N} \in \mathbb{M}_2$ we define the cross variation $[\mathsf{M}, \mathsf{N}]$ as

$$[\mathsf{M}, \mathsf{N}] = \frac{1}{4}\big([\mathsf{M} + \mathsf{N}] - [\mathsf{M} - \mathsf{N}]\big)\,. \tag{5.2}$$

5.2.2. Definition. The set of simple predictable processes with values in $\mathbb{R}$ is called $\mathbb{L}_0$. Any X in $\mathbb{L}_0$ is of the form

$$\mathsf{X}_t = \mathsf{C}_0\chi_{\{0\}}(t) + \sum_{j=0}^{l} \mathsf{C}_j\chi_{\{\tau_j < t \leq \tau_{j+1}\}}(t) \tag{5.3}$$

for some $l \in \mathbb{N}$, where $\{\tau_j\}_{j=1}^{l}$ is a set of increasing, a.s. finite stopping times starting with $\tau_0 = 0$, and each random variable $\mathsf{C}_j : \Omega \to \mathbb{R}$ is bounded and measurable with respect to the σ-algebra $\mathcal{F}_{\tau_j}$ of events determined by information up to time τ_j. More explicitly, $F \in \mathcal{F}_{\tau_j}$ means that for all $t \in \mathbb{T} : F \cap \{\tau_j \leq t\} \in \mathcal{F}_t$.

5.2.3. Definition. The Itô integral of $\mathsf{X} \in \mathbb{L}_0$ with respect to $\mathsf{M} \in \mathbb{M}_2$ is the $\mathbb{R}$-valued martingale $\mathsf{I} := \int \mathsf{X} d\mathsf{M} := \sum_{j=0}^{l} \mathsf{C}_j(\mathsf{M}^{\tau_{j+1}} - \mathsf{M}^{\tau_j})$. Note that this definition is only superficially dependent on the different ways to write $\mathsf{X} \in \mathbb{L}_0$ as a finite linear combination according to (5.3).

The immediate goal is now to replace $\mathbb{L}_0$ and $\mathbb{M}_2$ with a wider class of admissible integrands and integrators. To this end, we will employ several limiting procedures in suitable topologies.

5.2.4. Definition. Given $\mathsf{M} \in \mathbb{M}_2$, we define $\mathbb{L}_2(\mathsf{M})$ as the space of predictable, real-valued processes X such that $\mathbb{E}[\int_0^\infty \mathsf{X}_s^2 d[\mathsf{M}]_s] < \infty$.

5.2.5. Remark. Consider the distance functions

$$\mathsf{d}(\mathsf{M},\mathsf{N}) := \sum_{j=1}^{\infty} \frac{1}{2^j} \min\Big(\big(\mathbb{E}\big[[\mathsf{M}-\mathsf{N}]_j\big]\big)^{1/2}, 1\Big) \tag{5.4}$$

for $\mathsf{M},\mathsf{N} \in \mathbb{M}_2$ and

$$\mathsf{d}_{\mathsf{M}}(\mathsf{X},\mathsf{Y}) := \sum_{j=1}^{\infty} \frac{1}{2^j} \min\bigg(\Big(\mathbb{E}\Big[\int_0^j (\mathsf{X}_s - \mathsf{Y}_s)^2 d[\mathsf{M}]_s\Big]\Big)^{1/2}, 1\bigg) \tag{5.5}$$

for $\mathsf{X},\mathsf{Y} \in \mathbb{L}_2(\mathsf{M})$, respectively. By inspection, d is positive definite on $\mathbb{M}_2$, that is, a metric. At first, d_{M} is only positive semidefinite, but it becomes a metric on $\mathbb{L}_2(\mathsf{M})$ once we identify two vectors X,Y whenever $\mathsf{d}_{\mathsf{M}}(\mathsf{X},\mathsf{Y}) = 0$.

5.2.6. Remark. After this identification, $\mathbb{L}_0$ becomes a dense subspace of $\mathbb{L}_2(\mathsf{M})$, because the ring of stochastic intervals $\{[\![0,\tau]\!],]\!]\tau_1,\tau_2]\!] : \tau, \tau_{1,2}$ stopping times$\}$ generates the predictable σ-algebra $\mathcal{P}$ [RY94, Exercise I.4.20]. Hereby, for two stopping times τ_1, τ_2 we set $[\![0,\tau_1]\!] := \{(t,\omega) \subset \mathbb{T} \times \mathbb{R}^d : t \le \tau_1(\omega)\}$ and $]\!]\tau_1,\tau_2]\!] := \{(t,\omega) \subset \mathbb{T} \times \mathbb{R}^d : \tau_1(\omega) < t \le \tau_2(\omega)\}$.

5.2.7. Definition. The class of local martingales $\mathbb{M}_2^{loc}$ consists of martingales M for which there is an increasing sequence of stopping times $\sigma_j \nearrow \infty$ a.s. such that the stopped processes M^{σ_j} are in $\mathbb{M}_2$.

For $\mathsf{M} \in \mathbb{M}_2^{loc}$, we let $\mathbb{L}_2^{loc}(\mathsf{M})$ denote the space of all predictable processes X for which there is an increasing sequence of stopping times $\tau_j \nearrow \infty$ a.s. such that each $\mathbb{E}\big[\int_0^{\tau_j} \mathsf{X}^2 d[\mathsf{M}]\big] < \infty$.

5.2.8. Definition. The inequality

$$\mathbb{E}\left[\mathsf{I}_t^2\right] = \mathbb{E}\Big[\int_0^t \mathsf{X}_s^2 d[\mathsf{M}]_s\Big] \le \mathbb{E}\Big[\int_0^\infty \mathsf{X}_s^2 d[\mathsf{M}]_s\Big] < \infty\,. \tag{5.6}$$

shows that for a fixed $\mathsf{M} \in M_2$, the stochastic integration $\mathsf{X} \mapsto \mathsf{I} := \int \mathsf{X} d\mathsf{M}$ defines an isometry between $\mathbb{L}_0$ and $\mathbb{M}_2$. Therefore, we can define the Itô integral as the unique continuous linear extension of the mapping $\mathsf{X} \in \mathbb{L}_0 \mapsto \int \mathsf{X} d\mathsf{M} \in \mathbb{M}_2$ to the case $\mathsf{X} \in \mathbb{L}_2(\mathsf{M})$.

If $\mathsf{M} \in \mathbb{M}_2^{loc}$ and $\{\sigma_j\}_{j\in\mathbb{N}}$ is a localizing sequence, we first perform this extension for the stopped processes M^{τ_j} and X^{τ_j} with $\tau_j := \inf\{\sigma_j, r : \int_0^r \mathsf{X}^2 d[\mathsf{M}] \geq j\}$ and then define $\int \mathsf{X} d\mathsf{M} := \lim_{j\to\infty} \int \mathsf{X}^{\tau_j} d\mathsf{M}^{\tau_j}$, valid for $\mathsf{X} \in \mathbb{L}_2^{loc}(M)$.

This extension is characterized by the property [KS91, Proposition 3.2.19] that if $\mathsf{M}, \mathsf{N} \in \mathbb{M}_2^{loc}$ and X is predictable and locally bounded, then we have $\int \left(\int \mathsf{X} d\mathsf{M}\right) d\mathsf{N} = \int \mathsf{X} d[\mathsf{M}, \mathsf{N}]$.

5.2.9. Definition. We say a real-valued process A is of locally bounded variation if there is a sequence $\{\sigma_j\}_{j\in\mathbb{N}}$ of a.s. finite stopping times growing to infinity, and each realization of the stopped process A^{σ_j} is of totally bounded variation on $\mathbb{T}$.

If A satisfies the stronger requirement that its realizations have a.s. finite variation, it is called a process of finite variation.

The property of locally bounded variation or finite variation applies to a process with values in $\mathbb{R}^d$ when it holds for each component.

5.2.10. Definition. An adapted $\mathbb{R}^d$-valued process X is called a continuous semimartingale if there is a decomposition $\mathsf{X} = \mathsf{M} + \mathsf{A}$ with continuous, adapted processes M and A being a martingale and a process of finite variation, respectively.

It is called a continuous local semimartingale if there is a decomposition $\mathsf{X} = \mathsf{M} + \mathsf{A}$ such that M has components $\mathsf{M}^{(k)} \in \mathbb{M}_2^{loc}, k \in \{1, \ldots d\}$, and A is a continuous, adapted process of locally bounded variation.

Remark. In fact, this decomposition is unique [RY94, Proposition IV.1.2].

5.2.11. Definition. Given $\mathsf{M} \in \mathbb{M}_2$ and a process A of locally bounded variation, we define

$$\mathbb{L}_{2,1}(\mathsf{M}, \mathsf{A}) := \left\{\text{predictable } \mathsf{X} : \int_0^\infty \mathsf{X}_s^2 d[\mathsf{M}]_s + \int_0^\infty |\mathsf{X}_s| \, |d\mathsf{A}|_s < \infty\right\}, \tag{5.7}$$

where $|d\mathsf{A}|$ denotes the total variation measure for each realization of A. After a suitable identification of vectors, the refined distance function

$$\mathsf{d}_{\mathsf{M},\mathsf{A}}(\mathsf{X},\mathsf{Y}) := \mathsf{d}_{\mathsf{M}}(\mathsf{X},\mathsf{Y}) + \sum_{j=1}^{\infty} \frac{1}{2^j} \min\Big(\mathbb{E}\Big[\int_0^j |\mathsf{X}_s - \mathsf{Y}_s|\, |d\mathsf{A}|_s\Big], 1\Big) \tag{5.8}$$

becomes a metric and the set of simple predictable processes forms a dense linear subspace of $\mathbb{L}_{2,1}(\mathsf{M},\mathsf{A})$.

5.2.12. Definition. Let X be a simple predictable process, M and A as in the preceding definition. By applying Doob's inequality to the martingale part of $\mathsf{I} = \int \mathsf{X} d\mathsf{M} + \int \mathsf{X} d\mathsf{A}$, we obtain

$$\mathbb{E}[\sup_{t\in[0,l]} |\mathsf{I}_t|] \leq 4\Big(\mathbb{E}\Big[\int_0^l \mathsf{X}^2 d[\mathsf{M}]_s\Big]\Big)^{1/2} + \mathbb{E}\Big[\int_0^l |\mathsf{X}_s|\, |d\mathsf{A}|_s\Big] \tag{5.9}$$

for any $l \in \mathbb{N}$. With the help of a limiting procedure we can define $\int \mathsf{X} d\mathsf{Y} := \int \mathsf{X} d\mathsf{M} + \int \mathsf{X} d\mathsf{A}$ for continuous semimartingales $\mathsf{Y} = \mathsf{M} + \mathsf{A}$ whenever $\mathsf{X} \in \mathbb{L}_{2,1}(\mathsf{M},\mathsf{A})$. In fact, by the above inequality, a convergent sequence of semimartingales in $\mathbb{L}_{2,1}(\mathsf{M},\mathsf{A})$ implies uniform convergence of the corresponding Itô integrals on bounded subsets of $\mathbb{T}$ and thus I is seen to be a continuous process.

Following a similar strategy as for the integral with respect to martingales, a local version of this definition is obtained for continuous local semimartingales decomposed as $\mathsf{M} + \mathsf{A}$ and predictable processes X with a localizing sequence of stopping times $\sigma_n \nearrow \infty$ such that each $\mathsf{X}^{\sigma_n} \in \mathbb{L}_{2,1}(\mathsf{M},\mathsf{A})$.

5.2.13. Fact (Itô's formula). Let $f \in C^2(\mathbb{R}^d)$, M a process with values in $\mathbb{R}^d$ and components $\mathsf{M}^{(k)} \in \mathbb{M}_2^{loc}$ denumerated by $k \in \{1, \dots d\}$, and A an adapted, $\mathbb{R}^d$-valued process of locally bounded variation. We denote by $\mathsf{Y} = \mathsf{M} + \mathsf{A}$ the resulting semimartingale in $\mathbb{R}^d$. For all $t \geq 0$ we have a.s.

$$f(\mathsf{Y}_t) - f(\mathsf{Y}_0) = \sum_{k=1}^{d} \int_0^t \partial_k f(\mathsf{Y}) d\mathsf{Y}^{(k)} + \frac{1}{2} \sum_{k,l=1}^{d} \int_0^t \partial_k \partial_l f(\mathsf{Y}) d[\mathsf{Y}^{(k)}, \mathsf{Y}^{(l)}]\,. \tag{5.10}$$

Hereby, components with finite variation are automatically discarded in the cross variation. By inspection, the process $f(\mathsf{Y}_t)$ is a semimartingale. Although the summation of the components happens after the integration, we will abbreviate the right-hand side of equation (5.10) as $\int_0^t \operatorname{grad} f(\mathsf{Y}) d\mathsf{Y} + \frac{1}{2}\int_0^t d[f(\mathsf{Y}), f(\mathsf{Y})]$.

Proof. See [KS91, Section 3.3]. □

5.2.14. Definition. A continuous martingale $\mathsf{B} = (\mathsf{B}^{(1)}, \ldots \mathsf{B}^{(d)})$ is called d-dimensional Brownian motion with diffusion constant $D > 0$ if it starts at the origin $\mathsf{B}_0 = 0$, the individual coordinate processes are independent and identically distributed, and their quadratic variation grows linearly in time with the constant of proportionality $2D$, $\left[\mathsf{B}^{(k)}\right]_t = 2Dt$, with $k \in \{1, \ldots d\}$.

5.2.15. Definition. Let B denote the d-dimensional Brownian motion with diffusion constant $D > 0$, adapted to a standard filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$. Suppose the mappings $\sigma : \mathbb{R}^n \to \mathbb{R}^{d\times n}$ and $a : \mathbb{R}^n \to \mathbb{R}^n$ are continuous. An adapted continuous n-dimensional process X is called a strong, non-explosive solution of the stochastic differential equation

$$d\mathsf{X}_t = \sigma(\mathsf{X}_t)d\mathsf{B}_t + a(\mathsf{X}_t)dt \tag{5.11}$$

with initial value $x \in \mathbb{R}^n$ if $\mathsf{X}_0 = x$ and $\mathsf{X}_t - \mathsf{X}_0 = \int_0^t \sigma(\mathsf{X})d\mathsf{B} + \int_0^t a(\mathsf{X}_s)ds$ hold with probability one.

5.2.16. Remark. If σ and a are differentiable with bounded derivatives, then a strong solution exists and there is a constant $C > 0$ with $\mathbb{E}[|\mathsf{X}_t|^2] \le Ce^{Ct}$ and any two solutions are equal for all $t \ge 0$ with probability one, see [KS91, Theorem 2.9]. Here and in similar expressions, $|\mathsf{X}|$ denotes the Euclidean norm of X.

5.2.17. Definition. Let $\overline{\mathbb{R}}^n$ be the one-point compactification of $\mathbb{R}^n$. We define the set of continuous paths that are stopped at infinity as $C^\dagger(\mathbb{T}, \overline{\mathbb{R}}^n) := \{w \in C(\mathbb{T}, \overline{\mathbb{R}}^n) : \text{if } w(t) = \infty$ for some t, then $w(t') = \infty$ for all $t' > t\}$.

A strong solution of the stochastic differential equation (5.11) is an $\overline{\mathbb{R}}^n$-valued, adapted process X with realizations in $C^\dagger(\mathbb{T}, \overline{\mathbb{R}}^n)$ and an explosion time $\tau^\dagger := \inf\{t \geq 0 : \mathsf{X}_t = \infty\}$ such that for $t < \tau^\dagger$ the corresponding integral equation holds.

5.2.18. Remark. If σ and a are differentiable, then the stochastic differential equation (5.11) has a unique strong solution.

5.2.19. Definition. The Stratonovich integral of a locally bounded, predictable process $\mathsf{X} \in \mathbb{L}_2^{loc}(\mathsf{M}, \mathsf{A})$ with respect to a continuous local semimartingale $\mathsf{Y} := \mathsf{M} + \mathsf{A}$ composed of the continuous local martingale M and the continuous process A of locally bounded variation is given by $\int \mathsf{X}\delta\mathsf{Y} := \int \mathsf{X}d\mathsf{Y} + \frac{1}{2}[\mathsf{X}, \mathsf{Y}]$.

5.2.20. Property. If X, Y, and Z are locally bounded, continuous local semimartingales, then $\int \mathsf{X}\mathsf{Y}\delta\mathsf{Z} = \int \mathsf{X}\delta\left(\int \mathsf{Y}\delta\mathsf{Z}\right)$ [Eme89, Exercise I.1.12].

5.2.21. Proposition. Let X be a continuous local semimartingale and $f \in C^3(\mathbb{R}^d)$. Then $f(\mathsf{X}_t) - f(\mathsf{X}_0) = \int \text{grad}(\mathsf{X})\delta\mathsf{X}$. In other words, Stratonovich integrals can be manipulated according to the usual rules of calculus.

Proof. Using the definition of the integral,

$$\int \text{grad}\, f(\mathsf{X})\delta\mathsf{X} = \int \text{grad}\, f(\mathsf{X})d\mathsf{X} + \frac{1}{2}\big[\text{grad}\, f(\mathsf{X}), \mathsf{X}\big] \tag{5.12}$$

$$= \int \text{grad}\, f(\mathsf{X})d\mathsf{X} + \frac{1}{2}\sum_{k,l=1}^{d}\int \partial_k\partial_l f(\mathsf{X})d\big[\mathsf{X}^{(k)}, \mathsf{X}^{(l)}\big] \tag{5.13}$$

$$= f(\mathsf{X}) - f(\mathsf{X}_0) \tag{5.14}$$

The second equality is obtained with the help of Itô's formula (5.10) by expressing the term $\text{grad}\, f(\mathsf{X}_t)$ that appears in the cross variation of equation (5.12) as a stochastic integral, and by considering that iterated cross variations vanish. The last equality is again Itô's formula read from right to left. ∎

5.3 Brownian Motion on Riemannian Manifolds

5.3.1. Definition. A continuous $\mathcal{M}$-valued process X adapted to a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ is called a semimartingale on the manifold $\mathcal{M}$ if for each smooth function $f : \mathcal{M} \to \mathbb{R}$ the composition $f \circ \mathsf{X}$ is a continuous local semimartingale.

5.3.2. Convention. Note that from now on, when dealing with processes on manifolds, we always assume their realizations are a.s. continuous.

5.3.3. Definition. Given a semimartingale X with values in a manifold $\mathcal{M}$, the Stratonovich integral of smooth one-forms along X is characterized by following properties:

- $\int_0^t \langle df, \delta\mathsf{X}\rangle = f(\mathsf{X}_t) - f(\mathsf{X}_0)$ for any smooth function $f \in C^\infty(\mathcal{M})$
- $\int_0^t \langle f\alpha, \delta\mathsf{X}\rangle = \int_0^t f(\mathsf{X}_r)\delta\left(\int_0^r \langle \alpha, \delta\mathsf{X}\rangle\right)$ for any smooth one-form α, where the outer integral on the right-hand side is a Stratonovich integral with respect to the real-valued semimartingale in parentheses.

5.3.4. Remark. According to Whitney's embedding theorem [Spi79], every smooth one-form α on $\mathcal{M}$ can be written as a formal series $\alpha = \sum_j b_j da_j$ where at each $x \in \mathcal{M}$ only finitely many of the functions a_j, b_j are nonzero. By its definition and Property 5.2.20, the Stratonovich integral of $\alpha = \sum_j b_j da_j$ with respect to a semimartingale X can then be written as a Stratonovich integral of a real-valued continuous local semimartingale $\int \langle \alpha, \delta\mathsf{X}\rangle = \sum_j \int b_j \circ \mathsf{X}\delta(a_j \circ \mathsf{X})$.

5.3.5. Definition. Given semimartingales X and Y with values in the base manifold $\mathcal{M}$ and the total space $\mathcal{L}$, respectively, and a lifting operator $L : \Upsilon(\mathcal{M}) \to \Upsilon(\mathcal{L})$, then Y is a stochastic horizontal transport

$$\mathsf{Y} = \widehat{\mathsf{X}} \tag{5.15}$$

if $\pi \circ \mathsf{Y} = \mathsf{X}$ and for all smooth one-forms β on $\mathcal{L}$ the following Stratonovich stochastic integrals are equal:

$$\int \langle \beta, \delta\mathsf{Y}\rangle = \int \langle L^*_{\mathsf{Y}}\beta, \delta\mathsf{X}\rangle\,. \tag{5.16}$$

Comment. In the case of smooth, deterministic processes X and Y, equation (5.16) is according to Remark 2.2.31 the defining property of the pull-back form $L^*\beta$. Here, it becomes the key to characterize a horizontal transport $\widehat{\mathsf{X}}$ for any semimartingale X that will in general not allow a construction as in Definition 2.2.27.

5.3.6. Remark. The expression of stochastic horizontal transport in a coordinate patch U_j is slightly more delicate than just replacing the integral of α_j appearing in equation (2.8) by a stochastic analogue according to Definition 5.2.19, because some paths of the semimartingale X might leave U_j before any given time $t > 0$.

Therefore, we define $\tau := \tau_{\mathcal{M}\setminus U_j}$ to be the exit time of X from the set U_j containing the starting point $\mathsf{X}_0 = x$. An argument as in Lemma C.1 of Appendix C then yields

$$H_{\mathsf{X},t}\psi(\mathsf{X}_0) = \psi_j(\mathsf{X}_0)H_{\mathsf{X},t}s_j(X_0) = \psi_j(X_0)\, e^{i\int_0^t \langle \alpha_j,\, \delta \mathsf{X}\rangle} s_j(\mathsf{X}_t)\,, \tag{5.17}$$

the stochastic analogue of equation (2.8), valid for (t, ω) in the stochastic interval $[\![0, \tau]\!]$.

5.3.7. Definition. A process B with values in a manifold $\mathcal{M}$ is called Brownian motion with diffusion constant $D > 0$ if for every smooth function $\phi \in C^\infty(\mathcal{M})$, the difference

$$\mathsf{M}_t := \phi \circ \mathsf{B}_t - \phi \circ \mathsf{B}_0 - \int_0^t D\Delta\phi \circ \mathsf{B}_s ds \tag{5.18}$$

is a real-valued continuous local martingale M.

Remark. If $\mathcal{M} = \mathbb{R}^d$, then this characterization gives indeed the usual Brownian motion with starting point B_0 on d-dimensional Euclidean space, see [RY94, Proposition VII.1.11].

5.3.8. Theorem. For any given d-dimensional Riemannian manifold $\mathcal{M}$ and any $x \in \mathcal{M}$, there is a Brownian motion B with diffusion constant $D > 0$ and a possible explosion time, starting at $\mathsf{B}_0 = x$.

Proof. We construct this process from a Brownian motion $\mathsf{W} = \left(\mathsf{W}^{(1)}, \mathsf{W}^{(2)}, \ldots, \mathsf{W}^{(d)}\right)$ in $\mathbb{R}^d$ that has the same diffusion constant D and is adapted to a standard filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$. To keep track of D and the starting point x, the image measure controlling the realizations of B will from now on be denoted as $\mathbb{P}_x^D$. Let $\{(U_j, \phi_j)\}$ be an atlas containing locally finite charts. We choose in each patch $TU_j \subset T\mathcal{M}$ a section of the orthonormal frame bundle, which means a set of vector fields $\{E_k\}_{k=1}^d$ with $g(E_k, E_l) = \delta_{kl}$ whenever $k, l \in \{1, 2, \dots d\}$.

Each chart $\phi : U \to V \subset \mathbb{R}^d$ pushes forward the vector fields E_k. We define $A_k := \phi_* E_k$ and view them as vector fields $A_k \in C^\infty(V, \mathbb{R}^d)$. After extending them to smooth, compactly supported vector fields $C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$, we solve the Stratonovich stochastic differential equation

$$\delta \mathsf{X} = \sum_{k=1}^{d} A_k(\mathsf{X}) \delta \mathsf{W}^{(k)} \tag{5.19}$$

with initial condition $\mathsf{X}_0 = \phi(x_1)$ on the set $t < \tau_{U^c}$, where τ_{U^c} is the time when X exits from U. The solution can be pulled back to $\mathsf{B}_t := \phi^{-1}(\mathsf{X}_t)$.

Now we have solutions for $x \in U_j$ up to exit time $\tau_{U_j^c}$. The global solution must be welded together from all the patches. In this inductive procedure the solution B is constructed together with a sequence of stopping times $\{\tau_n\}_{n \in \mathbb{N}}$ indicating whenever B exits from a domain of a chart.

Starting at x_1 in some U_{j_1}, we have with τ_1 the exit time from U_{j_1}. After exiting we start the Brownian motion W anew, which means, we replace $\{\mathsf{W}_t\}_{t \geq 0}$ by the shifted process $\{\mathsf{W}_{t+\tau_1} - \mathsf{W}_{\tau_1}\}_{t \geq 0}$ and define the continuation of B_t to be given by the solution of the initial value problem (5.19) for the choice $U = U_{j_2}$ containing $x_2 = \mathsf{B}_{\tau_1}$. This procedure is repeated and defines a solution B up to the time $\tau_\infty := \lim_{n \to \infty} \tau_n$, which is the explosion time of the process. □

5.3.9. Definition. A Riemannian manifold $\mathcal{M}$ is called Brownian complete if for a fixed diffusion constant $D > 0$, a Brownian motion B starting at any $x \in \mathcal{M}$ has an infinite explosion time.

5.4 Heat Kernels and Variations of the Feynman-Kac Formula

5.4.1. Definition. Given a second-order differential operator N on a complex line bundle $\mathcal{L}$ with a Hermitian structure h and a measure m on the base manifold $\mathcal{M}$, we define a heat kernel for Z as a one-parameter family of Schwartz kernels $\{\mathcal{S}_t(x,y) : \mathcal{L}_y \to \mathcal{L}_x\}$ with $t > 0$ and $x, y \in \mathcal{M}$ satisfying the following properties:

- The kernel $\mathcal{S}_t(x,y)$ is jointly continuous in x, y, and t.
- The adjoint of $\mathcal{S}_t(x,y)$ is given by $\mathcal{S}_t(y,x)$.
- For every v with base point $y = \pi(v)$, the integrability condition $\mathcal{S}_t(\cdot,y)v \in L^1_{\mathcal{L}}(h\mu) \cap L^\infty_{\mathcal{L}}(h\mu)$ is satisfied. Moreover, $\mathcal{S}_t(\cdot,y)$ is at least twice continuously differentiable and satisfies the heat equation

$$\frac{\partial}{\partial t}\mathcal{S}_t(\cdot,y) = N\mathcal{S}_t(\cdot,y) \tag{5.20}$$

- Near $t = 0$, the kernel approximates the identity

$$\psi(y) = \lim_{t\to 0}\int_{\mathcal{M}} \mathcal{S}_t(y,x)\psi(x)dm(x) \tag{5.21}$$

  for all continuous, L^p-integrable sections ψ, $p \geq 1$.

5.4.2. Definition. By convolution with the heat kernel, we define a strongly continuous semigroup of operators $\{\mathcal{S}_t\}_{t\geq 0}$ acting on sections $\psi \in L^p(hm)$ by

$$\mathcal{S}_t\psi(y) := \int_{\mathcal{M}} \mathcal{S}_t(y,x)\psi(x)dm(x)\,. \tag{5.22}$$

In particular, for the case $p = 2$ this is a strongly continuous semigroup of bounded, self-adjoint operators. According to a standard argument [Rud91, Theorem 13.38], the semigroup $\{\mathcal{S}_t\}$ then has a self-adjoint generator which is an extension of N.

5.4.3. Fact. On complete Riemannian manifolds, the Laplacians Δ and $\Delta^{\mathcal{L}}$ possess unique heat kernels $\{p_t\}$ and $\{p_t^{\mathcal{L}}\}$, respectively.

One fundamental ingredient in the uniqueness is the essential self-adjointness shown in Theorem 4.2.5: In each case, there is only one way to have a generator of a strongly-continuous, self-adjoint semigroup that coincides with the Laplacian on compactly supported, smooth functions/sections. Thus, the associated semigroups $\{P_t\}$ and $\{P_t^{\mathcal{L}}\}$ are unique.

The transition from the semigroups to their kernels is also without ambiguity because we require that $\{p_t\}$ and $\{p_t^{\mathcal{L}}\}$ be jointly continuous. In fact, the process of constructing the kernels according to Appendix B shows that they are even smooth.

If $\mathcal{M}$ is complete with Ricci curvature bounded below, then $\mathcal{M}$ is also Brownian complete [Eme89, Remark 5.38] and the heat kernel of Δ preserves probabilities [Dav85] in the sense that for all $t > 0$, $\int_{\mathcal{M}} p_t(x,y)dm(y) = 1$. This is an indication of the connection between the heat kernel of the Laplacian and stochastic processes that we will pursue in the remaining part of this chapter.

5.4.4. Definition. Let $\mathcal{M}$ be a Brownian-complete Riemannian manifold and $\{\mathbb{P}_x^D\}_{x\in\mathcal{M}}$ a family of Brownian-motion measures with a fixed diffusion constant $D > 0$. A real-valued function $q : \mathcal{M} \to \mathbb{R}$ belongs to the Kato class $\mathcal{K}(\mathbb{P}^D)$ if the following condition is satisfied:

$$\lim_{t\searrow 0} \sup_{x\in\mathcal{M}} \int_0^t \mathbb{E}_x^D[|q|\,(\mathsf{B}_s)]\,ds = 0\,. \tag{5.23}$$

Whenever this property holds only locally, which means $\chi_\Lambda q \in \mathcal{K}(\mathbb{P}^D)$ for all compact sets Λ in $\mathcal{M}$, we write $q \in \mathcal{K}_{loc}(\mathbb{P}^D)$.

If a real-valued function q satisfies $q^+ \in \mathcal{K}_{loc}(\mathbb{P}^D)$ and $q^- \in \mathcal{K}(\mathbb{P}^D)$ then it is called Kato decomposable, symbolized as $q \in \mathcal{K}_\pm(\mathbb{P}^D)$.

Remark. If a function has the global or local Kato property for one choice of $D > 0$, then this holds for any $D > 0$. The reason to include D in the definition is merely for the consistency of notation.

5.4.5. Fact. If the Ricci curvature of a Riemannian manifold is bounded from below, then the local Kato property implies local integrability with respect to the volume measure, $\mathcal{K}_{loc}(\mathbb{P}^D) \subset L^1_{loc}(m)$. This fact may be derived using a strictly positive, lower bound $c > 0$ for the kernel of the semigroup $p_r(x,y) \geq c$, uniform in $x,y \in \Lambda$ and $\epsilon < r < t$ for some arbitrary $\epsilon > 0$ and any compact set $\Lambda \subset \mathcal{M}$ [Dav89, Stu92]. Thus, if $q : \mathcal{M} \to \mathbb{R}$ satisfies

$\chi_\Lambda q \in \mathcal{K}(\mathbb{P}^D)$, then the local integrability follows from the lower bound of the kernel and the finiteness of $\sup_{x\in\mathcal{M}} \int_0^t \mathbb{E}_x^D[|q|(\mathsf{B}_r)\chi_{\{\mathsf{B}_r\in\Lambda\}}]dr$.

The following lemma goes back to Khaśminskii [Kha59]. Our discussion of the Kato class follows some ideas of the nice exposition in [Szn98]. The sole purpose of the following passage is to show that functions from the Kato class can be viewed as infinitesimally form-bounded perturbations of the Dirichlet and Bochner Laplacians.

5.4.6. Lemma. Suppose $0 \le q \in \mathcal{K}(\mathbb{P}^D)$, and $t > 0$ is chosen such that

$$\kappa := \sup_{x\in\mathcal{M}} \mathbb{E}_x^D\left[\int_0^t q(\mathsf{B}_r)dr\right] < 1\,, \tag{5.24}$$

then

$$\sup_{x\in\mathcal{M}} \mathbb{E}_x^D\left[e^{\int_0^t q(\mathsf{B}_r)dr}\right] \le \frac{1}{1-\kappa}\,. \tag{5.25}$$

Proof. Given $t > 0$ satisfying condition (5.24), we select $x \in \mathcal{M}$ and use the strong Markov property to write

$$\mathbb{E}_x^D\left[e^{\int_0^t q(\mathsf{B}_r)dr}\right] = \sum_{l=0}^\infty \frac{1}{l!}\mathbb{E}_x^D\Big[\Big(\int_0^t q(\mathsf{B}_r)dr\Big)^l\Big] \tag{5.26}$$

$$= 1 + \sum_{l=1}^\infty \int\limits_{\{0<r_1<r_2<\cdots<r_l<t\}} \mathbb{E}_x^D\big[q(\mathsf{B}_{r_1})\cdots q(\mathsf{B}_{r_l})\big]\,dr_1\,dr_2\cdots dr_l \tag{5.27}$$

$$= 1 + \sum_{l=1}^\infty \int\limits_{\{0<r_1<r_2<\cdots<r_{l-1}<t\}} \mathbb{E}_x^D\Big[q(\mathsf{B}_{r_1})\cdots q(\mathsf{B}_{r_{l-1}})\mathbb{E}^D_{B_{r_{l-1}}}\Big[\int_0^{t-r_{l-1}} q(\mathsf{B}_r)dr\Big]\Big]dr_1\,dr_2\cdots dr_{l-1}\,. \tag{5.28}$$

Now we may estimate the inner expectation with condition (5.24) and repeat this procedure inductively, yielding

$$\begin{aligned}\mathbb{E}_x^D\left[e^{\int_0^t q(\mathsf{B}_r)dr}\right] &\le 1+\kappa\sum_{l\ge 1}\int_{\{0<r_1<r_2<\cdots<r_{l-1}<t\}}\mathbb{E}_x^D\Big[q(\mathsf{B}_{r_1})\cdots q(\mathsf{B}_{r_{l-1}})\Big]dr_1\,dr_2\cdots dr_{l-1}\\ &\le \sum_{l=0}^{\infty}\kappa^l=\frac{1}{1-\kappa}\,. \end{aligned}\tag{5.29}$$

□

5.4.7. Consequence. The above lemma implies that for $q\in\mathcal{K}(\mathbb{P}^D)$, the mapping Q_t given by

$$Q_t\phi(x):=\mathbb{E}_x^D\left[e^{\int_0^t q(\mathsf{B}_r)dr}\phi(\mathsf{B}_t)\right]\tag{5.30}$$

has a bound $\|Q_t\|_{\infty,\infty}:=\sup_{x\in\mathcal{M},\|\phi\|_\infty=1}|Q_t\phi(x)|\le e^{Ct}/(1-\kappa)$ on $L^\infty(m)$ with exponential growth in t.

Proof. Without loss of generality, we consider $q\ge 0$ and $\phi(x)=1$ for all $x\in\mathcal{M}$. To begin with, we choose t_0 such that $\kappa=\sup_{x\in\mathcal{M}}\mathbb{E}_x\left[\int_0^{t_0}q(\mathsf{B}_r)dr\right]<1$. Given any $t\ge 0$, we can split $[0,t]$ into $k+1$ subintervals of length at most t_0 such that $0\le t-kt_0<t_0$. Inductively using the preceding lemma gives

$$\mathbb{E}_x^D\left[e^{\int_0^t q(\mathsf{B}_r)dr}\right]\le\mathbb{E}_x^D\left[e^{\int_0^{(k+1)t_0}q(\mathsf{B}_r)dr}\right]\tag{5.31}$$

$$\le\mathbb{E}_x^D\left[e^{\int_0^{kt_0}q(\mathsf{B}_r)dr}\mathbb{E}_{\mathsf{B}_{kt_0}}^D\left[e^{\int_0^{t_0}q(\mathsf{B}_r)dr}\right]\right]\tag{5.32}$$

$$\le\frac{1}{1-\kappa}\mathbb{E}_x^D\left[e^{\int_0^{kt_0}q(\mathsf{B}_r)dr}\right]\tag{5.33}$$

$$\le\left(\frac{1}{1-\kappa}\right)^{k+1}.\tag{5.34}$$

Therefore, if we define $C:=-\frac{\ln(1-\kappa)}{t_0}$ then for any $t\ge 0$

$$\mathbb{E}_x^D\left[e^{\int_0^t q(\mathsf{B}_r)dr}\right]\le\frac{1}{1-\kappa}e^{Ct}\,,\tag{5.35}$$

since $k \leq \frac{t}{t_0}$. This shows the claimed bound of Q_t on $L^\infty(m)$. □

5.4.8. Lemma. Given a non-negative function $q \in \mathcal{K}(\mathbb{P}^D)$, then for any $c_1 > 0$ there is a $c_2 > 0$ such that

$$\int_{\mathcal{M}} q\,|\phi|^2\,dm \leq c_1 D\mathcal{E}(\phi,\phi) + c_2\|\phi\|_2^2 \tag{5.36}$$

whenever $\phi \in \mathcal{Q}(\mathcal{E})$. In other words, functions from the Kato class act as infinitesimally form-bounded perturbations of $-D\Delta$.

Proof. The proof is split into two steps.

Step 1. For $q \in \mathcal{K}(\mathbb{P}^D)$, the expression (5.30) defines a strongly continuous semigroup $\{Q_t\}_{t\geq 0}$ of bounded, self-adjoint operators Q_t on $L^2(m)$.

The preceding lemma together with the Jensen and Cauchy Schwarz inequalities establish the boundedness,

$$\int_{\mathcal{M}} \left|\mathbb{E}_x^D[e^{\int_0^t q(\mathsf{B}_s)ds}\phi(\mathsf{B}_t)]\right|^2 dm(x) \tag{5.37}$$

$$\leq \int_{\mathcal{M}} \mathbb{E}_x^D[e^{2\int_0^t q(\mathsf{B}_s)ds}]\mathbb{E}_x^D[|\phi(\mathsf{B}_t)|^2]dm(x) \leq \frac{1}{1-\kappa}e^{Ct}\|\phi\|_2^2\,. \tag{5.38}$$

Because of the Markovian semigroup property, it is enough to show strong continuity at $t = 0$. To this end, we consider

$$\lim_{t\searrow 0}\int_{\mathcal{M}} \left|\mathbb{E}_x^D[(e^{\int_0^t q(\mathsf{B}_s)ds}-1)\phi(\mathsf{B}_t)]\right|^2 dm(x) \tag{5.39}$$

$$\leq \lim_{t\searrow 0}\int_{\mathcal{M}} \mathbb{E}_x^D[(e^{\int_0^t q(\mathsf{B}_s)ds}-1)^2]\mathbb{E}_x^D[|\phi(\mathsf{B}_t)|^2]dm(x) \tag{5.40}$$

$$\leq \lim_{t\searrow 0}\sup_{x\in\mathcal{M}} \mathbb{E}_x^D[e^{2\int_0^t |q|(\mathsf{B}_s)ds}-1]\int \mathbb{E}_x^D[|\phi(\mathsf{B}_t)|^2]dm(x)\,. \tag{5.41}$$

The last step involves Hölder's inequality and the elementary estimate $(e^c-1)^2 \leq e^{2|c|}-1$ for any real number $c \in \mathbb{R}$. Using the definition of the Kato class in this estimate shows that the limit of the L^2-norm in (5.39) vanishes.

Moreover, by the time reversal invariance of Brownian motion each Q_t is seen to be self-adjoint, and according to the Hille-Yosida theorem, there is a semibounded, self-adjoint generator of the semigroup. Given any $c_1 > 0$, we choose a suitable constant c_2 such that replacing q by $\tilde{q} := q/c_1 - c_2/c_1$ in the above procedure yields a contraction semigroup.

Step 2. If we approximate $\tilde{q} \in \mathcal{K}(\mathbb{P}^D)$ with a sequence of semibounded functions $\tilde{q}_l := \min\{\tilde{q}, l\}$, then for any $\phi \in \mathcal{Q}(\mathcal{E})$ the generator of the contraction semigroup $\widetilde{Q}_t$ associated with the function $\tilde{q}_l$ gives rise to a quadratic form

$$\lim_{t \searrow 0} \frac{1}{t}(\phi, \widetilde{Q}_t \phi - \phi) = -D\mathcal{E}(\phi, \phi) - \frac{c_2}{c_1}\|\phi\|_2^2 + (\phi, \min\{\frac{q}{c_1}, l + \frac{c_2}{c_1}\}\phi) \leq 0 \tag{5.42}$$

because $-D\Delta$ is essentially self-adjoint and the multiplication by $\tilde{q}_l$ is a bounded operator. The contractivity of the semigroup furnishes the last inequality, which in turn yields the form-boundedness condition (5.36) by monotone convergence in the limit $l \to \infty$. □

5.4.9. Proposition. Let $\mathcal{L}$ be a Hermitian line bundle with a connection and a length-preserving horizontal transport H. Suppose the base manifold $\mathcal{M}$ is Riemannian and Brownian complete, equipped with a family of Brownian motion measures $\{\mathbb{P}_x^D\}_{x \in \mathcal{M}}$ having a common diffusion constant $D > 0$. Then $q \in \mathcal{K}(\mathbb{P}^D)$ is also a form-bounded perturbation of the negative Bochner Laplacian $-\Delta^{\mathcal{L}}$.

Proof. To make contact with the preceding lemma, we fix $\psi \in L^2(hm)$ and define a function $\phi \in L^2(m)$ with values $\phi(x) := \sqrt{h_x(\psi(x), \psi(x))}$.

We may now verify the L^2-boundedness of the operators $Q_t^{\mathcal{L}}$ given by

$$Q_t^{\mathcal{L}} \psi(x) := \mathbb{E}_x^D \left[e^{\int_0^t q(\mathsf{B}_r) dr} H_{\mathsf{B},t}^{-1} \psi(\mathsf{B}_t) \right] \tag{5.43}$$

with an estimate using that horizontal transport preserves the Hermitian metric and the same strategy as in the preceding lemma,

$$\sqrt{h_x(Q_t^{\mathcal{L}}\psi(x), Q_t^{\mathcal{L}}\psi(x))} \leq \mathbb{E}_x^D \left[e^{\int_0^t q(\mathsf{B}_r) dr} \sqrt{h_x(H_{\mathsf{B},t}^{-1}\psi(\mathsf{B}_t), H_{\mathsf{B},t}^{-1}\psi(\mathsf{B}_t))} \right] \tag{5.44}$$

$$= \mathbb{E}_x^D \left[e^{\int_0^t q(\mathsf{B}_r) dr} \sqrt{h_{\mathsf{B}_t}(\psi(\mathsf{B}_t), \psi(\mathsf{B}_t))} \right] \tag{5.45}$$

$$= \mathbb{E}_x^D \left[e^{\int_0^t q(\mathsf{B}_r) dr} \phi(\mathsf{B}_t) \right] = e^{-tS_{D,q}} \phi(x) \, . \tag{5.46}$$

A similar estimate gives

$$(\psi, Q_t^{\mathcal{L}}\psi - \psi) \leq (\phi, Q_t\phi - \phi) \tag{5.47}$$

and thus together with the preceding lemma the desired form-boundedness. □

Consequence. Therefore, any $q \in \mathcal{K}(\mathbb{P}^D)$ may be used to define a form-bounded perturbation of $-D\Delta$ or $-D\Delta^{\mathcal{L}}$ in order to define a self-adjoint Schrödinger operator.

One may also use $\mathfrak{S}_{D,q^+}$ or $\mathfrak{S}^{\mathcal{L}}_{D,q^+}$ as the unperturbed forms and thus extend this construction to define a Schrödinger operator $\mathfrak{S}_{D,q}$ with $q \in \mathcal{K}_\pm(\mathbb{P}^D)$.

5.4.10. Fact (Feynman-Kac formula). Let $\mathcal{L}$ be a Hermitian line bundle with a connection and a length-preserving horizontal transport H. Suppose the base manifold $\mathcal{M}$ is Riemannian and Brownian complete. Denote by m the natural volume measure on $\mathcal{M}$ and by $\mathbb{P}^D$ a family of Brownian-motion measures having a common diffusion constant $D > 0$.

If $q \in \mathcal{K}_\pm(\mathbb{P}^D)$, then the image of a section $\psi \in L^2(hm)$ under the semigroup $e^{-tS^{\mathcal{L}}_{D,q}}$ generated by the self-adjoint Schrödinger operator $S^{\mathcal{L}}_{D,q}$ has the probabilistic representation

$$e^{-tS^{\mathcal{L}}_{D,q}}\psi(x) = \mathbb{E}^D_x\left[e^{-\int_0^t q(\mathsf{B}_r)dr}H^{-1}_{\mathsf{B},t}\psi(\mathsf{B}_t)\right]. \tag{5.48}$$

The inverse of the stochastic horizontal transport appearing in this equation can either be understood by appealing to the local expression (5.17), or one interprets (5.48) as a shorthand for

$$h_x(s_j(x), e^{-tS^{\mathcal{L}}_{D,q}}\psi(x)) = \mathbb{E}^D_x\left[e^{-\int_0^t q(\mathsf{B}_r)dr}h_{\mathsf{B}_t}(H_{\mathsf{B},t}s_j(x), \psi(\mathsf{B}_t))\right] \tag{5.49}$$

with a local reference section s_j.

Proof. The proof of this fact is given in Appendix C. □

5.4.11. Consequence. Let $\mathcal{L}$ be a holomorphic Hermitian line bundle, assume its base manifold $\mathcal{M}$ is Kähler, and let g denote the real part of the Kähler metric. Let $\{Z_k\}_{k=1}^{d/2}$ denote a local holomorphic orthonormal frame of $T^{(1,0)}\mathcal{M}$. If the curvature term $\rho =$

$\sum_{k=1}^{d/2} R_{\overline{Z}_j,Z_j}$ determined by the connection of the line bundle $\mathcal{L}$ is Kato decomposable, then $-\Delta^{\mathcal{L}}$ and $-\Delta^{(0,\bullet)}$ have the same domain and are essentially self-adjoint on $C^\infty_{c\mathcal{L}}(\mathcal{M})$. If f is also Kato decomposable, then the Feynman-Kac formula (5.48) with $q = D\rho + f$ gives an expression for the Schwartz kernel of the semigroup generated by $S^{(0,\bullet)}_{2D,f}$.

5.4.12. Remark. With the help of the distance function on $\mathcal{M}$, the space $C_{\mathcal{M}}([0,t])$ of continuous paths in $\mathcal{M}$ parametrized by an interval $[0,t]$ can be turned into a complete, separable metric space. In this setting, one may construct a regular conditional probability measure of $\mathbb{P}^D_x$ given B_t [KS91, Theorem 5.3.19].

5.4.13. Definition. We call the family $\{\mathbb{P}^{D,t}_{x,y}\}_{y\in\mathcal{M}}$ a regular conditional probability distribution of $\mathbb{P}^D_x$ on $\mathcal{F}_t$ given B_t if it satisfies the following conditions [Par67, pp. 146–150]:

- For each $y \in \mathcal{M}$, $\mathbb{P}^{D,t}_{x,y}$ is a probability measure on the sets in $\mathcal{F}_t$.
- For each $F \in \mathcal{F}_t$, the mapping $y \mapsto \mathbb{P}^{D,t}_{x,y}(F)$ is measurable.
- For each $F \in \mathcal{F}_t$, $\mathbb{P}^D_x(F) = \int_{\mathcal{M}} \mathbb{P}^{D,t}_{x,y}(F) dw(y)$, where w is the image measure induced by B_t on $\mathcal{M}$, meaning $w(\Lambda) = \mathbb{P}^D_x(\{\mathsf{B}_t \in \Lambda\})$ for all measurable subsets $\Lambda \subset \mathcal{M}$.

Often, $\mathbb{P}^{D,t}_{x,y}$ is called the Brownian bridge measure, or the probability measure of the Brownian motion which starts at x and is conditioned to arrive at $\mathsf{B}_t = y$.

5.4.14. Consequence. With the assumptions of Consequence 5.4.11, the Feynman-Kac formula (5.48) can be modified to give an expression for the integral kernel of the Schrödinger semigroup

$$e^{-tS^{(0,\bullet)}_{2D,f}}(x,y) = p_{D,t}(x,y)\mathbb{E}^{D,t}_{x,y}\Big[e^{-\int_0^t (D\rho(\mathsf{B}_r) + f(\mathsf{B}_r))dr} H^{-1}_{\mathsf{B},t}\Big] \tag{5.50}$$

generated by $S^{(0,\bullet)}_{2D,f}$. Hereby, we use the heat kernel $\{p_{D,t}\}_{t>0}$ of $D\Delta$ and the expectation with respect to the Brownian bridge measure $\mathbb{P}^{D,t}_{x,y}$ given above, and the inverse of the stochastic horizontal transport is understood as a linear mapping from $\mathcal{L}_y$ to $\mathcal{L}_x$.

5.4.15. Definition. In the following results, we always consider a fixed symbol $f \in \mathcal{K}_\pm(\mathbb{P}^D)$. To simplify notation, we choose a reference diffusion constant $D_0 > 0$ and abbreviate for $c \geq 0, v \in \mathcal{L}$ the section

$$\eta_{c,v} := e^{-S^{(0,\bullet)}_{2D_0,cf}}(\cdot, \pi(v))v \tag{5.51}$$

obtained by keeping one end of the Schwartz kernel fixed.

5.4.16. Lemma. For any $v \in \mathcal{L}$ and $c \geq 0$, the section $\eta_{c,v}$ is contained in $L^2(hm)$.

Proof. Due to the linearity in v, it is enough to consider a vector of length $\|v\| = 1$. The L^2-norm of $\eta_{c,v}$ can then be estimated by repeatedly using the Cauchy Schwarz and Hölder inequalities:

$$\begin{aligned}
\|\eta_{c,v}\|_2 &\leq \sup_{\substack{v\in\mathcal{L}_y,\|v\|=1\\ \psi\in L^2(hm),\|\psi\|_2=1}} |(\eta_{c,v}, \psi)| && (5.52)\\
&\leq \sup_{\substack{v\in\mathcal{L},\|v\|=1\\ \psi\in L^2(hm),\|\psi\|_2=1}} \left|h_{\pi(v)}(e^{-S^{(0,\bullet)}_{2D_0,cf}}\psi(\pi(v)), v)\right| && (5.53)\\
&\leq \sup_{\|\psi\|_2=1, x\in\mathcal{M}} \left\|\mathbb{E}^{D_0}_x\left[e^{-\int_0^1 (D_0\rho + cf)(\mathsf{B}_t)dt} H^{-1}_{\mathsf{B},1}\psi(\mathsf{B}_1)\right]\right\| && (5.54)\\
&\leq \sup_{\|\psi\|_2=1, x\in\mathcal{M}} \left(\mathbb{E}^{D_0}_x\left[e^{-2\int_0^1 (D_0\rho + cf)(\mathsf{B}_t)dt}\right]\mathbb{E}^{D_0}_x\left[\|\psi(\mathsf{B}_t)\|^2\right]\right)^{1/2} && (5.55)\\
&\leq \left(\left\|e^{-S^{\mathcal{L}}_{D_0,2D_0\rho+2cf}}\right\|_{\infty,\infty}\right)^{1/2} \|p_1(\cdot, x)\|_\infty < \infty\,. && (5.56)
\end{aligned}$$

The finiteness results from the Kato decomposability of ρ and f and from the boundedness of the heat kernel p_1. □

5.4.17. Lemma. For a fixed $v \in \mathcal{L}$, the mapping $c \mapsto \eta_{c,v}$ is strongly continuous.

Proof. For simplicity, we again assume v to be normalized and consider two non-negative coupling constants c and c'. If $\psi \in L^2(hm)$ also has the L^2-norm $\|\psi\|_2 = 1$, then we may estimate

$$\begin{aligned}
\big|(\eta_{c,v}-\eta_{c',v},\psi)\big| &= \Big|\mathbb{E}^{D_0}_{\pi(v)}\Big[\Big(e^{-\int_0^1(D_0\rho+cf)(\mathsf{B}_t)dt} \\
&\qquad -e^{-\int_0^1(D_0\rho+c'f)(\mathsf{B}_t)dt}\Big)h_{\mathsf{B}_1}(H_{\mathsf{B},1}v,\psi(\mathsf{B}_1))\Big]\Big| \\
&\le \Big(\mathbb{E}^{D_0}_{\pi(v)}\big[|h_{\mathsf{B}_1}(v,\psi(\mathsf{B}_1))|^2\big]\Big)^{1/2}\Big(\mathbb{E}^{D_0}_{\pi(v)}\Big[\Big(e^{-\int_0^1(D_0\rho+cf)(\mathsf{B}_t)dt} \\
&\qquad -e^{-\int_0^1(D_0\rho+c'f)(\mathsf{B}_t)dt}\Big)^2\Big]\Big)^{1/2} \qquad (5.57) \\
&\le \Big(\mathbb{E}^{D_0}_{\pi(v)}\big[h_{\mathsf{B}_1}(\psi(\mathsf{B}_1),\psi(\mathsf{B}_1))\big]\Big)^{1/2}\Big(\mathbb{E}^{D_0}_{\pi(v)}\Big[\Big(e^{-\int_0^1(D_0\rho+cf)(\mathsf{B}_t)dt} \\
&\qquad -e^{-\int_0^1(D_0\rho+c'f)(\mathsf{B}_t)dt}\Big)^2\Big]\Big)^{1/2}. \qquad (5.58)
\end{aligned}$$

Taking the supremum over $\psi, \|\psi\|_2 = 1$ on both sides together with the L^2-contraction property of the unperturbed heat semigroup generated by $D_0\Delta^{\mathcal{L}}$ yields

$$\begin{aligned}
\big\|\eta_{c,v}-\eta_{c',v}\big\|_2 &= \sup_{\psi\in L^2(hm),\|\psi\|_2=1}\big|(\eta_{c,v}-\eta_{c',v},\psi)\big| \\
&\le \Big(\mathbb{E}^{D_0}_{\pi(v)}\Big[\Big(e^{-\int_0^1(D_0\rho+cf)(\mathsf{B}_t)dt}-e^{-\int_0^1(D_0\rho+c'f)(\mathsf{B}_t)dt}\Big)^2\Big]\Big)^{1/2} \qquad (5.59) \\
&\le 2\Big(\mathbb{E}^{D_0}_{\pi(v)}\Big[e^{2\int_0^1(D_0\rho^-+c_0f^-)(\mathsf{B}_t)dt}\Big]\Big)^{1/2}. \qquad (5.60)
\end{aligned}$$

The purpose of the last estimate is to show that with the help of some large c_0, dominated convergence applies to (5.59) in the limit $c' \to c$. □

5.4.18. Theorem. Let $\mathcal{L}$ be a holomorphic line bundle with a Hermitian metric h, suppose its base manifold $\mathcal{M}$ is equipped with a Kähler metric, denote its real part by g and the natural volume measure by m. Let $\mathcal{M}$ be Riemannian complete with Ricci curvature bounded below, to ensure Brownian completeness. Let the real-valued function $f:\mathcal{M}\to\mathbb{R}$ be Kato decomposable with respect to the Brownian motion measure $\mathbb{P}^D$ on $\mathcal{M}$, where the diffusion constant $D>0$ is arbitrary. In addition, suppose the curvature term ρ defined in Proposition 4.2.9 is also Kato decomposable. To include the case when T_f is not densely defined, we denote by K_f the orthogonal projector onto the closure of the form domain $\mathcal{Q}(\mathcal{T}_f)$ in $L^2_{hol}(hm)$.

Then the integral kernel of the Berezin-Toeplitz semigroup $e^{-tT_f}K_f$ on $L^2_{hol}(hm)$ is for $t > 0$ given by the pointwise limit

$$\big(e^{-tT_f}K_f\big)(x,y) = \lim_{D\to\infty} e^{-tS^{(0,\bullet)}_{D,f}}(x,y)\,, \tag{5.61}$$

where $S^{(0,\bullet)}_{D,f}$ is the semibounded Schrödinger operator defined by equation (4.39).

Proof. The proof borrows the strategy of [BLW99a] which is accommodated here to the manifold situation and the case of unbounded f. The key to the present generalization is the use of monotone form convergence.

We have to show that for $u, v \in \mathcal{L}$ with base points $x, y \in \mathcal{M}$ the equation

$$\lim_{D\to\infty} h_x(u, e^{-tS^{(0,\bullet)}_{D,f}}(x,y)v) = (e_u, e^{-tT_f}e_v) \tag{5.62}$$

holds, which by Consequence 3.2.8 characterizes the continuous integral kernel of e^{-tT_f} on $L^2_{hol}(hm) \subset L^2(hm)$.

To see (5.62), we use the semigroup property and express the integral kernel in a scalar product

$$h_x(u, e^{-tS^{(0,\bullet)}_{D,f}}(x,y)v) = \Big(\eta_{D_0/D,u}, \exp\big(-tS^{(0,\bullet)}_{D-2D_0,(D-2D_0)f/D}\big)\eta_{D_0/D,v}\Big) \tag{5.63}$$

which converges in the limit $D \to \infty$ to

$$\lim_{D\to\infty} h_x(u, e^{-tS^{(0,\bullet)}_{D,f}}(x,y)v) = (\eta_{0,u}\,, e^{-tT_f}K_f\eta_{0,v})\,. \tag{5.64}$$

This can be deduced from the strong continuity of $\eta_{c,w}$ in c for any $w \in \mathcal{L}$ and the strong convergence stated in Proposition 4.3.7 together with the uniform boundedness (according to the Banach-Steinhaus theorem) of the operators $\exp\big(-tS^{(0,\bullet)}_{D-2D_0,(D-2D_0)f/D}\big)$ in $D > 2D_0$.

To finish the proof, we observe that the right-hand side of (5.64) is an integral kernel for $\exp(-tT_f)K_f$ on $L^2_{hol}(hm)$ that is, in addition, continuous in x and y and therefore coincides with the right-hand side of (5.62). The continuity of (5.64) is guaranteed by the continuity of the heat kernel derived in Appendix B, and with $K_f \exp(-S^{(0,\bullet)}_{D_0,0}) = K_f$ it can be checked that it indeed constitutes an integral kernel. □

5.4.19. Consequence (Daubechies-Klauder formula). In conjunction with the probabilistic representation given in Consequence 5.4.14, the integral kernel of the Berezin-Toeplitz semigroup $\{e^{-tT_f}\}_{t\geq 0}$ is for $t > 0$ given by the formula

$$e^{-tT_f}(x, y) = \lim_{D\to\infty} p_{D,t}(x, y)\mathbb{E}^{D,t}_{x,y}\Big[e^{-\int_0^t (D\rho(\mathsf{B}_r) + f(\mathsf{B}_r))dr} H^{-1}_{\mathsf{B},t}\Big] . \qquad (5.65)$$

In particular, we obtain the reproducing kernel of $L^2_{hol}(hm)$ as a special case of this formula when $f = 0$.

5.4.20. Remarks. The probabilistic representation of Berezin-Toeplitz semigroups according to formula (5.65) has also been called a Wiener-regularized path integral, because it gives meaning to similar, non-rigorous versions of such path integrals.

With the particular choices of line bundles as in Examples 2.2.21, Numbers 1a, 1b and 3, the formula (5.65) yields an analogue of the situations considered by Daubechies, Klauder, and Paul [DK85, DKP87]. In each case, the complex dimension of the manifold $\mathcal{M}$ is set to $n = 1$, the Riemannian metric g governing the Brownian motion is the real part of the Hermitian metric on $T\mathcal{M}$, and the Kähler form of the Hermitian metric and the curvature of the line bundle are in the prequantum relation (3.22). For an explicit result that does not satisfy this relation, see the treatment of the setting in Examples 2.2.21, Number 3 by [BLW99a] or [BLW99b]. This result differs from that of Daubechies and Klauder [DK85] by a conformal rescaling of the Kähler metric on the base manifold as described in Remarks 3.3.11.

It is worth pointing out that with a suitable analyticity argument, one could obtain from formula (5.61) the probabilistic expression for the Schwartz kernel of the unitary group e^{-itT_f}, which was a primary motivation for [DK82, KD82, KD84, DK85, DK86, DKP87]. The case of bounded f may be treated according to [BLW99a]. The techniques in [Wit00] appear suitable for a generalization to $f \in \mathcal{K}(\mathbb{P}^D)$, but the Kato decomposable case seems to require an additional effort.

CHAPTER 6
A RELATION BETWEEN RESOLVENTS OF BEREZIN-TOEPLITZ OPERATORS BY AN INVARIANCE PROPERTY OF BROWNIAN MOTION

The idea for the following result is derived from a transformation formula relating resolvents of certain Schrödinger operators [DK79, Bla82, CS90]. Recently, Wittich [Wit00] proved this formula and a generalization in the setting of Riemannian manifolds with the help of an invariance property of Brownian motion under harmonic morphisms.

The same strategy applied to the probabilistic representation of the preceding chapter gives a relationship between resolvents of different Berezin-Toeplitz operators whenever the base manifolds of two holomorphic line bundles have Kähler structures that are conformally equivalent. Unfortunately, the rigidity of harmonic morphisms does not allow a nontrivial result when the base manifolds have a higher dimension [Fug78].

6.1 Conformal Metrics on Riemann Surfaces

6.1.1. Definition. A Riemann surface is a complex manifold $\mathcal{M}$ of dimension one, symbolically $\dim_{\mathbb{C}} \mathcal{M} = 1$. Any metric g on $\mathcal{M}$ that is compatible with the almost complex structure J is called conformal.

Remark. In a local coordinate system $z : U \to \mathbb{C}$, the compatibility requirement implies that g has the form

$$g = \frac{\gamma^2(z)}{2}(dz \otimes d\overline{z} + d\overline{z} \otimes dz) \tag{6.1}$$

with a conformal scaling function $\gamma : \mathbb{C} \to \{r > 0\}$. The associated Dirichlet-Laplacian Δ is locally expressed as

$$\Delta = \frac{4}{\gamma^2(z)} \frac{\partial^2}{\partial z \partial \overline{z}}, \tag{6.2}$$

where the abbreviations $\partial/\partial z := \frac{1}{2}(\partial/\partial z_1 - i\partial/\partial z_2)$ and $\partial/\partial\overline{z} := \frac{1}{2}(\partial/\partial z_1 + i\partial/\partial z_2)$ have been used. By inspection of (6.2), any linear combination of a holomorphic or antiholomorphic function is harmonic and vice versa.

As an aside, we remark that any smooth metric g on an oriented surface allows a complex analytic atlas for which g is conformal [Jos97, Theorem 3.11.1].

6.1.2. Definition. Let $\mathcal{M}$ and $\mathcal{M}'$ be Riemann surfaces with conformal metrics g and g', respectively. A mapping $\Phi : \mathcal{M}' \to \mathcal{M}$ is called conformal if it is a local diffeomorphism and $g_{\Phi(x')}(\Phi_* X', \Phi_* X') = \lambda^2(x') g'(X', X')$ holds for all $X' \in T_{x'}\mathcal{M}', x' \in \mathcal{M}'$ with a strictly positive dilatation function $\lambda : \mathcal{M}' \to \{r > 0\}$.

Comment. In the light of this definition, the coordinate charts of a Riemann surface are conformal mappings into the complex plane that is equipped with the standard metric.

6.1.3. Proposition. Given two Riemann surfaces $\mathcal{M}$ and $\mathcal{M}'$ with conformal metrics g and g' and a holomorphic map Φ from $\mathcal{M}'$ onto $\mathcal{M}$, then Φ is conformal on the set where Φ_* is non-zero. In addition, Φ is a harmonic morphism. This means, a local harmonic function $f : U \to \mathbb{C}$ defined on a chart domain $U \subset \mathcal{M}$ pulls back to a harmonic function on $\Phi^{-1}(U)$.

Proof. To prove this local property, it is enough to consider the special case when both domains U and $\Phi^{-1}(U)$ are subsets of the complex number plane. The conformality of Φ results from that of the metrics and because Φ satisfies the Cauchy-Riemann differential equations. By the chain rule

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f \circ \Phi = \left(\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f\right)(\Phi(z)) \frac{\partial \Phi}{\partial z}\frac{\partial \overline{\Phi}}{\partial \overline{z}} \tag{6.3}$$

and the local form (6.2) of the Dirichlet Laplacians associated with g and g', Φ is a harmonic morphism. □

6.1.4. Definition. Let $\mathcal{M}$ and $\mathcal{M}'$ be Riemann surfaces and suppose $\mathcal{M}$ is the base manifold of a holomorphic line bundle $\mathcal{L}$. Given a holomorphic map Φ from $\mathcal{M}'$ onto $\mathcal{M}$, then the pull-back operation creates a bundle $\mathcal{L}'$ with fibers $\mathcal{L}'_x := \pi^{-1}(\Phi(x))$ over $\mathcal{M}'$. The sections $\psi : \mathcal{M} \to \mathcal{L}$ transfer to $\mathcal{L}'$ by $\Phi^*\psi : x' \mapsto \psi(\Phi(x'))$.

The Hermitian structure h on $\mathcal{L}$ pulls back to the fibers of $\mathcal{L}'$ by $\Phi^* h_{x'} := h_{\Phi(x')}$, and the connection by $(\Phi^*\nabla)_X \Phi^*\psi := \Phi^*(\nabla_{\Phi^* X}\psi)$, which makes the curvature form of $\Phi^*\nabla$ the pull-back of the curvature on $\mathcal{L}$.

6.2 An Invariance Property of Brownian Motion under Conformal Mappings of Riemann Surfaces

6.2.1. Definition. Let B be a Brownian motion on a Riemann surface $\mathcal{M}$. An additive functional of Brownian motion is a stochastic process A given in the form

$$\mathsf{A}_\sigma := \int_0^\sigma q(\mathsf{B}_s)ds \tag{6.4}$$

with a non-negative function $q : \mathcal{M} \to \mathbb{R}^+$.

If A is everywhere finite, increasing without jumps, and if $\lim_{\sigma\to\infty} \mathsf{A}_\sigma = \infty$ with probability one, then we define a stochastic time change by the inverse $\tau : \mathbb{T} \times \Omega \to \mathbb{T}$ of A, in other words, $\mathsf{A}_{\tau(t)} = t$ for all $t \geq 0$.

6.2.2. Fact. Let $\mathcal{M}$ and $\mathcal{M}'$ be two Riemann surfaces with conformal metrics g and g'. Suppose B' is a Brownian motion on $\mathcal{M}'$ with an infinite explosion time. If $\Phi : \mathcal{M}' \to \mathcal{M}$ is a surjective holomorphic mapping having the dilatation function $\lambda : \mathcal{M}' \to \mathbb{R}^+$, and if the additive functional $\mathsf{A}_\sigma := \int_0^\sigma \lambda^2(\mathsf{B}'_s)ds$ satisfies the finitenes and limit conditions in the preceding definition of the stochastic time change τ, then $\{\mathsf{B}_t := \Phi(\mathsf{B}'_{\tau(t)})\}_{t\geq 0}$ defines a Brownian motion B on $\mathcal{M}$ that also has an infinite explosion time.

Proof. Essential to the proof is that since Φ is holomorphic, its singular points are isolated and thus by an argument of Fuglede [Fug78] polar. Therefore, $\lambda(\mathsf{B}')$ is an a.s. strictly positive, continuous process and A is increasing. By assumption, the other conditions needed to define the time change are satisfied, so $\{\Phi(\mathsf{B}'_{\tau(t)})\}_{t\geq 0}$ is some stochastic process with values in $\mathcal{M}$. Its Brownian motion character follows from Φ being a harmonic morphism, in essence because it relates the generators of Brownian motion by a conformal scaling operation [CØ83, Theorem 1] as in the localized version (6.3). □

6.3 A Formula Relating Resolvents of Berezin-Toeplitz Operators

6.3.1. Definition. Let $\mathcal{L}$ be a holomorphic line bundle with a smooth, non-degenerate Hermitian metric h and the unique compatible connection ∇. Denote by $L^2_{hol}(hm)$ the Hilbert space of holomorphic sections in $\mathcal{L}$ that are square-integrable with respect to a measure m obtained from a Riemannian metric g on the base manifold.

The resolvent of a self-adjoint, semibounded Berezin-Toeplitz operator T_f with symbol $f : \mathcal{M} \to \mathbb{R}$ is denoted as $G^{\mathcal{L}}_{f-z} := (T_f - z)^{-1}$ for any $z \in \mathbb{C}$ outside of the spectrum of T_f. For such z in the resolvent set, $G^{\mathcal{L}}_{f-z}$ is by definition a bounded operator and via its integral kernel it extends according to Consequence 3.2.8 to all of $L^2(hm)$. In addition, we define $G^{\mathcal{L}}_{f-z}$ to be zero outside the closure of $\mathcal{Q}(\mathcal{T}_f)$ in $L^2_{hol}(hm)$ if T_f is not densely defined as mentioned in Remarks 4.3.6. In short, this extension is characterized by $G^{\mathcal{L}}_{f-z} = G^{\mathcal{L}}_{f-z} K_f$.

6.3.2. Lemma. Let $\mathcal{M}$ and $\mathcal{M}'$ be two Riemann surfaces equipped with conformal metrics g and g', respectively, and suppose $\mathcal{M}$ is the base manifold of a complex line bundle with a connection ∇. In addition, let $\Phi : \mathcal{M}' \to \mathcal{M}$ be a local diffeomorphism that puts the Riemannian metrics g and g' in a conformal relationship $g' = \lambda^2 \Phi^* g$ with a dilatation function $\lambda : \mathcal{M}' \to \mathbb{R}^+$. Given in terms of local frames of normal vectors Z and Z' spanning $T^{(0,1)}\mathcal{M}$ and $T^{(0,1)}\mathcal{M}'$, the curvature terms $\rho = R_{\overline{Z},Z}$ and $\rho' = R_{\overline{Z}',Z'}$ satisfy $\rho'(x') = \lambda^2(x')\rho(\Phi(x'))$.

Proof. The conformal relationship between g and g' implies that the local frames given by Z and Z' can be chosen to satisfy $\Phi_*|_{x'} Z' = \lambda(x') Z|_{\Phi(x')}$. Now, the claimed relationship

$$\rho'(x') = (\Phi^* R)_{\overline{Z}',Z'}|_{x'} \tag{6.5}$$

$$= R_{\Phi_*\overline{Z}',\Phi_* Z'}|_{\Phi(x')} \tag{6.6}$$

$$= \lambda^2(x') R_{\overline{Z},Z}|_{\Phi(x')} = \lambda^2(x')\rho(\Phi(x')) \tag{6.7}$$

follows from the definition of the pull back of R. □

6.3.3. Fact. If for some $D>0$, $\mathfrak{Re}\, z < S^{(0,\bullet)}_{D,f}$, then by the spectral representation

$$(S^{(0,\bullet)}_{D,f} - z)^{-1}\psi = \int_0^\infty e^{-tS^{(0,\bullet)}_{D,f} + tz}\psi\, dt\,. \tag{6.8}$$

Since the monotone convergence of the quadratic forms associated with $S^{(0,\bullet)}_{D,f}$ implies strong resolvent convergence [Sim78], in the limit $D\to\infty$ we obtain

$$G^{\mathcal{L}}_{f-z}\psi = \lim_{D\to\infty}\int_0^\infty e^{-tS^{(0,\bullet)}_{D,f} + tz}\psi\, dt\,. \tag{6.9}$$

6.3.4. Proposition. The integral representation for the resolvent $G^{\mathcal{L}}_{f-z}$ holds even pointwise, that is,

$$G^{\mathcal{L}}_{f-z}\psi(x) = \lim_{D\to\infty}\int_0^\infty e^{-tS^{(0,\bullet)}_{D,f} + tz}\psi(x)\, dt\,. \tag{6.10}$$

Proof. To see this, we apply the point-evaluation functional $\vartheta_u = h_x(u,\cdot)$ to the integrand, use the self-adjointness and semigroup property to obtain

$$h_x(u, e^{-tS^{(0,\bullet)}_{D,f}}\psi(x)) = (e^{-\frac{t}{D}S^{(0,\bullet)}_{D,f}}(\cdot,x)u, e^{-t(1-\frac{1}{D})S^{(0,\bullet)}_{D,f}}\psi)\,, \tag{6.11}$$

and integrate over $t\in[0,\infty]$, which yields a pointwise expression for the image of ψ under the resolvent of $S^{(0,\bullet)}_{D,f}$:

$$h_x(u, (S^{(0,\bullet)}_{D,f} - z)^{-1}\psi(x)) = \int_0^\infty (e^{-\frac{t}{D}S^{(0,\bullet)}_{D,f}}(\cdot,x)u, e^{-t(1-\frac{1}{D})S^{(0,\bullet)}_{D,f} + tz}\psi)\, dt\,. \tag{6.12}$$

The uniform boundedness of $e^{-t(1-\frac{1}{D})S^{(0,\bullet)}_{D,f}}$ in D and the strong convergence of the function $e^{-\frac{t}{D}S^{(0,\bullet)}_{D,f}}(\cdot,x)u$ to $e^{\Delta^{(0,\bullet)}}(\cdot,x)u$ imply that the limit $D\to\infty$ gives

$$h_x(u, G^{\mathcal{L}}_{f-z}\psi(x)) = h_x(e^{\Delta^{(0,\bullet)}}(\cdot,x)u, G^{\mathcal{L}}_{f-z}\psi)\,. \tag{6.13}$$

Therefore, since identity (6.9) holds in the weak sense we conclude that it also holds pointwise. □

6.3.5. Theorem. Let $\mathcal{M}$ and $\mathcal{M}'$ be two Riemann surfaces equipped with conformal metrics g and g', and suppose $\mathcal{M}$ is the base manifold of a holomorphic line bundle $\mathcal{L}$. Furthermore, let $\Phi : \mathcal{M}' \to \mathcal{M}$ be a holomorphic, surjective mapping with dilatation function λ. The pull-back bundle $\mathcal{L}' := \Phi^*\mathcal{L}$ is thought of as being equipped with the Hermitian metric $h' := \Phi^* h$. Assume that $\mathcal{M}$ is both complete and Brownian complete and let the functions $f : \mathcal{M} \to \mathbb{R}$ and $\lambda^2 f \circ \Phi : \mathcal{M}' \to \mathbb{R}$ be Kato decomposable such that the corresponding Berezin-Toeplitz operators can be defined via semibounded quadratic forms on $L^2_{hol}(hm)$ and $L^2_{hol}(h'm')$, where m' is the natural volume with respect to g'.

If $\psi \in L^2_{hol}(hm)$ and $\lambda^2 \psi \circ \Phi \in L^2(h'm')$, then there is a relationship between the resolvents

$$\left(G^{\mathcal{L}}_{f-z}\psi\right)(\Phi(x')) = \left(G^{\mathcal{L}'}_{\lambda^2(f\circ\Phi - z)}\lambda^2\psi\circ\Phi\right)(x') \tag{6.14}$$

whenever $T_f > \mathfrak{Re}\, z$ and $T_{\lambda^2 f\circ\Phi} > \mathfrak{Re}\, z\, T_{\lambda^2}$. Hereby, the extension convention is implicit, since $\lambda^2\psi\circ\Phi$ is not holomorphic unless λ is constant. If additionally $\psi\circ\Phi \in L^2_{hol}(h'm')$, then this equation reads

$$\left(G^{\mathcal{L}}_{f-z}\psi\right)(\Phi(x')) = \left(G^{\mathcal{L}'}_{\lambda^2(f\circ\Phi - z)}T_{\lambda^2}\psi\circ\Phi\right)(x')\,. \tag{6.15}$$

Proof. Since we can always absorb the constant z into the definition of f, we will for convenience assume $z = 0$. Using the probabilistic representation, we have

$$G^{\mathcal{L}}_f\psi(\Phi(x')) = \lim_{D\to\infty}\int_0^\infty \mathbb{E}^D_{\Phi(x')}\left[e^{-\int_0^t (D\rho + f)(\mathsf{B}_s)ds}\psi(\mathsf{B}_t)\right]dt\,. \tag{6.16}$$

By the invariance property of Brownian motion, we can replace B_s with $\Phi(\mathsf{B}_{\tau(s)})$,

$$G^{\mathcal{L}}_f\psi(\Phi(x')) = \lim_{D\to\infty}\int_0^\infty \mathbb{E}^D_{x'}\left[e^{-\int_0^t (D\rho\circ\Phi + f\circ\Phi)(\mathsf{B}_{\tau(s)})ds}\psi\circ\Phi(\mathsf{B}_{\tau(t)})\right]dt\,. \tag{6.17}$$

Interchanging the integration with the expectation and substituting gives

$$G_f^{\mathcal{L}}\psi(\Phi(x')) = \lim_{D\to\infty} \mathbb{E}_{x'}^{D}\Big[\int_0^\infty e^{-\int_0^\tau \lambda^2(\mathsf{B}_\sigma)(D\rho\circ\Phi + f\circ\Phi)(\mathsf{B}_\sigma)d\sigma}\lambda^2(\mathsf{B}_\tau)\psi\circ\Phi(\mathsf{B}_\tau)d\tau\Big]\,. \tag{6.18}$$

After reversing the order of integration again, the relation between the resolvents

$$G_f^{\mathcal{L}}\psi(\Phi(x')) = G_{\lambda^2 f\circ\Phi}^{\mathcal{L}'}\lambda^2\psi\circ\Phi(x') \tag{6.19}$$

can be verified using Lemma 6.3.2. □

Comment. Unfortunately, one does not gain more generality by moving to higher dimensions, because if $\dim_{\mathbb{C}}\mathcal{M}' = \dim_{\mathbb{C}}\mathcal{M} \geq 2$, then Φ is necessarily a global rescaling by a constant. The case $\dim_{\mathbb{C}}\mathcal{M}' > \dim_{\mathbb{C}}\mathcal{M}$ is ruled out by the quantization context because the pull back of the curvature would be degenerate.

CHAPTER 7
SUMMARY AND OUTLOOK

In this work, we have studied a coordinate-independent quantization prescription in the spirit of Berezin and its representation by Wiener-regularized path integrals according to an idea of Daubechies and Klauder. In the present version, these path integrals express semigroups that are generated by self-adjoint semibounded Berezin-Toeplitz operators T_f on a weighted Bergman space $L^2_{hol}(h\mu)$.

A first step towards this result concerned conditions that guarantee self-adjointness and semiboundedness of Berezin-Toeplitz operators. The use of quadratic forms provided a convenient framework to develop such conditions, which in the course of Chapters 3 to 5 evolved from rather abstract form-boundedness to the more concrete requirement in terms of the Kato class. The Dirichlet Laplacian is fundamental to the definition of this class, which therefore possesses a natural geometric characterization.

Besides the Kato class, the holomorphic Laplacian proved central to our implementation of the concept by Daubechies and Klauder on Kähler manifolds. More specifically, we considered perturbations of the holomorphic Laplacian in conjunction with a limiting procedure and the Feynman-Kac formula to construct Wiener-regularized path integrals.

One implication of this construction was that the reproducing kernel of a space of holomorphic, square-integrable sections in a holomorphic Hermitian line bundle over a Kähler manifold could be expressed in purely geometric terms. To this end, we let the Kähler metric govern a Brownian motion on the base manifold and used the connection of the bundle to lift the process horizontally into the fibers. In the ultra-diffusive limit, the expectation of the reverse horizontal transport with respect to the conditional Brownian motion gave the desired reproducing kernel. A similar coordinate-independent expression resulted for the integral kernel of a Berezin-Toeplitz semigroup.

As an application of the path-integral formulation, we used an invariance property of Brownian motion in the probabilistic representation of the semigroups in order to relate the resolvents of certain Berezin-Toeplitz operators.

The fundamental idea behind all those results was the relation between Berezin-Toeplitz operators and Schrödinger operators, which enabled us to transfer all the relevant analytic and probabilistic techniques.

As to further developments, one may ask whether Wiener-regularized path integrals can also be found for continuous representations without underlying complex structures. A step in this direction this has been pointed out by [AK96] with the use of Dirac operators and spin^c structures. Indeed, the completeness argument for the Hilbert space in Appendix A could be applied to a space of merely harmonic functions, since all that is required are mean-value and continuity properties. In addition, the context of Dirac operators may provide enough analytic tools to replace techniques that so far relied on the presence of complex structures.

Finally, it may be worthwhile to study the use of Wiener-regularized path integrals to extend the correspondence principle from the compact Kähler case to non-compact manifolds. Probabilistic representations have often been useful to bridge between different function spaces. In this case, a suitable procedure of approximating non-compact Kähler manifolds by compact ones in the path-integral representation could help enlarging the validity of the correspondence principle.

APPENDIX A
COMPLETENESS OF THE TRADITIONAL BERGMAN SPACE

In this part of the appendix, we show that the space of holomorphic, square-integrable functions on the unit ball studied by Bergman [Ber70] forms a Hilbert space.

Let $d^{2n}z$ denote the Lebesgue measure on $\mathbb{C}^n$. We recall that the open ball $B(x,r) := \{y \in \mathbb{C}^n : \|y-x\|^2 := \sum_{k=1}^n |y_k - x_k|^2 < r^2\}$ centered at $x \in \mathbb{C}^n$ with radius $r \geq 0$ has the volume $\int_{\mathbb{C}^n} \chi_{B(x,r)}(z)\, d^{2n}z = \pi^n r^{2n}/n!$, where $n! := 1 \cdot 2 \cdots n$ denotes the factorial of n.

A.1. Proposition. The inner-product space $L^2_{hol}(B(0,1))$ of holomorphic functions that are square-integrable with respect to the Lebesgue measure on $B(0,1)$ is complete in the norm-topology induced by the usual L^2-inner product.

Proof. Let $(f_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $L^2_{hol}(B(0,1))$. First we show uniform convergence on all compact sets C inside $B(0,1)$. For any such set C, we can find a nonzero safety radius smaller than the distance from C to the boundary of $B(0,1)$, $0 < r < \inf\{\|y-x\| : x \in C, y \in \mathbb{C}^n, \|y\| = 1\}$. Using the mean value property for holomorphic functions and Jensen's inequality in conjunction with the convexity of the square-modulus function $c \mapsto |c|^2$ on $\mathbb{C}$, we estimate

$$\sup_{x\in C} |f_j(x) - f_k(x)|^2 = \sup_{x\in C} \frac{n!}{\pi^n r^{2n}} \Big| \int_{B(x,r)} (f_j(z) - f_k(z))\, d^{2n}z \Big|^2 \tag{A.1}$$

$$\leq \frac{n!}{\pi^n r^{2n}} \sup_{x\in C} \int_{B(x,r)} |f_j(z) - f_k(z)|^2\, d^{2n}z \tag{A.2}$$

$$\leq \frac{n!}{\pi^n r^{2n}} \int_{B(0,1)} |f_j(z) - f_k(z)|^2\, d^{2n}z \tag{A.3}$$

$$= \frac{n!}{\pi^n r^{2n}} \|f_j - f_k\|_2^2 \,. \tag{A.4}$$

The right-hand side can be made arbitrarily small and thus the sequence $(f_j)_{j\in\mathbb{N}}$ converges uniformly on C. By a standard argument in complex analysis, we conclude that the pointwise limit defines a holomorphic function $f : f(z) = \lim_{j\to\infty} f_j(z)$ in $B(0,1)$.

That the convergence $f_j \to f$ is also in the sense of the L^2-norm follows from the Cauchy property of the sequence and from the inequality

$$\|f - f_k\|_2 \leq \liminf_{j\to\infty} \|f_j - f_k\|_2 \tag{A.5}$$

due to pointwise convergence and Fatou's lemma. □

APPENDIX B
EXISTENCE AND SMOOTHNESS OF HEAT KERNELS

The crucial idea used in the construction of the heat kernel is that the index $m \in \mathbb{N}$ of a Sobolev space $W^{m,2}(\mathbb{R}^d)$ controls the regularity properties of the functions it contains.

To simplify the notation, we introduce a customary d-dimensional multi-index $j = (j_1, j_2, \dots j_d)$ with non-negative components $j_1, j_2, \dots j_d \in \mathbb{Z}^+$ and define its degree by $|j| := \sum_{k=1}^{d} j_k$. For $k = (k_1, k_2, \dots, k_d) \in \mathbb{R}^d$, we abbreviate $k^j := k_1^{j_1} k_2^{j_2} \dots k_d^{j_d}$.

B.1. Definition. The Sobolev space $W^{m,2}(\mathbb{R}^d)$ with $m \in \mathbb{N}$ consists of square-integrable functions $f : \mathbb{R}^d \to \mathbb{C}$ having Fourier transforms $\tilde{f} : k \mapsto \int_{\mathbb{R}^d} e^{-ik\cdot x} f(x) d^d x$ that render the Sobolev norm $\int_{\mathbb{R}^d} |\tilde{f}(k)|^2 (1+k^2)^m d^d k$ finite. Equipped with this norm, $W^{m,2}(\mathbb{R}^d)$ is complete.

B.2. Lemma. Given a fixed maximal degree $0 \leq l < m - d/2$, the linear functionals

$$\delta_x^{(j)} : f \longmapsto \int_{\mathbb{R}^d} k^j e^{ik\cdot x} \tilde{f}(k) \frac{d^d k}{(2\pi)^d} \tag{B.1}$$

with $x \in \mathbb{R}^d$ and $|j| \leq l$ are uniformly bounded on $W^{m,2}(\mathbb{R}^d)$. Moreover, in this case any function $f \in W^{m,2}(\mathbb{R}^d)$ has an l-times continuously differentiable representative $x \mapsto \delta_x^{(0)}(f)$. This statement is a rearrangement of [CFKS87, Theorem 12.29].

Proof. The Cauchy-Schwarz inequality

$$\left| \int k^j e^{ik\cdot x} \tilde{f}(k) d^d k \right| \leq \left(\int (1+k^2)^m |\tilde{f}(k)|^2 d^d k \right)^{1/2} \left(\int k^{2j} (1+k^2)^{-m} d^d k \right)^{1/2} \tag{B.2}$$

provides the claimed bound, because the last integral converges if $m > d/2 + l \geq d/2 + |j|$. Moreover, the case $|j| = 0$ shows that $\tilde{f}$ is integrable and its inverse Fourier transform is defined everywhere. By a dominated convergence argument and the integrability of

$k \mapsto k^j \tilde{f}(k)$ in the general case $|j| \leq l$, the function $x \mapsto \delta_x^{(0)}(f)$ is seen to be l-times continuously differentiable. □

B.3. Proposition. Given a complex line bundle $\mathcal{L}$ with a Riemannian base manifold $\mathcal{M}$, the semigroup generated by the self-adjoint Bochner-Laplacian $-\Delta^{\mathcal{L}}$ as defined in (4.6) has a Schwartz kernel $\{p_t^{\mathcal{L}}(x,y) : \mathcal{L}_y \to \mathcal{L}_x\}_{t>0;x,y\in\mathcal{M}}$ that is smooth in the parameters t, x and y.

Proof. As a first step, we establish properties of point-evaluation functionals on Sobolev-type spaces of sections in $\mathcal{L}$.

A section $\sigma = \sigma_j s_j$ with compact support in the domain of a chart $\phi_j : U_j \to V_j \subset \mathbb{R}^d$ can be identified with $\sigma_j \circ \phi_j^{-1}$, and because of its compact support canonically extends by zero on the remaining part of $\mathbb{R}^d$. Due to the smoothness and non-degeneracy of the metric, its eigenvalues obtain a maximum and a nonzero minimum on the support of σ. Therefore, $\Delta^{\mathcal{L}}$ acts locally as a uniformly elliptic operator and allows estimating $(\sigma, (1-\Delta^{\mathcal{L}})\sigma)$ from above and below by multiples of the Sobolev-norm of the function σ_j in $W^{1,2}(\mathbb{R}^d)$. By an inductive procedure, the same technique gives estimates for $(\sigma, (1-\Delta^{\mathcal{L}})^m\sigma)$ in terms of norms in $W^{m,2}(\mathbb{R}^d)$. From now on, we refer to the Sobolev-type space of sections ψ having the finite norm $|||\psi||| := \left\|(1-\Delta^{\mathcal{L}})^{m/2}\psi\right\|$ as $\mathcal{W}_{\mathcal{L}}^{m,2}(\mathcal{M})$.

In analogy to the Sobolev spaces on $\mathbb{R}^d$, the linear functional $\vartheta_u : \psi \mapsto h_x(u, \psi(x))$ evaluating sections at $x = \pi(u)$ is for sufficiently large m bounded in $\mathcal{W}_{\mathcal{L}}^{m,2}(\mathcal{M})$. At first, the bound is only valid on the closed subspace of sections σ with support in a sufficiently small compact set C containing x. However, the sections in the orthogonal complement of the subspace vanish on C and thus the bound of ϑ_u passes unchanged to the whole of $\mathcal{W}_{\mathcal{L}}^{m,2}(\mathcal{M})$ [Hed86]. By a similar localization argument and the preceding lemma, $u \mapsto \vartheta_u$ is seen to be smooth, and so are all the sections ψ in $\mathcal{W}_{\mathcal{L}}^{m,2}(\mathcal{M})$.

The next step of the proof makes use of these smoothness properties to construct the heat kernel.

In the spectral representation we see that for fixed $m \in \mathbb{N}$ and $t_0 > 0$, the operators $(1-\Delta^{\mathcal{L}})^{m/2}e^{t\Delta^{\mathcal{L}}}$ are uniformly in $t \geq t_0$ bounded on $L^2(h\mu)$. In consequence, the semigroup $e^{t\Delta^{\mathcal{L}}}$ is bounded as a mapping from $L^2(h\mu)$ into all Sobolev-type spaces $\mathcal{W}^{m,2}(\mathcal{M})$, and choosing a sufficiently large m proves that the functional $\psi \mapsto \vartheta_u(e^{t\Delta^{\mathcal{L}}}\psi)$ is bounded and linear in $\psi \in L^2(h\mu)$. By the Riesz Representation Theorem and due to the linearity of $u \mapsto \vartheta_u$ in the fibers, there is a vector $q_t(\cdot, \pi(u))u$ in $L^2(h\mu)$ such that $\vartheta_u(e^{t\Delta^{\mathcal{L}}}\psi) = (q_t(\cdot, \pi(u))u, \psi)$ for all $\psi \in L^2(h\mu)$. By the smoothness of $u \mapsto (q_t(\cdot, \pi(u))u, \psi)$ and a uniform boundedness argument, the map $u \mapsto q_t(\cdot, \pi(u))u$ is smooth in the strong sense.

In addition, the map $t \mapsto q_t(\cdot, \pi(u))u$ is also smooth, because the operator $(1 - \Delta^{\mathcal{L}})^{m/2} e^{t\Delta^{\mathcal{L}}}$ is real analytic in $t > 0$.

In the last step, we define a smooth kernel $p_t(x, y)$ by

$$h_x(u, p_t(\pi(u), \pi(v))v) := (q_{t/2}(\cdot, \pi(u))u, q_{t/2}(\cdot, \pi(v))v) \tag{B.3}$$

and claim that it is a Schwartz kernel for $e^{t\Delta^{\mathcal{L}}}$. Using the definition (B.3), the equation

$$\int_{\mathcal{M}\times\mathcal{M}} h_x(\psi(x), p_t(x, y)\sigma(y)) d\mu(x)\, d\mu(y) = (e^{-t\Delta^{\mathcal{L}}/2}\psi, e^{-t\Delta^{\mathcal{L}}/2}\sigma) \tag{B.4}$$

follows for sections $\psi, \sigma \in C^{\infty}_{c\mathcal{L}}(\mathcal{M})$. The self-adjointness and boundedness of $e^{-t\Delta^{\mathcal{L}}/2}$ then completes the proof. □

B.4. Remark. For the L^p-boundedness of $p_t(\cdot, x)$ that we require in Definition 5.4.1, we refer to the literature [Dav88, Dav89, Stu92].

APPENDIX C
A VERSION OF THE FEYNMAN-KAC FORMULA FOR PERTURBATIONS OF THE BOCHNER LAPLACIAN

This appendix is concerned with a proof of formula (5.48). The strategy followed here is a combination of ideas as presented by Simon [Sim79, Chapter V], Bismut [Bis81, Chapitre IX], and Wittich [Wit00]. The core portion of the proof is a version of Itô's formula for sections in line bundles, which will be derived first. The remaining part is an approximation argument.

We will use the same notation as in the main text, so $\mathcal{L}$ is a Hermitian line bundle with a connection ∇ and a metric-preserving horizontal transport H. The base manifold $\mathcal{M}$ is complete with respect to the topology induced by a Riemannian metric. As usual, the Brownian motion in $\mathcal{M}$ with the diffusion constant $D > 0$ and the starting point x is denoted by B, and the underlying probability measure by $\mathbb{P}_x^D$. Moreover, $\mathcal{M}$ is assumed to be Brownian-complete and its Riemannian curvature bounded from below.

The Bochner Laplacian is denoted by $\Delta^{\mathcal{L}}$. An additive perturbation to $-D\Delta^{\mathcal{L}}$ by a function q as discussed in Definition 4.3.3 results in the Schrödinger operator $S^{\mathcal{L}}_{D,q}$. At first, we focus on the unperturbed case.

C.1. Lemma. Given a smooth section ψ in $\mathcal{L}$, then for $t \geq 0$ the equation

$$H^{-1}_{\mathsf{B},t}\psi(\mathsf{B}_t) = \psi(\mathsf{B}_0) + \sum_{k=1}^{d} \int_0^t H^{-1}_{\mathsf{B},r} \nabla_{E_k} \psi(\mathsf{B}_r) \langle E_k^\flat, \delta \mathsf{B}\rangle_r \tag{C.1}$$

relates the horizontal transport to the connection in a stochastic analogue of formula (2.9).

Proof. By localization [Sch80], it suffices to check this on a stochastic interval $[\![0,\tau]\!]$, where τ is the exit time of B from the chart domain U_j containing the starting point B_0. The proof is accomplished using the local formulation of horizontal transport according to equation (5.17).

To simplify the notation, we define semimartingales Y and Z on $[\![0,\tau]\!]$ by

$$\mathsf{Y}_t := \psi_j(\mathsf{B}_t) \quad \text{and} \quad \mathsf{Z}_t := e^{-i\int_0^t \langle \alpha_j, \delta \mathsf{B}\rangle}\,, \tag{C.2}$$

and use the shorthand $\delta \mathsf{W}^{(k)} = \langle E_k^\flat, \delta\mathsf{B}\rangle$, which indeed represents the components of a Brownian motion W in $\mathbb{R}^d$ that is restricted to the stochastic interval. In conjunction with Stratonovich stochastic integrals, an integration by parts rule applies,

$$\mathsf{Z}_t\mathsf{Y}_t - \mathsf{Y}_0 = \int_0^t \mathsf{Z}_r \delta \mathsf{Y}_r + \int_0^t \mathsf{Y}_r \delta \mathsf{Z}_r \tag{C.3}$$

$$= \sum_{k=1}^d \int_0^t \mathsf{Z}_r E_k(\psi_j)(\mathsf{B}_r)\delta\mathsf{W}_r^{(k)} - i\sum_{k=1}^d \int_0^t \mathsf{Z}_r\mathsf{Y}_r\alpha_j(E_k)(\mathsf{B}_r)\delta\mathsf{W}_r^{(k)} \tag{C.4}$$

$$= \sum_{k=1}^d \int_0^t \mathsf{Z}_r \big(E_k(\psi_j) - i\alpha_j(E_k)\psi_j\big)(\mathsf{B}_r)\delta\mathsf{W}_r^{(k)}\,, \tag{C.5}$$

and after reinserting the definitions of X and Y, we obtain the identity

$$e^{-i\int_0^t \langle \alpha_j, \delta \mathsf{B}\rangle}\psi_j(\mathsf{B}_t) - \psi_j(\mathsf{B}_0) = \sum_{k=1}^d \int_0^t e^{-i\int_0^r \langle \alpha_j, \delta \mathsf{B}\rangle}(E_k(\psi_j) - i\alpha_j(E_k)\psi_j)(\mathsf{B}_r)\delta\mathsf{W}_r^{(k)}\,. \tag{C.6}$$

By the localized form of the reverse horizontal transport in accordance with (5.17), this identity shows that the scalars multiplying $s_j(\mathsf{B}_0)$ on both sides of equation (C.1) coincide.

□

C.2. Proposition. With the same notation as in the preceding lemma, a version of the Itô formula in fiber bundles is expressed as

$$H_{B,t}^{-1}\psi(\mathsf{B}_t) = \psi(\mathsf{B}_0) + \sum_{k=1}^d \int_0^t H_{\mathsf{B},r}^{-1}\nabla_{E_k}\psi(\mathsf{B}_r)d\mathsf{W}_r^{(k)} + \int_0^t H_{\mathsf{B},r}^{-1} D\Delta^{\mathcal{L}}\psi(\mathsf{B}_r)dr\,. \tag{C.7}$$

Proof. As the first step of the proof, we repeat the calculation in the preceding lemma, with Y_t replaced by $\mathsf{Y}_t^{(k)} := \big(E_k(\psi_j) - i\alpha_j(E_k)\psi_j\big)(\mathsf{B}_t)$, which yields

$$\mathsf{Z}_t\mathsf{Y}_t^{(k)} - \mathsf{Y}_0^{(k)} = \sum_{l=1}^{d}\int_0^t \mathsf{Z}_r\Big(E_l(E_k(\psi_j)) - (\mathrm{Cov}_l E_k)\psi_j - iE_l(\alpha_j(E_k)) - i\alpha_j(\mathrm{Cov}_l E_k) \\ - i\alpha_j(E_l)(E_k(\psi_j) - i\alpha_j(E_k)\psi_j)\Big)(\mathsf{B}_r)\delta\mathsf{W}_r^{(l)}\,. \tag{C.8}$$

The covariant derivative of the frame vectors enters because those do not get horizontally transported along B.

Now we convert equations (C.5) and (C.8) to Itô differentials and insert the stochastic integral expression for $\mathsf{Z}\mathsf{Y}^{(k)}$ into the cross variation emerging in (C.5),

$$\mathsf{Z}_t\mathsf{Y}_t - \mathsf{Y}_0 = \sum_{k=1}^{d}\int_0^t \mathsf{Z}_r\mathsf{Y}_r^{(k)}\delta\mathsf{W}^{(k)} + \frac{1}{2}\sum_{k=1}^{d}[\mathsf{Z}\mathsf{Y}^{(k)},\mathsf{W}^{(k)}]_t \tag{C.9}$$

$$= \sum_{k=1}^{d}\int \mathsf{Z}_r\mathsf{Y}_r^{(k)}d\mathsf{W}^{(k)} + \frac{1}{2}\sum_{k,l=1}^{d}\int_0^t \mathsf{Z}_r\Big(\big(E_l(E_k(\psi_j)) - (\mathrm{Cov}_l E_k)\psi_j \\ -iE_l(\alpha_j(E_k)) - i\alpha_j(\mathrm{Cov}_l E_k)\big)(\mathsf{B}_r) - i\alpha_j(E_l)\mathsf{Y}_r^{(k)}\Big)d[\mathsf{W}^{(l)},\mathsf{W}^{(k)}]_r\,. \tag{C.10}$$

After contracting the summation indices with the cross variation $\big[\mathsf{W}^{(l)},\mathsf{W}^{(k)}\big]_r = 2D\delta_{lk}r$, a similar identification as in the preceding lemma and the differential operator expression for $\Delta^{\mathcal{L}}$ according to Proposition 4.2.3 proves formula (C.7). □

C.3. Consequence. If $\psi \in L^2(hm)$ and $\mathbb{E}_x^D[\bullet]$ denotes the expectation with respect to the Brownian motion starting at $x \in \mathcal{M}$, then the semigroup generated by $D\Delta^{\mathcal{L}}$ can be represented as

$$e^{tD\Delta^{\mathcal{L}}}\psi(x) = \mathbb{E}_x^D[H_{\mathsf{B},t}^{-1}\psi(\mathsf{B}_t)] \tag{C.11}$$

Proof. First, we assume $\psi \in C^\infty_{c\mathcal{L}}(\mathcal{M})$ and denote $P^{\mathcal{L}}_{D,t}\psi(x) := \mathbb{E}_x(H^{-1}_{\mathsf{B},t}\psi(\mathsf{B}_t)]$. Since $H^{-1}_{\mathsf{B},t}$ preserves the Hermitian metric h, each $P^{\mathcal{L}}_{D,t}$ is seen to be a bounded operator. Moreover, by the time reversal invariance of Brownian motion it is self-adjoint. Finally, the family

$\{P^{\mathcal{L}}_{D,t}\}_{t\geq 0}$ forms a semigroup due to the Markov property

$$P^{\mathcal{L}}_{D,t+s}\psi(x) = \mathbb{E}^D_x[H^{-1}_{\mathsf{B},t+s}\psi(\mathsf{B}_{t+s})] \tag{C.12}$$

$$= \mathbb{E}^D_x[H^{-1}_{\mathsf{B},t}\mathbb{E}^D_{\mathsf{B}_t}[H^{-1}_{\mathsf{B},s}\psi(\mathsf{B}_s)]] = P^{\mathcal{L}}_{D,t}(P^{\mathcal{L}}_{D,s}\psi)(x) \tag{C.13}$$

valid for $s,t \geq 0$. To verify that both sides of (C.11) are identical, we note that the generators agree on $\psi \in C^\infty_{c\mathcal{L}}(\mathcal{M})$, because $P^{\mathcal{L}}_{D,t}\psi$ satisfies the same integral equation as $e^{tD\Delta^{\mathcal{L}}}\psi$,

$$P^{\mathcal{L}}_{D,t}\psi(x) = \mathbb{E}^D_x[H^{-1}_{\mathsf{B},t}\psi(\mathsf{B}_t)] \tag{C.14}$$

$$= \mathbb{E}^D_x\left[\psi(\mathsf{B}_0) + \int_0^t H^{-1}_{\mathsf{B},s}D\Delta^{\mathcal{L}}\psi(\mathsf{B}_s)ds\right] \tag{C.15}$$

$$= \psi(x) + \int_0^t P^{\mathcal{L}}_{D,s}D\Delta^{\mathcal{L}}\psi(x)ds\,. \tag{C.16}$$

By the definition of $P^{\mathcal{L}}_{D,t}$, the semigroup can be defined on all $\psi \in L^2(hm)$. Therefore, its generator defines a self-adjoint extension of $D\Delta^{\mathcal{L}}|_{C^\infty_{c\mathcal{L}}(\mathcal{M})}$, but this is necessarily $D\Delta^{\mathcal{L}}$, because the latter is essentially self-adjoint on $C^\infty_{c\mathcal{L}}(\mathcal{M})$. $\square$

C.4. Theorem. If the assumptions listed at the beginning of this appendix are satisfied, $\psi \in L^2(hm)$, and $q \in \mathcal{K}_\pm(\mathbb{P}^D)$, then the semigroup $e^{-tS^{\mathcal{L}}_{D,q}}$ generated by the Schrödinger operator $S^{\mathcal{L}}_{D,q}$ has the probabilistic representation

$$e^{-tS^{\mathcal{L}}_{D,q}}\psi(x) = \mathbb{E}^D_x\left[e^{-\int_0^t q(\mathsf{B}_s)ds}H^{-1}_{\mathsf{B},t}(\mathsf{B}_t)\right]\,, \tag{C.17}$$

valid for m-almost every $x \in \mathcal{M}$.

Proof. First, we suppose q is continuous, ψ is a smooth section, and both are bounded. Then, along the lines of (C.7) and with the integration by parts rule,

$$e^{-\int_0^t q(\mathsf{B}_s)ds}H^{-1}_{\mathsf{B},t}\psi(\mathsf{B}_t) = \psi(\mathsf{B}_0) + \sum_{k=1}^d \int_0^t e^{-\int_0^r q(\mathsf{B}_s)ds}H^{-1}_{\mathsf{B},r}\nabla_{E_k}\psi(\mathsf{B}_r)d\mathsf{W}^{(k)}_r$$
$$+ \int_0^t e^{-\int_0^r q(\mathsf{B}_s)ds}H^{-1}_{\mathsf{B},r}(D\Delta^{\mathcal{L}}\psi - q\psi)(\mathsf{B}_r)dr\,. \tag{C.18}$$

Since q is bounded, the modification of the heat semigroup defined by inserting (C.18) in the expectation value of (C.11) has as its generator a self-adjoint extension of $(D\Delta^{\mathcal{L}} - q)|_{C^\infty_{c\mathcal{L}}(\mathcal{M})}$. Again, by essential self-adjointness, this is seen to be the difference $-S^{\mathcal{L}}_{D,q} = D\Delta^{\mathcal{L}} - q$.

In the last step, we approximate the general case $q \in \mathcal{K}_\pm(\mathbb{P}^D)$ by truncation. We define the net $\{q_k^{(l)}\}$ of bounded functions

$$x \mapsto q_k^{(l)}(x) := \min\{\max\{q(x), -k\}, l\} \tag{C.19}$$

indexed by $k, l \in \mathbb{N}$. Truncating $q \in \mathcal{K}_\pm(\mathbb{P}^D)$ in this manner gives

$$e^{-tS^{\mathcal{L}}_{D,\, q_k^{(l)}}}\psi(x) = \mathbb{E}^D_x\left[e^{-\int_0^t q_k^{(l)}(\mathsf{B}_s)ds} H^{-1}_{\mathsf{B},t}\psi(\mathsf{B}_t)\right] \tag{C.20}$$

valid for m-almost every x by the above argument. Now, by monotone form convergence we obtain strong convergence on the left when consecutively first $l \to \infty$ and then $k \to \infty$, whereas on the right dominated convergence applies to both limits, because

$$\mathbb{E}^D_x\left[e^{\int_0^t q^-(\mathsf{B}_s)ds} h_{\mathsf{B}_t}(\psi(\mathsf{B}_t), \psi(\mathsf{B}_t))\right] < \infty \tag{C.21}$$

since the negative part $q^- \in \mathcal{K}(\mathbb{P}^D)$ and $x \mapsto h_x(\psi(x), \psi(x))$ is m-integrable. □

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BIOGRAPHICAL SKETCH

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