CONGESTION TOLL PRICING MODELS AND METHODS FOR VARIABLE DEMAND NETWORKS

By

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I dedicate this work to God and my family.
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CONGESTION TOLL PRICING MODELS AND METHODS FOR VARIABLE DEMAND NETWORKS

By

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Major Department: Industrial and Systems Engineering

General variable demand (GVD) traffic assignment problems estimate the traffic volumes on each link and the flows and demands between each origin destination pair. In GVD models, demand is a function of the generalized trip cost on the network. We define two optimization models for GVD traffic assignment problems. In the user equilibrium, each user maximizes his/her own net user benefit. However, this results in the suboptimal utilization of the transportation network. When the total net user benefit is maximized, the most efficient utilization of the network is achieved and the system optimal flows and demands are realized. Our goal is to characterize the set of all tolls which will make the tolled user equilibrium problem have the same flows and demands as the system optimal ones. The set of all such toll vectors is defined as the first best toll set.
Given that there are constraints on tolling some portions of the network, a first best toll pricing scheme might not be feasible. Under such circumstances, a second best toll pricing scheme can be utilized. The appropriate models for second best pricing schemes are mathematical programs with equilibrium constraints (MPECs). Since MPECs are very difficult to solve, obtaining alternative toll vectors by adding side constraints and resolving the MPEC is not desirable. Given a second best solution, we show that there exists a non-empty second best toll set.

Given the first or second best toll set, one can define additional objectives and constraints and solve either linear programs or mixed integer programs to obtain alternative toll vectors. This process is known as the “toll pricing framework” (TPF). We generalize the TPF for the first best fixed demand traffic assignment models to GVD models in order to find alternative first best and second best toll vectors. Two natural choices of such vectors are MINTB and MINMAX toll vectors. The MINTB problem minimizes the number of toll booths whereas the MINMAX problem minimizes the maximum toll on the network.
1.1 Overview

In this thesis, we study several toll pricing schemes for variable demand
network models including the first best pricing and second best pricing schemes.
In particular, we study variable demand models such as traffic assignment (TA)
models with elastic demand, combined distribution assignment, and combined
travel choice models. We extend the first best toll pricing idea introduced for fixed
demand traffic assignment problems by Hearn and Ramana [49] to variable demand
models. Our goal is to synthesize congestion pricing approaches by the economists
and the operations researchers.

In this chapter, we give the background for road pricing by which the
individuals can be charged on the transportation network for using the links
(roads) and increasing the congestion level. Congestion is so costly that transportation
authorities propose to charge the users to reduce the congestion level on the traffic
network. By charging the individuals, the authorities want to achieve several
objectives like cost recovery, demand management, and retrieval of the negative
externalities. We present several traffic management tools such as area licensing;
road pricing; parking pricing; taxes on fuel, tires and spare parts; and inhibiting
vehicle ownership. Since we are particularly interested in road pricing, we explain
the most well-known road pricing concept, Marginal Social Cost Pricing (MSCP),
which is a first best pricing scheme. We describe the electronic road pricing
technology and then present successful implementations of road pricing projects
along with failures.
In Chapter 2, we first introduce the traffic planning process and state Wardrop’s Principles that define the user and system problems in traffic assignment. We summarize the mathematical programming literature on traffic assignment and toll pricing. In particular, we summarize the results on first best toll pricing for a fixed demand traffic equilibrium problem from Bergendorff et al. [11] and Hearn and Ramana [49] as these works are the point of departure for the planned research.

Chapter 3 introduces first best toll pricing for traffic assignment problems with variable demand. After stating the optimality conditions for user and system problems, we characterize the set of all link tolls which cause user equilibrium flows to be the same as the system optimal ones. We extend the fixed demand toll pricing framework to variable demand. Then we show that traffic assignment problems with elastic demand, combined distribution and assignment models, and combined travel choice models are special cases of the variable demand model. We present alternative descriptions of first best elastic demand toll set. Finally, we give some computational results for $\epsilon$-approximation of the toll pricing framework.

In Chapter 4, second best toll pricing models are introduced. These models address the situation where it is not possible to levy tolls on some parts of the transportation network. We define the first best, constrained first best and second best pricing schemes and propose equivalent second best models by using alternative characterizations of the tolled user equilibrium conditions. Further, we extend the first best toll pricing framework to second best pricing.

In Chapter 5, future research directions are presented. One direction is applying computational methods used to solve fixed demand toll pricing problems to also solve variable demand ones. Another direction is to develop specialized algorithms for toll pricing problems where the goal is to minimize the number of toll booths or the maximum toll charge on the network.
1.2 Congestion and Road Pricing

Congestion is becoming an inevitable part of everyday life in most metropolitan areas all over the world. Increasing population and wealth result in more automobiles than current transportation networks can handle. Due to limited expansion possibilities of the transportation network, congestion has increased drastically over the last decade.

In this section, we describe costs of congestion and then list some efficient means of traffic control. Next, we present the objectives of road pricing and finally state alternative traffic management methods.

Arnott and Small [4] estimate that one-third of the vehicular movement occurs in congested areas in 39 metropolitan areas of the U.S. with a population of one million or more. Even without taking into account additional factors such as lost work and leisure time, health problems, stress, discomfort, cost of extra fuel, accidents or air pollution, the annual cost of driving in congested areas is around $48 billion or $640 per driver. Japan’s international co-operation agency calculated that Bangkok loses one third of its potential output due to congestion [19].

Schiller [90] reports results from a study for determining the cost of congestion to drivers in California. In 1996, the congestion cost for an average driver in San Bernardino was $1090 (the highest in the nation) whereas this average was $950 (the third highest in the nation) in the San Francisco Bay area, $920 (the fourth highest in the nation) for Los Angeles and $750 (the fifth highest in the nation) in San Jose. In 1995, 75% of San Francisco’s, 66.5% of Los Angeles’s, 63.8% of San Bernardino and Riverside’s and 60% of San Jose’s rush hour traffic was under congested conditions. Due to congestion, the amount of pollutants released into the air was 250% higher than from free-flowing traffic.

Traffic planners often charge users in order to restrain the number of travellers on the transportation network. By charging the users, transportation planners
try to achieve several objectives. Lo et al. [72] and May [80] state some of these objectives as:

Cost recovery for planning, construction, operation and maintenance. This type of objective is widely used by many transportation agencies to recover costs of building new highways, tunnels, and bridges. In the context of congestion pricing, users can be charged by the operation agency to the extent of the congestion that they are causing on the transportation network. A good example for this case is the privately owned toll road along side SR 91 in Orange County, California [72].

Retrieval of negative externalities. The users are expected to pay not only the direct costs like fuel costs and travel time, but also the costs that he or she imposes on the others, such as additional congestion and travel delay that the driver creates, air pollution, and accidents. All such additional costs are defined to be negative externalities.

Demand management. When the use of the transportation resources is not efficient, that is, when some parts of the network are over-congested compared to other points, those segments can be tolled in order to shift the demand from the over utilized areas. This might increase overall efficiency of the system.

We now describe some alternatives to road pricing. Armstrong-Wright [3], Lo et al. [72], and Morrison [84] review effective demand management tools like area licensing, parking restraints, user taxes, and vehicle ownership restraints. Note that these control measures do not directly affect the root cause of congestion. In addition, they do not improve the quality of driving on the congested roads.

In area licensing, low occupancy vehicles are charged for entering the congested areas during the rush period. Thus, the transportation planning agency aims to reduce the number of private cars during rush hour by encouraging the use of shared vehicles and public transportation. For this system to be successful, there should be adequate public transport for handling the extra demand generated.
Users should be able to pay for the permits in advance; otherwise, unnecessary congestion might occur. There should be few entry points to the system in order to have better control, and enforcement must be sufficiently effective to avoid abuse.

Parking restraint is another effective tool for controlling the traffic in congested areas. With differential parking charges, the objective is to reduce the number of cars entering the congested areas while not interfering with business activities and shopping. For an initial short amount of time, users are charged relatively smaller amounts, and then for longer periods they are charged more. Thus users who need short term parking benefit from such a parking scheme while others are encouraged to use public transportation and commuting. However, for this to be effective, businesses must not subsidize the parking restraint charges of employees.

When proper taxes are imposed on fuel, tires, and spare parts, traffic might decrease substantially on the transportation network. Obviously, these taxes are based on the distance traveled and are easily administered. They do not, however, differentiate between usage during peak and off peak periods or in congested or noncongested areas [58].

Another way to restrain traffic is to inhibit vehicle ownership through high import duties, purchase taxes and annual licensing fees. This is not a distance based restraint, but is easily administered. Similar to user taxes, vehicle ownership restraints are insensitive to location and time of use.

In this dissertation, our primary focus is on road pricing, which is a traffic management tool that can be employed to charge the individuals based on the time, distance and congestion level of the links they are using. For example, *marginal social cost pricing* (MSCP) tolls, which are the most common road pricing concept, can be used to charge the users for the negative externalities they impose on others. In the next section, we briefly summarize MSCP.
1.3 Marginal Social Cost Pricing (First Best Pricing)

Economic theory argues that to achieve economic efficiency in a market, the price of a good or a service should be at its full cost to society. The social cost of travel includes both private and external costs. The private costs are direct ones such as gasoline costs, travel time and automobile usage; the external costs are those that travelers impose on other people, such as congestion, travel delays, air pollution, and accidents. Since the external costs are underpriced, the network is not used efficiently. Pigou [89], Armstrong-Wright [3], Beckmann et al. [7], Elliot [28], Johnson [58], Luk and Chung [74], and Arnott and Small [4] recommend MSCP tolls that are equal to the negative externalities imposed on other users in order to have an efficient utilization of the transportation system.

Figure 1.1 shows the relationship between demand for travel and price. The demand curve illustrates how much road users are willing to pay for road use. The area below the demand curve is the total user benefit gained for the road travel. Any individual user entering the road will face only the costs he or she bears which are usually a combination of vehicle operation cost and the cost of travel time. The cost rises with the number of trips. Congestion increases the travel time and decreases vehicle operating efficiency and travel speeds.

However, the marginal cost represents the effect of adding one extra vehicle or trip to the traffic flow. Marginal costs include all private costs plus external ones, such as delay, pollution, and accidents. Since the extra user affects all other users, the marginal cost is always higher than the average cost. When the road demand approaches the capacity of the road, there is a substantial increase both in travel time and delay to other vehicles.

In Figure 1.1, users have an incentive to take the next trip until the total demand, $t$, reaches $t_U$. For $t < t_U$ the benefit gained is higher than the cost incurred. When there is no incentive for the next user to take the trip, i.e., demand
is at $t_U$, the system reaches user equilibrium. However, at this demand level economic efficiency in the market is not achieved. When the level of road use is equal to $t_S$, the marginal social cost of road travel is equal to the marginal user benefit given by the demand curve. At point $t_S$ the economic efficiency is achieved. This is the system optimal point. In order to have the user equilibrium at $t_S$, economists argue that that the users should be charged a toll in the amount of $C_B - C_A$, the MSCP toll.

The MSCP tolls are easy to compute by a formula (to be given in Section 2.2) which prices the time value of an additional user on the system. MSCP tolls are optimal in the sense of changing user behavior to system optimal behavior. This makes MSCP tolls one of the most popular tools for road pricing applications. Economists define the MSCP tolls as the “first best” tolls since they achieve the optimal utilization of the transportation system. However, Bergendorff [10], Bergendorff et al. [11], Hearn and Ramana [49] and Hearn and Yildirim [51] prove that other first best toll vectors there exist which will achieve the optimal usage of

Figure 1.1: Supply-Demand Equilibrium in Traffic Networks
the transportation system. They show that cheaper and more implementable toll vectors compared to MSCP tolls can be found among such solutions.

1.4 Electronic Road Pricing

Due to the technological advancements in the last decade, it is now possible to implement MSCP and other tolling schemes without causing more congestion by collecting tolls electronically. An electronic toll collection system uses various technologies to automate the toll collection process in such a manner that the subscribers do not need to stop or slow down to pay toll charges. Luk and Chung [74] summarize the components of an electronic toll collection system: roadside readers, automatic vehicle identification or tags, automatic vehicle classification, video enforcement system, lane controller, and a host computer.

Each lane contains an antenna connected to a reader that uses radio frequencies to communicate with a tag mounted on each car. The reader sends a signal to initialize the communication. Then the tag responds with an identification number. For charging the correct amount, the type of car is identified by using sensors such as inductive loops, ultrasonic scanners, and infra-red cameras. A video system records the license plates of the vehicles using the facility to enforce using a valid tag. The information gathered is sent to the lane controller, and then transferred to the host computer for making the necessary accounting transactions.

1.5 Applications of Road Pricing

There have been several attempts for implementing road pricing projects. Some of them were successful while others failed because of strong public opposition. It seems the most important thing in the implementation phase is the public acceptance of the road pricing project. The users should be informed and persuaded that tolling the roads will make life easier in terms of congestion and public transportation. We now summarize some road pricing projects.
1.5.1 Successful Applications

When the Singapore government realized that the city lost one third of its potential city product due to travel delays, they began to enforce an area licensing scheme for restraining the traffic [91]. The pricing program, combined with other vehicle fees, significantly reduced traffic and improved air pollution when implemented in 1975. Singapore plans to implement pricing programs on all its major highways due to the program success. For a similar implementation, certain prerequisites such as public acceptability, appropriate technical features, and complementary transport strategies have to be met. Local governments should discourage businesses from changing their current locations so that a shift in demand does not occur.

In Orange County, California, a privately owned, variable priced, toll road called SR 91 Express Lanes is placed next to a heavily congested freeway. Over 25000 commuters daily use electronic toll pricing technology to pay for using the express lanes. To increase car pooling and to keep traffic moving on the expressway, tolls are adjusted according to the number of vehicle occupants, time of the day, and the amount of traffic. As a result, congestion on the Highway 91 free lanes dropped to levels not seen in more than 15 years. Car and van pooling also increased significantly.

When 407 Express Toll Road, Toronto, Canada, opened in 1997, nobody was expecting a demand of 11000 to 12000 cars in peak hours. The 36 kilometer toll road will be paid in 25 years, five years earlier than expected. The technology used for the collection of tolls is similar to the one on SR 91, California. The average speed is about double that of the nearby congested public highways.

Norway had success with road pricing in three urban centers: Oslo, Bergen and Trondheim. For instance, in Trondheim where toll booths operate with an
electronic card system, the rush hour traffic has decreased by 10% while non-peak period traffic has increased by 9%.

In 1996, San Diego I-15 Express Lanes introduced high occupancy toll (HOT) lanes. Single occupancy vehicles are allowed to use the HOT lanes if they pay $50 for a special permit, and the spare capacity of HOT lanes are used more efficiently. Many users are willing to pay a toll for gaining an improved level of service.

1.5.2 Failures In Implementation of Road Pricing Projects

Although the extensive scientific research and experimentation on the effects of pricing on the Hong Kong traffic network showed encouraging results, a proposed road pricing project in 1986 was not implemented (Dawson and Catling [24] and Harrison et al. [43]). There were several reasons: First, the timing was not good. There was noticeably less congestion due to a new elevated highway in 1985-86. Second, a recession occurred during the mid 1980s in Hong Kong. The political situation was not clear due to negotiations with China, and community participation and consultation on the project were not well handled. Furthermore the Government did not have a well-defined revenue redistribution program for the toll revenue.

Similarly, there had been many obstacles for implementing road pricing projects in the United States, mainly political and institutional problems. Consequently, this idea has not made progress beyond the discussion stage (Higgins [55] and Elliot [28]). Jones and Harvey [59] and Evans [29] argue that the opposition for not implementing road pricing is not caused by technical reasons but rather a political one: a concern about the public acceptability. In the case of road pricing, public knowledge about what such schemes might achieve, or how they might operate, is rather low. The public has several objections to the road pricing idea. First of all, the public sees road pricing as an additional burden, i.e.,
an additional tax. It not only discriminates against the low income driver, but also causes other problems such as invasion of privacy.

Several authors [4, 59] suggest that road pricing would be more acceptable if it is introduced in the form of a road charge in which the money raised will be used for transport and environmental improvements.

1.6 Summary

Congestion is an inevitable part of everyday life in most metropolitan areas. It is clear from the examples given in Section 1.2 that congestion imposes a high burden on the people using the transportation system in those areas. Thus, transportation planning agencies use several traffic management tools like road pricing and parking pricing in order to restrain the traffic and reduce the congestion level on the transportation network. In this chapter, we have reviewed the road pricing concept along with some successful implementations that use the electronic road pricing technology to charge users in an efficient way. We have given the theory of marginal social cost pricing, which is the most common road pricing concept and is also known as a first best pricing scheme. For the fixed demand networks, tolling schemes other than MSCP tolls which can be utilized to create an efficient transportation network. In the subsequent chapters, we present alternative toll pricing schemes which will provide better utilization of variable demand transportation networks.
CHAPTER 2
TOLL PRICING REVIEW

2.1 Introduction

The purpose of this chapter is to summarize the main results for mathematical models of toll pricing. We first introduce the transportation planning process and traffic assignment problems and then present Wardrop’s principles which define the user and system problems. We review mathematical programming approaches to toll pricing in the literature, especially the toll pricing theory and the toll pricing framework for traffic assignment problems with fixed demand.

2.2 Transportation Planning Process and Traffic Assignment Models

In this section we introduce the transportation planning process, and then elaborate on a particular step of this process known as traffic assignment. We define the user and system problems and then present the optimality conditions for these models.

City planners require a careful investigation of the impact of new projects on the transportation infrastructure. It is essential to facilitate an efficient, smooth and stable transportation system in order to have a good quality of life. Thus, traffic planning is one of the most important components of the planning process for the future investments and operating policies.

The traffic planning process analyzes individual travelers’ reactions and responses to alternative scenarios so that a macroscopic description of the traveler behavior can be obtained. In this process, it is assumed that the transportation system is modeled in a steady state and that past experience such as travel times
over particular routes are still valid. Each traveler has complete and precise
information about all routes available and about their characteristics.

The traffic planning process has four stages. In the trip generation stage,
the objective is to calculate the number of trips that start and end in each of the
previously defined zones as a function of population, land use, socio-economic and
locational characteristics of these areas. Using the aggregated trip numbers from
the previous step, trip distribution models estimate the number of trips between
each origin-destination zone. In the modal split stage, the goal is to determine the
number of trips using different modes of transportation. Finally, traffic assignment
models take as input a complete description of the transportation system and an
origin-destination matrix of trip demands or travel demand formulas, and output
an estimate of traffic volumes, travel costs, and travel times on each link and
demand between each origin-destination pair. Traffic assignment models are used
in real life applications because they provide a description of the real world traffic
flows with sufficient accuracy. The reader is referred to Florian and Hearn [33],
Patriksson [87] and Sheffi [92] for background on traffic assignment problems.

Certain traffic assignment models are used to predict users’ behavior in
the transportation system. It is assumed that users choose the routes which are
perceived as the shortest under the current traffic conditions. In other words, users
try to minimize their own travel times between an origin and a destination. This
results in a situation in which no driver can unilaterally reduce his or her journey
time by choosing an alternative route. This is known as user equilibrium.

However, individuals do not necessarily utilize the transportation system in
the most efficient way when they try to optimize their travel times on the network.
A better utilization can be achieved by minimizing the total system cost. The
resulting travel pattern is known as system optimal.
To introduce the fixed demand user and system problems formally, we now give the notation for traffic assignment problems with fixed demand. Let $G$ denote the network model of a transportation system which consists of streets, $\mathcal{A}$, and intersections, $\mathcal{N}$. Mathematically, this is a network, $G = (\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ is the node set and $\mathcal{A}$ is the link set. Let $A$ be the node-arc incidence matrix of $G$. We define a commodity by an origin, $p$, and a destination, $q$. Let $\mathcal{K}$ represent the set indexing all such origin-destination pairs $k = (p, q)$. The demand between an origin-destination pair is expressed as $\bar{t}_k$. The $k$th commodity flow vector is denoted by $x^k$ and the sum of all the commodity flow vectors is the aggregate flow vector $v$.

We assume that a continuously differentiable cost map $s : \mathcal{A} \to \mathcal{A}$ is given. $\nabla s$ denotes the Jacobian of $s$. When the aggregate flow on the network is $v$, the travel time for a user on arc $a$ is given by $s_a(v)$. The system defining the feasible flows is given by the following:

$$v = \sum_{k \in \mathcal{K}} x^k$$

$$Ax^k = \bar{t}_k E_k \quad \forall k \in \mathcal{K}$$

$$x^k \geq 0 \quad \forall k \in \mathcal{K}$$

where $E_k = e_p - e_q$, a column incidence vector for commodity $k$, and $e_p$ and $e_q$ are unit vectors. The first constraint is the aggregate flow and the second constraint is the network balance constraint. Define $V$ as the set of all feasible flows.

British traffic engineer Wardrop [99] proposed two principles to model the user behavior and the system optimal flows. We state Wardrop’s principles and then give mathematical formulations for the fixed demand models.

Wardrop’s first principle. “The journey times on all of the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.” This is known as user equilibrium.

Wardrop’s first principle can be stated as a variational inequality problem [33]. A feasible aggregate flow vector $\bar{v}$ is a user equilibrium (UOPT-FD) solution if and
only if
\[ s(\bar{v})^T(v - \bar{v}) \geq 0 \quad \forall v \in V. \]

Bergendorff et al. [11] prove that a feasible aggregate flow vector \( \bar{v} \) is a user equilibrium (UOPT-FD) flow only if there exists \( \rho^k \) such that
\[
\begin{align*}
    s(\bar{v}) & \geq A^T \rho^k \quad \forall k \in K \\
    s(\bar{v})^T \bar{v} & = \sum_{k \in K} \bar{t}_k E^T_k \rho^k.
\end{align*}
\]
Wardrop’s second principle. “The average journey time is minimum.”
Wardrop’s second principle is used to define the system problem (SOPT-FD) which can be mathematically stated as
\[
\min s(v)^T v \\
\text{s. t.} \quad v \in V
\]
Using the Karush-Kuhn-Tucker (KKT) conditions, we can conclude that for any given \( \bar{v} \in V \), there exists a local minimum \( \bar{v} \) for SOPT-FD problem only if there exists \( \rho^k \) such that
\[
\begin{align*}
    s(\bar{v}) + \nabla s(\bar{v}) \bar{v} & \geq A^T \rho^k \quad \forall k \in K \\
    (s(\bar{v}) + \nabla s(\bar{v}) \bar{v})^T \bar{v} & = \sum_{k \in K} \bar{t}_k E^T_k \rho^k
\end{align*}
\]
It is well known from examples such as the Braess Paradox [33] that the user equilibrium and system optimal flows usually differ.

Note that the optimality conditions given for the user and system problems differ only by the term \( \nabla s(v) v \). Assume that we define a modified user problem, UOPT-FD-M, with a cost map \( \tilde{s}(v) = s(v) + \nabla s(v) v \), which is the derivative of the SOPT-FD objective function. Then in the UOPT-FD-M model, individuals pay not only \( s(v) \) but also \( \nabla s(v) v \), the externalities they impose on others. Thus \( \nabla s(v) v \) is a vector of MSCP tolls.
Comparing the results above, it is clear that UOPT-FD-M has equilibrium conditions that are exactly the same as the optimality conditions for the system problem. A main goal in this research is to investigate the toll vectors other than MSCP tolls which have this property.

2.3 Review of Mathematical Models of Toll Pricing

In this section, we summarize mathematical programming approaches to toll pricing. We review several models including fixed demand, elastic demand and stochastic demand traffic assignment models. The toll pricing idea was introduced for the first time in the 1920s. Pigou [89] was one of the first to propose a taxation scheme (marginal social cost pricing) for achieving the optimal usage of scarce resources. He examined the case where a company would like to make an investment and observed that total freedom in a decision process will force the companies to build factories with higher marginal investment costs. Knight [60] extended Pigou’s ideas to traffic models. Although Knight proposed the route choice behavior of the users on a traffic network, it was Wardrop [99] who formalized the notion of equilibrium by the two principles known as Wardrop’s principles. Later, Beckmann et al. [7] formulated and analyzed a mathematical model of traffic equilibrium.

In the fixed demand traffic assignment model, the demand between OD pairs is constant and in the simplest case, the constraints are network balance constraints. However, frequently the trip rates between origin-destination pairs are modeled as functions of the least travel cost between the origin and destination. This case is known as the traffic assignment problem with elastic demand. The traveler has a number of choices available and is motivated by economical conditions in his/her decisions. Beckmann et al. [7], Gartner [38, 39], LeBlanc and Farhangian [68] and Yang and Huang [110] define the user equilibrium and system optimum concepts for the elastic demand traffic assignment problem.
and Mattson [57] and Yang and Huang [110] show that when the cost vector is perturbed by the MSCP toll vector, the resulting equilibrium flows are the same as the system optimal ones both in fixed and elastic demand models.

When there are capacities on the links, Hearn [46], Ferrari [31] and Larsson and Patriksson [62, 63, 64, 65] show that the user equilibrium flows in a capacitated fixed demand network will be the same as the uncapacitated user equilibrium flows when the cost vector is perturbed in a specific manner. Hearn [46], Payne and Thompson [88], Yang [107] and Yang and Lam [111] define this adjustment factor as the queuing toll. Furthermore, Hassin [44], Lippman and Stidham [70] and Stidham [94] show that the MSCP toll pricing scheme is also valid for the queuing systems.

When the route choice is based on “perceived” travel times rather than the measured ones, a stochastic user equilibrium model can be defined. In the stochastic user equilibrium, the perceived travel times are equal on the utilized routes for an OD-pair. Bell [9], Damberg et al. [23], Dial [26], Smith et al. [93] and Watling [100] show that for several stochastic user equilibrium models, if the system cost is perturbed by MSCP tolls, system optimal flows, i.e., a first best solution, will be attained.

In the dynamic traffic assignment problems, the travel demands and costs vary over time and thus congestion tolls need to be time varying (Merchant and Nemhauser [82, 83] and Wie [101]). Note that the MSCP scheme is dynamic since it only depends on the instantaneous flows. Henderson [53], Agnew [2]), Carey and Srinivasan [18] and Huang and Yang [54] show that the MSCP scheme makes the user problem have the same flows as the system solution for a bottleneck or a single-destination network. Furthermore, this property is also observed by Friesz et al. [35, 36], Masuda and Hang [79], Wie et al. [102, 104, 105] and Wie and Tobin [103] for general networks.
There are also models which combine two or more stages of the traffic planning process. For these models, MSCP achieves the goal of optimal system usage. For example, Boyce et al. [12] show that MSCP tolls are valid for the combined distribution, mode and route choice model.

Up to now, we have discussed the fact that MSCP tolls are valid for various traffic assignment models. However, other toll vectors there exist along with MSCP tolls which make the user equilibrium flows the same as the system optimal ones.

Define the *First Best Toll Set* as the set of all tolls which makes the user problem have the same optimal flows as the system problem. Define all such tolls as *valid first best* tolls. The toll set concept is introduced by Bergendorff et al. [11] who also propose the notion of toll pricing for fixed demand networks. Hearn and Ramana [48, 49] define the *Toll Pricing Framework*, which is a procedure to identify the toll set and solve a toll pricing problem with various objectives. Larsson and Patriksson [66, 67] also propose a toll set when there are constraints on flows. However, as we show in Chapter 3, the toll set that they define is smaller than the toll sets Bergendorff et al. [11] and Hearn and Ramana [48, 49] define for the fixed demand traffic assignment problems as well as the one Hearn and Yildirim [51] propose for elastic demand traffic assignment problems.

Assuming that the toll set has been characterized, the traffic planners can define secondary objectives and additional constraints and try to solve a toll pricing problem. In Section 2.5, we present toll pricing problems defined by Hearn and Ramana [49] and Hearn and Yildirim [51]. Tolls other than MSCP tolls can be found by solving either a linear or mixed integer program using mathematical programming softwares like OSL [56] or CPLEX [20]. More specialized algorithms such as cutting plane methods and bilevel programming [52, 61, 109, 111] can be utilized to obtain the toll vectors.
One of the most well studied toll pricing problems for fixed demand networks is the one where the objective is minimization of the total toll revenue when tolls are constrained to be nonnegative. We refer to this as the MINSYS-FD [11, 49] problem. Dial [26, 25, 27] and Hagstrom [42] show that the tolls which minimize the total toll revenue for the single origin-destination traffic assignment problem can be found by solving a max-path problem. However, when there are multiple OD-pairs, the problem becomes a multicommodity network flow problem [26].

Although the first best toll pricing schemes that Bergendorff et al. [11], Hearn and Ramana [48, 49], and Hearn and Yildirim [51] propose cause the user equilibrium flows to be the same as the system optimal ones, usually with fewer number of toll booths compared to the MSCP tolling scheme, it might still be the case that some of the restrictions imposed by the system planner might not be satisfied. Therefore, the most efficient usage of the system might not be achievable. Under such circumstances, a “second best” toll pricing scheme can be utilized. Marchand [78] shows how the MSCP toll pricing scheme should be modified for a network with two parallel routes where only one route can be tolled. Braid [15, 16] examines peak load pricing for the two parallel routes problem and compares the first best and second best toll pricing alternatives. Verhoef et al. [98] analyze the effects of various demand and cost parameters on the relative efficiency of one-route tolling using a two-link network simulation model. The two parallel routes problem has been extended in various ways. For example, Liu and McDonald [71] propose a two period, two parallel routes problem. Verhoef [97, 96] proposes a general network second best pricing scheme in which a system of equations for finding the second best tolls is derived. A common goal in all of these studies is to express the second best tolls as functions of MSCP tolls.

In the next section, we summarize results from Bergendorff et al. [11] and Hearn and Ramana [49] on the notion of toll pricing for fixed demand networks.
2.4 Toll Pricing Theory for Fixed Demand Networks

Define $\beta$ as a *toll vector*, and consider the perturbed user optimality problem:

$$(s(\bar{v}) + \beta)^T(v - \bar{v}) \geq 0, \quad \forall v \in V. \quad \text{(UOPT-\beta)}$$

The optimality conditions for UOPT$_\beta$-FD can be summarized as follows:

$$s(\bar{v}) + \beta \geq A^T \rho^k \quad \forall k \in K$$

$$(s(\bar{v}) + \beta)^T \bar{v} = \sum_{k \in K} \bar{t}_k E_k^T \rho^k.$$ 

Denote by $U^*_\beta$, the set of *Tolled Equilibrium Solutions*, i.e., those $\bar{v}$ that satisfy UOPT$_\beta$-FD, and let $S^*$ be the set of optimal solutions to the (untolled) SOPT-FD problem. The *Toll Set* of valid toll vectors is denoted by

$$\mathcal{T} := \{\beta|\emptyset \neq U^*_\beta \subseteq S^*\}.$$ 

For a given vector $\bar{v} \in V$ (typically chosen to be a system optimal solution), we define $W_{FD}(\bar{v})$ to be the set of all tolls that ensure that $\bar{v}$ is a solution of UOPT$_\beta$-FD:

$$W_{FD}(\bar{v}) = \{\beta|\bar{v} \in U^*_\beta\}.$$ 

Hearn and Ramana [49] prove the following lemma:

**Lemma 2.1** For $\bar{v} \in V$, $W_{FD}(\bar{v})$ is the polyhedron given by the $\beta$ part of the following linear inequality system in $\beta$ and $\rho$ variables:

$$s(\bar{v}) + \beta \geq A^T \rho^k \quad \forall k \in K$$

$$(s(\bar{v}) + \beta)^T \bar{v} = \sum_{k \in K} \bar{t}_k E_k^T \rho^k.$$ 

It follows from the optimality conditions of SOPT-FD and UOPT-FD and Lemma 2.1 that for any KKT point of SOPT-FD, there exists a toll vector, MSCP tolls, $\beta_{MSCP} = \nabla s(\bar{v})\bar{v}$, which are in $W_{FD}(\bar{v})$. 
The following is a result from Bergendorff et al. [11] expressing $\mathcal{T}$ in terms of the polyhedra $W_{FD}(\bar{v})$:

**Theorem 2.1** $\mathcal{T} = \bigcup_{\bar{v} \in \mathcal{S}^*} \{ \beta \in W_{FD}(\bar{v}) | U^*_\beta \subseteq S^* \}$. If $s$ is strictly monotonic, then $\mathcal{T} = \bigcup_{\bar{v} \in \mathcal{S}^*} W_{FD}(\bar{v})$. If $s$ is both strictly monotonic and $s(v)^T v$ is strictly convex, there exists a unique optimal solution $v^*$ to the system optimum problem and the first best toll set $\mathcal{T}$ equals the polyhedron $W_{FD}(v^*)$.

### 2.5 First Best Toll Pricing Framework for Fixed Demand Networks

Assume that $s$ is both strictly monotonic and $s(v)^T v$ is strictly convex so that there are unique solutions for both user and system problems. The toll pricing framework (TPF) can be summarized by Hearn and Ramana [49] as follows:

1. Solve the system optimum problem to obtain the unique optimal solution $v^*$ and $s(v^*)$.
2. Define the first best toll set $W_{FD}(v^*)$ by
   \[
   s(v^*) + \beta \geq A^T \rho^k \quad \forall k \in \mathcal{K} \\
   (s(v^*) + \beta)^T v^* = \sum_{k \in \mathcal{K}} \bar{t}_k E_k^T \rho
   \]
3. Define and optimize an objective function over the toll set, possibly intersected with other constraints.

In Step 3, the problem to solve will depend on the choice of the objective. The formulations which follow all assume that $W_{FD}(v^*)$ defines the full set of possible tolls. The tolls and their objectives can be summarized as follows:

- **MINSYS**: Minimize the total (nonnegative) tolls collected.
- **MINMAX**: Minimize the largest nonnegative toll collected.
- **Targeted Revenues (TR)**: Total of tolls and subsidies is $\theta$, a desired amount of tolls to be collected. In the case of $\theta = 0$, these tolls are called Robinhood (RH) tolls. The TR tolls can be found by linear programming or the
following equation [49]:

\[ \beta_{RH} = -s(v^*) + \frac{\theta + s(v^*)^Tv^*}{v^T\nabla s(v^*)v^* + s(v^*)^Tv^*}(\nabla s(v^*)v^* + s(v^*)) \]

- MINTB: Minimize the number of toll booths.
- MINTB/TR: Minimize the number of toll booths while constraining the total tolls collected to \( \theta \). When \( \theta = 0 \) this is designated MINTB/RH.

<table>
<thead>
<tr>
<th>TOLL</th>
<th>Objective ((\bar{Z}))</th>
<th>Extra Constraints ((\hat{T}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>MINSYS</td>
<td>( \beta^Tv^* )</td>
<td>( \beta \geq 0 )</td>
</tr>
<tr>
<td>MINMAX</td>
<td>( z )</td>
<td>( z \geq \beta_a, \forall a \in \mathcal{A}, \beta \geq 0 )</td>
</tr>
<tr>
<td>Targeted Revenues</td>
<td>( \beta^Tv^* )</td>
<td>( \beta^Tv^* = \theta )</td>
</tr>
<tr>
<td>MINTB</td>
<td>( \sum_{a \in \mathcal{A}} y_a \beta_a \leq My_a \forall a \in \mathcal{A}, y_a \in {0,1}, \beta \geq 0 )</td>
<td></td>
</tr>
<tr>
<td>MINTB/TR</td>
<td>( \sum_{a \in \mathcal{A}} y_a \beta_a \leq My_a \forall a \in \mathcal{A}, y_a \in {0,1}, \beta^Tv^* = \theta )</td>
<td></td>
</tr>
</tbody>
</table>

As shown in Table 2.1, let \( \hat{T} \) be the set of additional (side) constraints and \( \bar{Z} \) be the objective function for each toll pricing problem. The feasible region for each toll problem is the intersection of \( \hat{T} \) with \( T = W_{FD}(v^*) \). The objective is linear in each case, thus each toll pricing problem is either a linear program or a linear integer program.

### 2.6 Summary

In this chapter, we reviewed the mathematical programming literature on toll pricing. We presented the notion of first best toll pricing and the toll pricing framework by Hearn and Ramana [49] for the fixed demand case. In Chapter 3, we extend the notion of first best toll pricing to traffic assignment models with variable demand.
CHAPTER 3
FIRST BEST PRICING IN VARIABLE DEMAND MODELS

3.1 Introduction

In the foregoing chapter, we assumed that the trip rate demand for every OD pair is fixed and known. However, in reality the demand between the OD pairs might change substantially based on the “service level,” which is usually a function of the travel time on the transportation network. When the service level varies, users might decide to take the trip, change the departure time of the trip, change the mode of transportation or not to take the trip at all. The underlying assumption throughout this chapter is that the demand between an origin-destination pair is a function of the least generalized cost between that pair.

Our primary goal is to generalize the first best fixed demand toll pricing framework to the variable demand models. We propose a general variable demand (GVD) model which can be reduced to several traffic assignment problems such as elastic demand (ED) traffic assignment problem, elastic demand with capacities on links (ED-C), elastic demand with side constraints (ED-G), combined distribution assignment models (CDAM), and combined travel choice models (CTCM). We show that the toll set for each of these models is an instance of the toll set of the GVD model. We also propose an $\epsilon$—approximation of the toll set for the case where the system solution is not exact.

3.2 General Variable Demand TA Models

We first give the notation that we utilize for the GVD model and then define the feasible region for this problem. Next, we present the system problem (SOPT-GVD) and state the optimality conditions for SOPT-GVD. Then, the system of
inequalities that describe the user equilibrium (UOPT-GVD) flows and demands follow. After defining the toll set, we present the toll pricing framework for the GVD model. Finally, we derive the toll set and illustrate the toll pricing framework for specific instances of the GVD model.

Let $c_k$ be the generalized cost of travel for commodity $k$, and $t_k$ denote a nonnegative invertible function of $c_k$. The demand for travel from some origin $p$ to destination $q$ is expressed as $t_k(c_k)$. It is generally assumed that the $t_k(c_k)$ are monotonically decreasing and bounded from above. Let $w_k(t_k)$ indicate the inverse of $t_k$. The vector $t$ has components $t_k$ and the vector function $w(t)$ has $w_k(t_k)$ as components. Then the set of inequalities which define all possible feasible flows and demands can be stated as

$$
v = \sum_{k \in K} x^k : \mu
$$

$$Ax^k = Ek_t \forall k \in K : \rho^k
$$

$$g_m(v) \leq 0 \quad \forall m \in M : \gamma_m
$$

$$h_n(t) = 0 \quad \forall n \in N : \theta_n
$$

$$-x^k \leq 0 \quad \forall k \in K : \tau^k
$$

$$-t_k \leq 0 \quad \forall k \in K : \phi_k
$$

where $g_m(v) \leq 0, m \in M$, are constraints on the aggregate flows, and $h_n(t) = 0, n \in N$, are constraints on the demand, and $(\mu, \rho, \gamma, \theta, \tau, \phi)$ are multipliers related to the constraints. $V$, the set of all feasible flows and demands, can be represented as

$$V = \{(v, t) : (v, t) satisfies the inequalities above\}.$$ 

The system and user objectives that we consider throughout this chapter are of the form $S(v) - W(t)$. We use $S(v) = \sum_{a \in A} s_a(v)v_a$ for the system problems and $S(v) = \sum_{a \in A} \int_0^{v_a} s_a(z)dz$ for the user problems. We assume that $S(v)$ and
Table 3.1: $W(t)$, $g(v)$ and $h(t)$ for the Special Cases of the GVD model.

<table>
<thead>
<tr>
<th>MODEL</th>
<th>$W(t)$</th>
<th>$g(v)$</th>
<th>$h(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ED</td>
<td>$\sum_{k \in K} \int_{0}^{t_k} w_k(z)dz$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
</tr>
<tr>
<td>ED-C</td>
<td>$\sum_{k \in K} \int_{0}^{t_k} w_k(z)dz$</td>
<td>$v_a - C'_a \leq 0$</td>
<td>$\cdot$</td>
</tr>
<tr>
<td>ED-G</td>
<td>$-\sum_{k \in K} \int_{0}^{t_k} w_k(z)dz$</td>
<td>$g_m(v) \leq 0$</td>
<td>$\cdot$</td>
</tr>
<tr>
<td>CDAM</td>
<td>$-\frac{1}{2} \sum_{k \in K} t_k \ln t_k$</td>
<td>$\sum_q t_k = O_p, \sum_p t_k = D_q, k = (p, q)$</td>
<td>$\cdot$</td>
</tr>
</tbody>
</table>

$W(t)$ are convex. $W(t)$, $g(v)$ and $h(t)$ for the models that we consider are listed in Table 3.1. We assume that $g(v)$ and $h(t)$ are linear. The resulting system and user problems have a convex nonlinear objective and linear constraints. Thus a constraint qualification\(^1\) automatically holds and multipliers will exist at the local optima.

3.2.1 System Problem

The system problem assumes that there is a central planning authority which controls the transportation system, thus an efficient system design can be implemented. In the general variable demand model, when the system objective is minimization of the total system cost, an efficient network utilization is achieved by having flows and demands which are consistent with Wardrop’s second principle.

We can define the system problem (SOPT-GVD) for the GVD model as

$$\min \quad s(v)^T v - \sum_{k \in K} \int_{0}^{t_k} w_k(z)dz$$

subject to $\ (v, t) \in V$.

\(1\) See Bazaraa et al. [6], Chapter 5.

**Lemma 3.1** Assume that $(\bar{v}, \bar{t})$ is an optimal solution to the SOPT-GVD problem. Then there exists $(\mu, \rho, \gamma, \theta, \tau, \phi)$ such that the following holds:

$$s_a(\bar{v}) + \frac{\partial s_a(\bar{v})}{\partial v_a} \bar{v}_a + \sum_{m \in M} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a} = \mu_a \quad \forall a \in A$$
\[
\mu_a + (\rho^k_i - \rho^k_j) = \tau^k_a \quad \forall k \in \mathcal{K}, \forall a = (i, j) \in \mathcal{A}
\]

\[
-w_k(\bar{t}_k) - \sum_{n \in \mathcal{N}} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} + (\rho^k_q - \rho^k_p) = \phi_k \quad \forall k = (p, q) \in \mathcal{K}
\]

\[
\sum_{k \in \mathcal{K}} \left( w_k(\bar{t}_k) + \sum_{n \in \mathcal{N}} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) \bar{t}_k = \sum_{a \in \mathcal{A}} \mu_a \bar{v}_a
\]

\[
\gamma_m g_m(\bar{v}) = 0 \quad \forall m \in \mathcal{M}
\]

\[
\tau, \phi, \gamma \geq 0.
\]

**Proof.** Since \((\bar{v}, \bar{t})\) is a KKT point for the SOPT-GVD problem, there exists \((\mu, \rho, \gamma, \theta, \tau, \phi)\) such that the following conditions hold:

\[
\frac{\partial L}{\partial v_a} = s_a(\bar{v}) + \frac{\partial s_a(\bar{v})}{\partial v_a} - \mu_a + \sum_{m \in \mathcal{M}} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a} = 0 \quad \forall a \in \mathcal{A}
\]

\[
\frac{\partial L}{\partial x^k_a} = \mu_a + (\rho^k_i - \rho^k_j) - \tau^k_a = 0 \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A}
\]

\[
\frac{\partial L}{\partial t_k} = -w_k(\bar{t}_k) + (\rho^k_q - \rho^k_p) - \sum_{n \in \mathcal{N}} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} - \phi_k = 0 \quad \forall k = (p, q) \in \mathcal{K}
\]

\[
\gamma_m g_m(\bar{v}) = 0 \quad \forall m \in \mathcal{M}
\]

\[
\tau^k_a \bar{x}_a^k = 0 \quad \forall k \in \mathcal{K}, \forall a \in \mathcal{A}
\]

\[
\phi_k \bar{t}_k = 0 \quad \forall k \in \mathcal{K}
\]

where \(L(x, v, t, \mu, \rho, \gamma, \theta, \tau, \phi)\) is the Lagrangian of the system problem. We can aggregate the complementarity conditions, \(\tau^k_a \bar{x}_a^k = 0\) and \(\phi_k \bar{t}_k = 0\) to

\[
\sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \tau^k_a \bar{x}_a^k = \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \mu_a \bar{x}_a^k - \sum_{k \in \mathcal{K}} (\bar{x}^k)T A^T \rho^k = 0
\]

and

\[
\sum_{k \in \mathcal{K}} \phi_k \bar{t}_k = -\sum_{k \in \mathcal{K}} \left( w_k(\bar{t}_k) + \sum_{n \in \mathcal{N}} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) \bar{t}_k + \sum_{k \in \mathcal{K}} \bar{t}_k E_k^T \rho^k
\]

\[
= -\sum_{k \in \mathcal{K}} \left( w_k(\bar{t}_k) + \sum_{n \in \mathcal{N}} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) \bar{t}_k + \sum_{k \in \mathcal{K}} (A \bar{x}^k)T \rho^k = 0
\]
Summing up these equations we get,

\[ \sum_{a \in A} \sum_{k \in K} \mu_a \bar{v}_a - \sum_{k \in K} (\bar{x}^k)^T A^T \rho^k - \sum_{k \in K} \left( w_k(\bar{t}_k) + \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) \bar{t}_k + \sum_{k \in K} (A\bar{x}^k)^T \rho^k = 0 \]

\[ \sum_{a \in A} \mu_a \bar{v}_a - \sum_{k \in K} \left( w_k(\bar{t}_k) + \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) \bar{t}_k = 0. \]

\[ \sum_{a \in A} \sum_{k \in K} \tau_a^k \bar{x}_a^k + \sum_{k \in K} \phi_k \bar{t}_k = 0 \]

3.2.2 User Problem

The user problem is defined by Wardrop’s first principle. The underlying assumption is that all users taking a trip between an OD pair have the same travel time which is less than or equal to the travel time on any unutilized path. Mathematically, the user problem (UOPT-GVD) can be formulated as a variational inequality. A given aggregate flow and demand vector \((\bar{v}, \bar{t})\) is a user equilibrium flow if and only if

\[ s(\bar{v})^T (v - \bar{v}) - w(\bar{t})^T (t - \bar{t}) \geq 0, \quad \forall (v, t) \in V. \]

When \(s\) and \(w\) are monotonic and separable in flows and demands, UOPT-GVD can be stated as a convex nonlinear program:

\[ \min \sum_{a \in A} \int_0^{v_a} s_a(z)dz - \sum_{k \in K} \int_0^{t_k} w_k(z)dz \]

subject to \((v, t) \in V.\]

Lemma 3.2 A feasible point \((\bar{v}, \bar{t})\) is an user equilibrium flow if and only if there exists \((\mu, \rho, \gamma, \theta, \tau, \phi)\) such that the following holds:

\[ s_a(\bar{v}) + \sum_{m \in M} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a} = \mu_a \quad \forall a \in A \]

\[ \mu_a + (\rho^k_i - \rho^k_j) = \tau^k_a \quad \forall k \in K, \forall a = (i, j) \in A \]

\[ -w_k(\bar{t}_k) - \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} + (\rho^k_p - \rho^k_q) = \phi_k \quad \forall k \in K \]
\[
\sum_{k \in K} \left( w_k(\bar{t}_k) + \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) \bar{t}_k = \sum_{a \in A} \mu_a \bar{v}_a \\
\tau, \phi, \gamma \geq 0.
\]

**Proof:** Let \((\bar{v}, \bar{t})\) be the equilibrium flows and demand, then

\[
s(\bar{v})^T v - w(\bar{t})^T t \geq s(\bar{v})^T \bar{v} - w(\bar{t})^T \bar{t}, \quad \forall (v, t) \in V
\]

holds. This implies

\[
\min_{(v, t) \in V} s(\bar{v})^T v - w(\bar{t})^T t \geq s(\bar{v})^T \bar{v} - w(\bar{t})^T \bar{t}.
\]

Consider now

\[
\min \quad s(\bar{v})^T v - w(\bar{t})^T t \\
\text{s. t.} \quad (v, t) \in V.
\]

From the inequality above it is clear that \((\bar{v}, \bar{t})\) solves this linear program. Therefore, by linear programming duality, there exists \((\mu, \rho, \gamma, \theta, \tau, \phi)\) such that the following conditions hold:

\[
s_a(\bar{v}) - \mu_a + \sum_{m \in M} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a} = 0 \quad \forall a \in A
\]

\[
\mu_a + (\rho^k_i - \rho^k_j) - \tau^k_a = 0 \quad \forall k \in K, \forall a \in A
\]

\[
-w_k(\bar{t}_k) + (\rho^k_q - \rho^k_p) - \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} - \phi^k = 0 \quad \forall k \in K
\]

\[
\gamma_m g_m(\bar{v}) = 0 \quad \forall m \in M
\]

\[
\tau_a \bar{t}_a = 0 \quad \forall a \in A, \forall k \in K
\]

\[
\phi_k \bar{t}_k = 0 \quad \forall k \in K.
\]

The last three equations are the complementary slackness conditions. Further, \(\tau^k_a \bar{t}_a = 0\) and \(\phi_k \bar{t}_k = 0\) can be aggregated to obtain

\[
\sum_{k \in K} \left( w_k(\bar{t}_k) + \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) \bar{t}_k = \sum_{a \in A} \mu_a \bar{v}_a.
\]
Then the lemma follows. □

We note that this would also hold when \( g \) and \( h \) are nonlinear, provided that some constraint qualification holds (see Bazaara et al. [6]). However, the GVD models that we discuss in this chapter have linear side constraints. Thus a constraint qualification holds automatically.

When the optimality conditions of the system and user problems are compared, it is clear that they differ only in the definition of \( \mu \). The difference between \( \mu \) for the system and user problems is \( \nabla s(v) v \), the MSCP toll vector.

Similar to the fixed demand case, we can conclude from this observation that if the individuals were charged with MSCP tolls, the user equilibrium flows would be the same as the system optimal flows.

### 3.2.3 Toll Pricing for the GVD Model

In this section, we generalize the notion of toll pricing to the GVD model. First, we give some definitions which are similar to Bergendorff et al. [11] and then characterize the toll set.

Suppose \( s_\beta(v) = s(v) + \beta \) is the perturbed cost map where \( \beta \) is a toll vector. Then the perturbed user problem can be stated as a variational inequality:

\[
\text{If } (\bar{v}, \bar{t}) \text{ is a tolled user equilibrium flow if and only if } \quad (s(\bar{v}) + \beta)^T(v - \bar{v}) - w(\bar{t})^T(t - \bar{t}) \geq 0, \quad \forall (v, t) \in V.
\]

Let \( U^*_\beta \) be the set of all *tolled equilibrium solutions*, i.e., those \( (\bar{v}, \bar{t}) \in V \) that solve the above variational inequality, and let \( S^* \) be the optimal solution set for SOPT-GVD. We would like to identify all toll vectors such that the resulting tolled user equilibrium problem has a system optimal solution, i.e.,

\[
\emptyset \neq U^*_\beta \subseteq S^*
\]
Any such $\beta$ is defined to be a valid toll vector. Let the toll set denoted by $T$ be the set of all such vectors, $T := \{\beta | \emptyset \neq U_\beta^* \subseteq S^*\}$. For a given vector $(\bar{v}, \bar{t}) \in V$,

$$W_{GVD}(\bar{v}, \bar{t}) = \{\beta | (\bar{v}, \bar{t}) \in U_\beta^*\},$$

is the set of all tolls which ensures that $(\bar{v}, \bar{t})$ is a solution of $U_{OPT_\beta}$-GVD. In fact, using Lemma 3.1 and Lemma 3.2, we can define $W_{GVD}(\bar{v}, \bar{t})$ as follows:

**Lemma 3.3** Given that $(\bar{v}, \bar{t}) \in V$, $W_{GVD}(\bar{v}, \bar{t})$ is the polyhedron given by the $\beta$ part of the linear inequality system defined in $(\beta, \rho, \gamma, \theta)$:

\[
\begin{align*}
\left(s_a(\bar{v}) + \beta_a + \sum_{m \in M} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a}\right) + (\rho^k_i - \rho^k_j) & \geq 0 \quad \forall k \in K, \forall a = (i, j) \in A \\
\left(w_k(\bar{t}_k) + \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k}\right) - (\rho^k_p - \rho^k_q) & \leq 0 \quad \forall k \in K \\
\sum_{a \in A} \left(s_a(\bar{v}) + \beta_a + \sum_{m \in M} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a}\right) \bar{v}_a & = \sum_{k \in K} \left(w_k(\bar{t}_k) + \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k}\right) \bar{t}_k \\
\gamma_m g_m(\bar{v}) & = 0 \quad \forall m \in M.
\end{align*}
\]

**Proof.** Using Lemma 3.2 and the perturbed cost vector $s_a(v_a) + \beta_a$, we observe that in the tolled user equilibrium the following holds:

\[
\begin{align*}
\left(s_a(\bar{v}) + \beta_a + \sum_{m \in M} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a}\right) = \mu_a = \rho^k_j - \rho^k_i + \tau_a^k, \quad \forall k \in K, \forall a \in A
\end{align*}
\]

and

\[
\begin{align*}
\rho^k_p - \rho^k_q & - \left(w_k(\bar{t}_k) + \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k}\right) = \phi^k, \quad \forall k \in K
\end{align*}
\]

hold. But $\tau_a^k \geq 0$ and $\phi^k \geq 0$. The lemma follows. $\square$

The following lemma gives an expression for $T$ in terms of the polyhedra $W_{GVD}(\bar{v}, \bar{t})$ when uniqueness assumptions in the system and user solutions are made.

**Lemma 3.4** If the user and system problems have unique solutions, then

$$T = W_{GVD}(\bar{v}, \bar{t}).$$
We omit the proof of this lemma since it is similar to the one by Hearn and Ramana [49] for the fixed demand case.

From Wardrop’s First Principle, we know that in the equilibrium, the utilized paths for an origin-destination pair have the same travel time and the unused ones do not have lower travel times. The next corollary verifies that the generalized cost version of this principle holds. Let \( r_k \) be a path for commodity \( k \), \( \chi_{ar_k} \) be 1 if link \( a \) is on path \( r_k \) and zero otherwise,

\[
s_a(\bar{v}_a) + \beta_a + \sum_{m \in M} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a}
\]

be the generalized cost of traveling on arc \( a \) and

\[
w_k(\bar{t}_k) + \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k}
\]

be the generalized benefit for commodity \( k \).

**Corollary 3.1** At the UOPT\(_{\beta}\)-GVD solution \((\bar{v}, \bar{t})\) the total generalized cost on any path is greater than or equal to the generalized benefit for any commodity \( k \), i.e.,

\[
\sum_{a \in A} \chi_{ar_k} \left( s_a(\bar{v}_a) + \beta_a + \sum_{m \in M} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a} \right) \geq w_k(\bar{t}_k) + \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k}
\]

For any path with positive flow, the inequality holds as an equality. Therefore the costs on the utilized paths are constant for any commodity.

**Proof.** For \( k = (p, q) \) and \( a = (i, j) \in r_k \),

\[
s_a(\bar{v}_a) + \beta_a + \sum_{m \in M} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a} \geq \rho_j^k - \rho_i^k
\]

holds as an equality when \( \bar{x}_a^k > 0 \) and similarly,

\[
w_k(\bar{t}_k) + \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} \leq \rho_q^k - \rho_p^k
\]

holds as an equality when \( \bar{t}_k > 0 \). That is, the complementarity conditions force the inequality to hold as an equality when there is flow on that path. Thus, when
they are summed over the arcs on the utilized paths, the inequality in the corollary holds as an equality. The conclusion of the corollary then follows. □

In other words, this corollary can be interpreted by saying that every commodity reaches an equilibrium exactly when the generalized path costs, including tolls, equals the generalized benefit at the final demand level.

We give an alternative toll set description when the commodity flows $\bar{x}_a^k$ are also known.

**Lemma 3.5** Given that $(\bar{x}, \bar{v}, \bar{t}) \in V$, $W_{GVD}(\bar{x}, \bar{v}, \bar{t})$ is the polyhedron given by the $\beta$ part of the linear inequality system defined in $(\beta, \rho, \gamma, \theta)$.

\[
\left( s_a(\bar{v}) + \beta_a + \sum_{m \in M} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a} \right) + (\rho_i^k - \rho_j^k) = 0 \quad \forall \bar{x}_a^k > 0, \; k \in K, \; a \in A
\]
\[
\left( s_a(\bar{v}) + \beta_a + \sum_{m \in M} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a} \right) + (\rho_i^k - \rho_j^k) \geq 0 \quad \forall \bar{x}_a^k = 0, \; k \in K, \; a \in A
\]
\[
\left( w_k(\bar{t}_k) + \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) - (\rho_q^k - \rho_p^k) \leq 0 \quad \forall k \in K
\]
\[
\sum_{a \in A} \left( s_a(\bar{v}) + \beta_a + \sum_{m \in M} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a} \right) \bar{v}_a = \sum_{k \in K} \left( w_k(\bar{t}_k) + \sum_{n \in N} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) \bar{t}_k
\]
\[
\gamma_m g_m(\bar{v}) = 0 \quad \forall m \in M.
\]

**Proof:** The complementarity constraint $\tau_a^k \bar{x}_a^k = 0$ implies that if $\bar{x}_a^k > 0$, then $\tau_a^k = 0$ and similarly if $\bar{x}_a^k = 0$, then $\tau_a^k \geq 0$. But $\tau_a^k = s_a(\bar{v}) + \beta_a + \sum_{m \in M} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a} + (\rho_i^k - \rho_j^k)$. The conclusion of the lemma follows. □

Now assume that $(\bar{\beta}, \bar{\rho}, \bar{\gamma}, \bar{\theta})$ satisfies the inequalities that define $W_{GVD}(\bar{x}, \bar{v}, \bar{t})$. An interesting question to address is if $\bar{\beta}$ is a unique toll vector for the given multipliers $(\bar{\rho}, \bar{\gamma}, \bar{\theta})$ or not. Let $A'$ denote the set of links that do not carry any flow, i.e. $\bar{v}_a = 0, \forall a \in A'$ (this also implies $\bar{x}_a^k = 0, \forall a \in A', \forall k \in K$). Note that for
A given \((\bar{\rho}, \bar{\gamma}, \bar{\theta}), \bar{\beta}, a \in \mathcal{A} \setminus \mathcal{A}'\) is unique and determined by

\[
\bar{\beta}_a = \bar{\rho}_j^k - \bar{\rho}_i^k - s_a(\bar{v}) - \sum_{m \in \mathcal{M}} \bar{\gamma}_m \frac{\partial g_m(\bar{v})}{\partial v_a}, \quad \forall x_a^k > 0, \quad k \in \mathcal{K}, a \in \mathcal{A} \setminus \mathcal{A}'.
\]

However, the toll amount is not unique on those links where \(v_a = 0\). In fact any

\[
\beta_a \geq \hat{\beta}_a = \max_{k \in \mathcal{K}} \left( \bar{\rho}_j^k - \bar{\rho}_i^k - s_a(\bar{v}) - \sum_{m \in \mathcal{M}} \bar{\gamma}_m \frac{\partial g_m(\bar{v})}{\partial v_a} \right), \quad \forall a \in \mathcal{A}'
\]

is a valid toll. Note that on link \(a = (i,j)\), \(\bar{\rho}_j^k - \bar{\rho}_i^k\) is constant for all commodities that carry flow.

### 3.2.4 Interpretation of the KKT Multipliers for the GVD Model

The toll set is defined by variables \((\beta, \rho, \gamma, \theta)\) and by parameters \((\bar{v}, \bar{t})\). \(\beta\) is the toll vector. \(\rho_i^k\) is the total travel cost from origin \(p\) to node \(i\) for the commodity \(k = (p,q)\) on the transportation network. \(\gamma_m \geq 0\) is the amount that users will pay if there were no \(g_m(v) \leq 0\) constraint and similarly, \(\theta_n\) is the amount that users will pay in the absence of experiencing the \(h_n(t) = 0\) constraint. We define all such costs resulting from having these constraints as the “constraint costs.”

Based on the interpretation above, an interesting result for the sum of the toll revenue and constraint costs is given by the following corollary:

**Corollary 3.2** The total amount of toll charges and constraint costs paid by the users over a variable demand traffic network is constant and defined by

\[
\sum_{a \in \mathcal{A}} \left( \beta_a + \sum_{m \in \mathcal{M}} \gamma_m \frac{\partial g_m(\bar{v})}{\partial v_a} \right) \bar{v}_a - \sum_{k \in \mathcal{K}} \sum_{n \in \mathcal{N}} \theta_n \frac{\partial h_n(\bar{t})}{\partial t_k} \bar{t}_k = \sum_{k \in \mathcal{K}} w_k(\bar{t}_k) \bar{t}_k - \sum_{a \in \mathcal{A}} s_a(\bar{v}) \bar{v}_a
\]

when \(W_{GVD}(\bar{v}, \bar{t})\) defines the toll set.

**Proof**: This is obvious from the definition of \(W_{GVD}(\bar{v}, \bar{t})\). \(\Box\)

What if the multipliers \((\rho, \gamma, \theta)\) were fixed to a particular value beforehand? For example, when the system problem is solved, the multipliers \((\bar{\rho}, \bar{\gamma}, \bar{\theta})\) are...
generally available. If \( \rho \) were fixed to \( \bar{\rho} \), we would force the users to pay exactly the same amount an individual is paying if he/she is traveling on a system optimized travel network. When \( \bar{\gamma}_m \) is fixed, users will perceive exactly the same amount of costs they are experiencing in the system optimal solution. A similar argument holds for the \( \bar{\theta} \) multipliers. Now, consider the special case where \( (\bar{\rho}, \bar{\gamma}, \bar{\theta}) \) are all fixed to define the toll set.

**Theorem 3.1** Assume that \( (\bar{\rho}, \bar{\gamma}, \bar{\theta}) \) are the optimal multipliers for the SOPT-GVD problem. Define the toll set \( W_{GVD}(\bar{v}, \bar{t}, \bar{\rho}, \bar{\gamma}, \bar{\theta}) \) as

\[
\left( s_a(\bar{v}) + \beta_a + \sum_{m \in M} \bar{\gamma}_m \frac{\partial g_m(\bar{v})}{\partial v_a} \right) + (\bar{\rho}_i - \bar{\rho}_j) \geq 0 \quad \forall k \in K, \forall a = (i, j) \in A \\
\left( w_k(\bar{t}_k) + \sum_{n \in N} \bar{\theta}_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) - (\bar{\rho}_q - \bar{\rho}_p) \leq 0 \quad \forall k \in K \\
\sum_{a \in A} \left( s_a(\bar{v}) + \beta_a + \sum_{m \in M} \bar{\gamma}_m \frac{\partial g_m(\bar{v})}{\partial v_a} \right) \bar{v}_a = \sum_{k \in K} \left( w_k(\bar{t}_k) + \sum_{n \in N} \bar{\theta}_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) \bar{t}_k.
\]

In this case, on links that carry flow, users can only be charged by an amount equal to the marginal social cost pricing toll, \( s'_a(\bar{v}_a)\bar{v}_a \).

**Proof.** Using Lemma 3.1 and Lemma 3.5, for those links where \( \bar{x}_a^k > 0 \)

\[
\beta_a = \bar{\rho}_j - \bar{\rho}_i - s_a(\bar{v}) - \sum_{m \in M} \bar{\gamma}_m \frac{\partial g_m(\bar{v})}{\partial v_a} = s'_a(\bar{v}_a)\bar{v}_a
\]

which is the MSCP toll amount. \( \square \)

Furthermore, on all other links (i.e., those links with zero flow), the minimum nonnegative toll amount users should be charged in order to prevent having traffic is \( \beta_a = 0 \). And this is also the MSCP toll amount.

In other words, on links that carry flow, the toll amount is equal to the MSCP toll amount, thus MSCP toll vector is the unique toll vector on those links. On all other links, users need not be charged as it is the case in the MSCP tolling scheme.
Note also that, for the toll set to be nontrivial, some feasibility (variability) in \((\rho, \gamma, \theta)\) is required.

### 3.2.5 A Restricted Toll Set

Now, assume that \((\bar{v}, \bar{t}, \bar{\gamma}, \bar{\theta})\) are obtained from the system problem. Then it is clear that the complementary slackness conditions \(\bar{\gamma}_m g_m(\bar{v}) = 0\) holds. Further, consider the case where SOPT-GVD and UOPT-GVD solutions are unique. In this case, the toll set can be characterized by the following lemma.

**Lemma 3.6** Assume that \((\bar{v}, \bar{t}, \bar{\gamma}, \bar{\theta})\) are optimal flows, demands and KKT multipliers of the SOPT-GVD. Then, when the user and system problems have unique solutions, \(W_{GVD}(\bar{v}, \bar{t}, \bar{\gamma}, \bar{\theta})\) can be defined as follows:

\[
\left( s_a(\bar{v}) + \beta_a + \sum_{m \in M} \bar{\gamma}_m \frac{\partial g_m(\bar{v})}{\partial v_a} \right) + (\rho^k_i - \rho^k_j) \geq 0 \quad \forall k \in K, \forall a = (i, j) \in A \\
\left( w_k(\bar{t}_k) + \sum_{n \in N} \bar{\theta}_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) - (\rho^p_q - \rho^p_p) \leq 0 \quad \forall k \in K \\
\sum_{a \in A} \left( s_a(\bar{v}) + \beta_a + \sum_{m \in M} \bar{\gamma}_m \frac{\partial g_m(\bar{v})}{\partial v_a} \right) \bar{v}_a = \sum_{k \in K} \left( w_k(\bar{t}_k) + \sum_{n \in N} \bar{\theta}_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) \bar{t}_k.
\]

**Proof:** Since \((\bar{v}, \bar{t})\) is the system optimal solution, the optimality conditions of the system problem ensure that the complementarity conditions \(\bar{\gamma}_m g_m(\bar{v}) = 0\) holds. As a result, the toll set reduces to \(W_{GVD}(\bar{v}, \bar{t}, \bar{\gamma}, \bar{\theta}).\)

Using this toll set, the traffic planner is only allowed to control the \(\beta\) tolls for the transportation network and is not allowed to control any costs/gains related to the side constraints, i.e., the transportation authority decides to have the same amount of incidental costs as the system problem. Further, a valid toll vector in this set generates a total toll revenue \(\beta^T \bar{v}\). Then the following corollary holds:

**Corollary 3.3** The toll revenue for the variable demand traffic assignment models is constant when the toll set is defined by \(W_{GVD}(\bar{v}, \bar{t}, \bar{\gamma}, \bar{\theta}).\)
Proof: This is obvious from the definition of $W_{GVD}(\bar{v}, \bar{t})$. The total toll revenue is,

$$\beta^T \bar{v} = \sum_{k \in K} \left( w_k(\bar{t}_k) + \sum_{n \in N} \bar{\theta}_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) \bar{t}_k - \sum_{a \in A} \left( s_a(\bar{v}) + \sum_{m \in M} \bar{\gamma}_m \frac{\partial g_m(\bar{v})}{\partial v_a} \right) \bar{v}_a. \square$$

However, note that this is not the case in traffic assignment problems with fixed demand [11, 49]. Further, as a result of Theorem 3.1 and Corollary 3.3, the following corollary holds:

**Corollary 3.4** Marginal social cost pricing tolls, $\beta_{MSCP-GVD} = \nabla s(\bar{v})\bar{v}$, are optimal for the GVD model when the objective is minimization of the total toll revenue.

### 3.2.6 First Best Toll Pricing Framework for GVD Models

After defining the toll set, the fixed demand first best toll pricing framework [49] can be extended to the GVD model, assuming uniqueness of the solutions for the SOPT-GVD and UOPT-GVD models. As can be seen in Figure 3.1, the framework can be summarized as follows:

1. Solve the SOPT-GVD to obtain the system optimal solution $(\bar{v}, \bar{t})$ and $(\bar{\gamma}, \bar{\theta})$. The algorithm of Larsson and Patriksson [64] can be used for this step.
2. Define a toll set which is the $\beta$ part of either $W_{GVD}(\bar{v}, \bar{t})$ or $W_{GVD}(\bar{v}, \bar{t}, \bar{\gamma}, \bar{\theta})$ based on the goal which the system planner plans to achieve.
3. Define and optimize an objective function over the toll set, possibly intersected with other constraints (See Table 2.1 for examples).

As in the fixed demand case, various objectives in step 3 lead to linear or mixed integer programs. For the variable demand models, the natural choices can be D-MINSYS, D-MINREV, MINTB and MINMAX objectives (see Table 2.1). The first of these aims to minimize the total tolls collected while constraining the toll vector to be nonnegative. D-MINREV is similar to D-MINSYS, but tolls
Solve the system problem to obtain $(\overline{\beta}, \overline{\gamma}, \overline{\theta})$

Should the users pay exactly the same constraint costs as in the system optimal problem?

Yes

Define the toll set by $\beta$
part of $W_{GVD}(\overline{\gamma}, \overline{\theta})$

Define and optimize an objective function over the toll set, possibly intersected with other constraints

No

Define the toll set by $(\beta, \gamma, \theta)$ part of $W_{GVD}(\overline{\gamma}, \overline{\theta})$

Figure 3.1: First Best Toll Pricing Framework.

are free to be negative as well as positive. MINTB minimizes the number of toll booths, and the MINMAX minimizes the maximum toll on the transportation network. Note that any valid toll vector including MSCP toll vector produces the same toll revenue. As a result, D-MINSYS and D-MINREV formulations are not very interesting for the GVD networks, however, having some alternative toll vectors might help the traffic planner to propose different toll pricing schemes for various scenarios.

In the following sections, we present special cases of the GVD model, define the toll set for each model, and illustrate the toll pricing framework by numerical examples.
3.2.7 Elastic Demand TA Models

The elastic demand (ED) model is a special case of the GVD model. Thus the toll set for the ED model can be derived from the toll set for the GVD model after some manipulations.

The system model with elastic demand assumes that the goal of transportation planners is to maximize the net user benefit \[32, 38, 51, 110\] which is the difference between the total network user benefit, \[\sum_{k \in K} \int_{t_k}^{t_k} w_k(z)dz\], and the system cost, \(s(v)^Tv\). We can state the system problem (SOPT-ED) as

\[
\begin{align*}
\min & \quad s(v)^Tv - \sum_{k \in K} \int_{0}^{t_k} w_k(z)dz \\
\text{subject to} & \quad (v, t) \in V.
\end{align*}
\]

where \(V\) denotes the set of all feasible flows and demands. The system which defines the feasible flows for the elastic demand traffic equilibrium problem has the aggregate flow, network balance and nonnegativity constraints, but there are no \(g\) and \(h\) constraints.

The elastic demand user equilibrium problem (UOPT-ED) models Wardrop’s first principle \[99\] in which users choose the routes from which there is no independent improvement available to any individual. Mathematically, the user problem can be stated as a variational inequality \[33\]:

Find \((\bar{v}, \bar{t}) \in V\) such that

\[
\begin{align*}
s(\bar{v})^T(v - \bar{v}) - w(\bar{t})^T(t - \bar{t}) & \geq 0 \quad \forall (v, t) \in V,
\end{align*}
\]

or a convex programming problem when \(s(v)\) and \(w(t)\) are monotonic and separable in flows and demand

\[
\begin{align*}
\min & \quad \sum_{a \in A} \int_{0}^{v_a} s_a(z)dz - \sum_{k \in K} \int_{0}^{t_k} w_k(z)dz \\
\text{subject to} & \quad (v, t) \in V.
\end{align*}
\]
Using Lemma 3.3 of the GVD model, we can extend the notion of toll pricing to the elastic demand traffic assignment models. For a given vector \((\bar{v}, \bar{t}) \in V\), define \(W_{ED}(\bar{v}, \bar{t})\) as the set of all tolls which ensures \((\bar{v}, \bar{t})\) is tolled user equilibrium flow and demands. Then we define the toll set for the ED model as follows:

\(W_{ED}(\bar{v}, \bar{t})\) is the polyhedron given by the \(\beta\) part of the following linear inequality system in \(\beta\) and \(\rho^k\) variables:

\[
\begin{align*}
& s(\bar{v}) + \beta \geq A^T \rho^k \quad \forall k \in K \\
& w_k(\bar{t}_k) \leq E_k^T \rho^k \quad \forall k \in K \\
& (s(\bar{v}) + \beta)^T \bar{v} = w(\bar{t})^T \bar{t}.
\end{align*}
\]

where \((\bar{v}, \bar{t}) \in V\).

It is clear that \(\beta^T \bar{v} = w(\bar{t})^T \bar{t} - s(\bar{v})^T \bar{v}\) is constant for the elastic demand model.

To illustrate the toll pricing framework, we have modified the nine node example from [11, 49, 50] which is depicted in Figure 3.2. This network has cost data similar to large-scale traffic assignment problems. Emphasis here will be on comparison of the alternative tolling schemes including MSCP tolls.

The nine node network has 18 links and all of the links have cost functions with the same structure: \(s_a(v) = s_a(v_a) = T_a (1 + 0.15 (v_a/C_a)^4)\) where \(T_a\) is a measure of travel time when there is zero flow and \(C_a\) is the practical capacity of link \(a\). In fact \(s(v)\) is strictly convex and separable. Therefore the user and system problems have unique optimal solutions. There are four OD-pairs. The demand functions between those OD-pairs are, \(t_{(1,3)}(c_{(1,3)}) = 10 - 0.5c_{(1,3)}\), \(t_{(1,4)}(c_{(1,4)}) = 20 - 0.5c_{(1,4)}\), \(t_{(2,3)}(c_{(2,3)}) = 30 - 0.5c_{(2,3)}\), and \(t_{(2,4)}(c_{(2,4)}) = 40 - 0.5c_{(2,4)}\) where \(c_{pq}\) is the generalized cost between OD-pair \((p, q)\).

As shall be seen in Table 3.2, SOPT-ED and UOPT-ED have different demands. The total demand is 57.411 in the system problem and 60.753 in the user problem, so there is a 5.5% difference. Table 3.2 also contains the user benefit for the optimal demand levels.
Table 3.2: Demand and User Benefit \((\bar{t}, w(\bar{t}))\) for the Nine Node Problem.

<table>
<thead>
<tr>
<th>OD-PAIR</th>
<th>UOPT-ED SOLUTION</th>
<th>SOPT-ED SOLUTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.151, 19.698)</td>
<td>(10.698, 18.605)</td>
</tr>
<tr>
<td>2</td>
<td>(20.672, 18.656)</td>
<td>(29.232, 21.537)</td>
</tr>
</tbody>
</table>

Table 3.3 provides the optimal flows for the SOPT-ED and UOPT-ED problems. Although the UOPT-ED solution has higher user benefit than the SOPT-ED solution, it also has a higher system cost due to more congestion on the network. Thus the net user benefit in the system problem is greater.

When the individual link flow values for SOPT-ED and UOPT-ED are compared, it is clear that they are within 10% of each other on almost all links of the network. However, link flows differ considerably on links (5,7), (5,9) and (9,7). The total flow between nodes 5 and 7 is within 10% for both user and system problems, but the link flows differ substantially. For example (5,7) is over utilized in the user problem while (5,9) and (9,7) are under utilized. So the system planner can charge the users on link (5,7) to make that link less attractive or reimburse on path 5-9-7, so that traffic can be diverted to the path 5-9-7.

Table 3.4 presents alternative toll vectors for the nine node problem obtained by Toll Pricing Framework using an optimization modeling package (GAMS/CPLEX) implementation. For comparison, the MSCP tolls are also listed. As expected, tolls
Table 3.3: Nine Node Problem-Optimal Solution to User and System Problems.

<table>
<thead>
<tr>
<th>Link</th>
<th>SOPT-ED SOLUTION</th>
<th>UOPT-ED SOLUTION</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( v_a ) s_a(v_a)</td>
<td>( v_a' s_a(v_a') )</td>
</tr>
<tr>
<td>1-5</td>
<td>5.000</td>
<td>5.000</td>
</tr>
<tr>
<td>1-6</td>
<td>9.696</td>
<td>6.076</td>
</tr>
<tr>
<td>2-5</td>
<td>31.715</td>
<td>3.303</td>
</tr>
<tr>
<td>2-6</td>
<td>15.999</td>
<td>9.059</td>
</tr>
<tr>
<td>5-6</td>
<td>9.000</td>
<td></td>
</tr>
<tr>
<td>5-7</td>
<td>17.978</td>
<td>4.140</td>
</tr>
<tr>
<td>5-9</td>
<td>13.738</td>
<td>8.094</td>
</tr>
<tr>
<td>6-5</td>
<td>4.000</td>
<td></td>
</tr>
<tr>
<td>6-8</td>
<td>25.696</td>
<td>6.331</td>
</tr>
<tr>
<td>6-9</td>
<td>7.000</td>
<td></td>
</tr>
<tr>
<td>7-3</td>
<td>19.476</td>
<td>3.166</td>
</tr>
<tr>
<td>7-4</td>
<td>12.239</td>
<td>6.061</td>
</tr>
<tr>
<td>7-8</td>
<td>2.000</td>
<td></td>
</tr>
<tr>
<td>8-3</td>
<td>8.000</td>
<td></td>
</tr>
<tr>
<td>8-7</td>
<td>4.000</td>
<td></td>
</tr>
<tr>
<td>9-7</td>
<td>13.738</td>
<td>4.047</td>
</tr>
<tr>
<td>9-8</td>
<td>8.000</td>
<td></td>
</tr>
</tbody>
</table>

User Benefit: 2544.75  2613.50
System Cost: 1005.474  1217.21
Net User Benefit: 1539.284  1396.285

on link (5,7) are high for the tolling schemes with positive tolls, namely, MSCP, D-MINSYS, MINMAX and MINTB. When negative tolls are allowed as in the case of D-MINREV tolls, users are rewarded with subsidies for traveling on the 5-9-7 path in order to achieve the SOPT-ED solution. By Corollary 3.3, all tolling schemes results in the same toll revenue, 268.519, which is 17.44% of net user benefit. D-MINSYS requires nine toll booths, while MSCP and D-MINREV tolls need 10 toll booths. MINMAX has an objective value of 8.00 with eight toll booths. It happens that MINTB obtains the same maximum toll with only five toll booths. Thus it could be argued that the MINTB solution is the best implementable tolling scheme if there is a high cost of building and maintaining the toll booths.

In Table 3.4, there is also information on the number of users who are tolled and the average toll users are paying in each tolling scheme. Since the toll revenue is constant, MSCP has the smallest average toll (1.44) because users are being
Table 3.4: Nine Node Problem-Alternative Tolls using SOPT-ED as the System Problem.

<table>
<thead>
<tr>
<th>Link</th>
<th>( \beta_{MSCP} )</th>
<th>( \beta_{D-MINSYS} )</th>
<th>( \beta_{MINMAX} )</th>
<th>( \beta_{MINTB} )</th>
<th>( \beta_{D-MINREV} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-5</td>
<td>15.000</td>
<td>8.000</td>
<td>8.000</td>
<td>8.000</td>
<td>8.000</td>
</tr>
<tr>
<td>1-6</td>
<td>0.303</td>
<td>2.086</td>
<td>2.086</td>
<td>0.067</td>
<td>2.086</td>
</tr>
<tr>
<td>2-5</td>
<td>1.214</td>
<td>2.018</td>
<td>2.018</td>
<td>-7.444</td>
<td>2.018</td>
</tr>
<tr>
<td>2-6</td>
<td>0.236</td>
<td>2.018</td>
<td>2.018</td>
<td>2.018</td>
<td>2.018</td>
</tr>
<tr>
<td>5-6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>6.218</td>
</tr>
<tr>
<td>5-7</td>
<td>8.561</td>
<td>8.000</td>
<td>8.000</td>
<td>8.000</td>
<td>8.000</td>
</tr>
<tr>
<td>5-9</td>
<td>0.374</td>
<td></td>
<td></td>
<td></td>
<td>-5.953</td>
</tr>
<tr>
<td>6-5</td>
<td></td>
<td>7.839</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6-8</td>
<td>1.323</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6-9</td>
<td></td>
<td>4.839</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7-3</td>
<td>0.663</td>
<td>0.420</td>
<td>2.438</td>
<td>0.420</td>
<td>17.882</td>
</tr>
<tr>
<td>7-4</td>
<td>0.243</td>
<td></td>
<td>2.018</td>
<td></td>
<td>17.462</td>
</tr>
<tr>
<td>7-8</td>
<td></td>
<td></td>
<td></td>
<td>15.408</td>
<td></td>
</tr>
<tr>
<td>8-3</td>
<td></td>
<td></td>
<td>2.320</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8-4</td>
<td>0.459</td>
<td></td>
<td>2.018</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8-7</td>
<td></td>
<td>0.047</td>
<td>0.716</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9-7</td>
<td>0.187</td>
<td></td>
<td></td>
<td>-4.047</td>
<td></td>
</tr>
<tr>
<td>9-8</td>
<td></td>
<td></td>
<td></td>
<td>9.408</td>
<td></td>
</tr>
<tr>
<td>( \beta^t \bar{v} )</td>
<td>268.519</td>
<td>268.519</td>
<td>268.519</td>
<td>268.519</td>
<td>268.519</td>
</tr>
<tr>
<td>( \beta^t \bar{v}/(NUB) ) (%)</td>
<td>17.44</td>
<td>17.44</td>
<td>17.44</td>
<td>17.44</td>
<td>17.44</td>
</tr>
<tr>
<td>Toll Booths</td>
<td>10</td>
<td>9</td>
<td>8</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>( \sum_{a \in A : \beta_a \neq 0} \bar{v}_a )</td>
<td>185.971</td>
<td>94.864</td>
<td>75.388</td>
<td>104.561</td>
<td>116.601</td>
</tr>
<tr>
<td>Average Toll</td>
<td>1.44</td>
<td>2.83</td>
<td>3.56</td>
<td>2.56</td>
<td>2.30</td>
</tr>
</tbody>
</table>

tolled on all utilized links. Individuals are paying an average toll of 3.56 in the MINMAX tolling scheme which is the scheme with the fewest users being charged.

Now, we give an example to illustrate Corollary 3.1. Consider the paths 2-6-8-4 and 2-5-7-4. The total cost of travel on path 2-6-8-4 is 23.523 including the 2.018 of toll costs and 21.505 of travel time costs. The total cost for the second path is 23.523 which includes 10.018 of the toll cost and 13.504 of the travel time cost. The total path cost for both paths is equal to the user benefit as shown in Table 3.2.

This example illustrates how the toll pricing framework can be utilized to generate alternative tolling schemes. Traffic engineers can employ this framework to solve different models by setting additional objectives to be realized while still achieving link flows that optimize the system objective.
3.2.8 Elastic Demand TA Models with Link Capacities

Traditionally, traffic planners handle capacities using special social cost functions like

\[ s_a(v_a) = T_a(1 + 0.15\left(\frac{v_a}{C_a}\right)^4). \]

However, using these functions will not guarantee that the attained flows do not exceed the capacity on each link. For example, in Figure 3.2, the capacity of link (5,7) is 11, but as it can be seen from Table 3.3, the user problem allows 26.44 users and the system problem has 17.98 users, both of which are far above the capacity. Thus it might be important to use the capacity constraints explicitly in some cases.

When the GVD model has the objective function as maximizing the net user benefit and constraints as aggregate flow, network balance and capacity constraints, \( g_a(v_a) = v_a - C_a \), where \( C_a \) is the capacity of link \( a \), the resulting model is the ED-C model.

Given the toll sets \( W_{ED-C}(\bar{v}, \bar{t}) \) and \( W_{ED-C}(\bar{v}, \bar{t}, \bar{\gamma}) \) for the ED-C model, a traffic planner has two options in order to achieve the most efficient utilization (i.e., the system optimal flows and demands) of the network: On the one hand, when \( W_{ED-C}(\bar{v}, \bar{t}) \) is used, the transportation planner can control both the delays (constraint costs) and the amount of toll charges on the network. On the other hand, when \( W_{ED-C}(\bar{v}, \bar{t}, \bar{\gamma}) \) is utilized (where \( \bar{\gamma} \) is the KKT multiplier vector for the capacity constraints of the system problem), users will perceive exactly the same constraint costs as those given by the system problem. As we have mentioned above, the toll revenue is constant for this case.

\( W_{ED-C}(\bar{v}, \bar{t}) \) is the set of all tolls which makes the tolled user equilibrium flows and demands \( (\bar{v}, \bar{t}) \) be same as the system optimal flows and demands. Based on the definitions and assumptions made in Section 3.2.3, we define the toll set as:
$W_{ED-C}(\bar{v}, \bar{t})$ is the polyhedron given by the $\beta$ part of the following linear inequality system in $\beta$, $\rho$ and $\gamma$ variables:

$$
s(\bar{v}) + \beta + \gamma \geq A^T \rho^k \quad \forall k \in K
$$
$$
\quad w_k(\bar{t}_k) \leq E^T_k \rho^k \quad \forall k \in K
$$
$$
(s(\bar{v}) + \beta + \gamma)^T \bar{v} = w(\bar{t})^T \bar{t}
$$
$$
\gamma^T (\bar{v} - C) = 0
$$
$$
\gamma \geq 0
$$

where $(\bar{v}, \bar{t}) \in V$.

Similarly, $W_{ED-C}(\bar{v}, \bar{t}, \bar{\gamma})$ is defined as

$$
s(\bar{v}) + \beta + \bar{\gamma} \geq A^T \rho^k \quad \forall k \in K
$$
$$
\quad w_k(\bar{t}_k) \leq E^T_k \rho^k \quad \forall k \in K
$$
$$
(s(\bar{v}) + \beta + \bar{\gamma})^T \bar{v} = w(\bar{t})^T \bar{t}.
$$

The last two constraints for $W_{ED-C}(\bar{v}, \bar{t})$ were dropped, since $\bar{\gamma}$ already satisfies those conditions. In this case, the total toll revenue is $\beta^T \bar{v} = w(\bar{t})^T \bar{t} - s(\bar{v})^T \bar{v} - \bar{\gamma}^T \bar{v}$ which is unique for a given $(\bar{v}, \bar{t}) \in V$.

In the literature, there have been several interpretations for the multiplier $\gamma$. For example, $\gamma$ is interpreted as the link toll (queuing toll) that the travelers will pay for being allowed to use the links at capacity. It is also a measure of time gained by users of routes filled with capacity compared to the fastest route. It might be also interpreted as delays in the steady state link queue.

In this section, we give two examples to exploit the effect of the queuing (capacities) on the total user delay. The first one is a single link (with a capacity of $C$) having $\bar{t} = \bar{v} > 0$ as the system optimal flow and demand. Using $W_{ED-C}(\bar{v}, \bar{t})$, we observe that the toll set simplifies to

$$
s(\bar{v}) + \beta + \gamma = \rho_2 - \rho_1 = w(\bar{t}).
$$
Actually, as shown in Figure 3.3, any combination of $\beta$ and $\gamma$ on the line

$$\beta + \gamma = w(\bar{t}) - s(\bar{v})$$

is a valid combination of toll charges and constraint costs. In fact, the total cost of tolling in terms of the waiting time is constant, i.e., $(\beta + \gamma)^T \bar{v} = w(\bar{t})\bar{t} - s(\bar{v})\bar{v}$. However, the transportation authority can make a decision on how much users will pay as a toll charge and as a constraint cost (waiting time in queues). For example, the authority has the choice not to toll the users at all (i.e. $\beta = 0$) as a result the users will have a constraint cost of $\gamma = w(\bar{t}) - s(\bar{v})$. Another alternative is tolling the users by an amount of $\beta = w(\bar{t}) - s(\bar{v})$ and have the constraint cost as zero. In fact, the MSCP toll amount $s(\bar{v})\bar{v}$ and the system optimal multiplier $\bar{\gamma}$ are also a valid toll and constraint cost combination. Note also that the MSCP toll vector that Yang and Huang propose is also a valid first best toll vector, i.e.,

$$\beta_{MSCP-ED-YH-C_a} = \beta_{MSCP-ED-C} + \bar{\gamma} = \begin{cases} \nabla s(\bar{v}_a)\bar{v}_a + \bar{\gamma} & \text{if } \bar{v}_a = C_a \\ \nabla s(\bar{v}_a)\bar{v}_a & \text{otherwise.} \end{cases}$$

In other words, $\beta_{MSCP-ED-YH-C_a}$ is a combination of the classical MSCP toll vector $\beta_{MSCP-ED-C}$ and the queueing toll $\bar{\gamma}$ (which is obtained from the system
optimal solution). But recall that Theorem 3.1 proves that the classical definition of the MSCP toll vector, $\nabla s(\bar{v})\bar{v}$, is valid for all GVD models. Another interesting property for a single link capacitated network is that when $\gamma > 0$ the system and user solutions are the same; they only differ in the amount of constraint cost that users incur.

Table 3.5: Demand and User Benefit $(\bar{t}_k, w_k(\bar{t}_k))$ for the ED-C Nine Node Problem.

<table>
<thead>
<tr>
<th>OD-PAIR</th>
<th>SOPT-ED-C SOLUTION</th>
<th>UOPT-ED-C SOLUTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0, 20.000)</td>
<td>(8.276, 23.447)</td>
</tr>
<tr>
<td>2</td>
<td>(15.745, 28.510)</td>
<td>(24.255,31.490)</td>
</tr>
</tbody>
</table>

Table 3.6: Nine Node Problem-Optimal Solution to User and System Problems.

<table>
<thead>
<tr>
<th>Link</th>
<th>SOPT-ED-C SOLUTION</th>
<th>UOPT-ED-C SOLUTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-5</td>
<td>3.216</td>
<td>3.497</td>
</tr>
<tr>
<td>1-6</td>
<td>5.060</td>
<td>4.9619</td>
</tr>
<tr>
<td>2-5</td>
<td>20.000</td>
<td>20.000</td>
</tr>
<tr>
<td>2-6</td>
<td>20.000</td>
<td>20.000</td>
</tr>
<tr>
<td>5-6</td>
<td>9.000</td>
<td>9.000</td>
</tr>
<tr>
<td>3-7</td>
<td>17.717</td>
<td>20.000</td>
</tr>
<tr>
<td>4-9</td>
<td>4.002</td>
<td>4.002</td>
</tr>
<tr>
<td>5-9</td>
<td>5.499</td>
<td>3.497</td>
</tr>
<tr>
<td>6-5</td>
<td>4.000</td>
<td>4.000</td>
</tr>
<tr>
<td>6-8</td>
<td>20.000</td>
<td>20.000</td>
</tr>
<tr>
<td>7-3</td>
<td>15.745</td>
<td>15.750</td>
</tr>
<tr>
<td>7-4</td>
<td>12.531</td>
<td>12.708</td>
</tr>
<tr>
<td>7-8</td>
<td>2.000</td>
<td>2.000</td>
</tr>
<tr>
<td>8-3</td>
<td>8.000</td>
<td>8.000</td>
</tr>
<tr>
<td>8-4</td>
<td>20.000</td>
<td>20.000</td>
</tr>
<tr>
<td>8-7</td>
<td>4.000</td>
<td>4.000</td>
</tr>
<tr>
<td>9-7</td>
<td>10.560</td>
<td>8.458</td>
</tr>
<tr>
<td>9-8</td>
<td>8.000</td>
<td>8.000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Link</th>
<th>User Benefit</th>
<th>Social Cost</th>
<th>Net User Benefit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-5</td>
<td>2311.444</td>
<td>851.010</td>
<td>1460.434</td>
</tr>
<tr>
<td>1-6</td>
<td>2315.666</td>
<td>862.093</td>
<td>1453.573</td>
</tr>
</tbody>
</table>

The second example is the Nine Node problem defined in Section 3.2.7 by adding capacities of 20 on each link. We will use this example to illustrate the toll pricing framework for the SOPT-ED-C problem. This results in substantial changes in the network flows when compared with the SOPT-ED optimal flows and
UOPT-ED equilibrium flows. The total number of users on the network, which is 48.458 for the UOPT-ED-C and 48.276 for SOPT-ED-C, is much less than the uncapacitated case (Table 3.5). It is interesting to note that the user and system problems do not have any demand between OD-pair (1,3), since individuals do not have any incentive for making a trip between this OD-pair.

Table 3.7: Nine Node Problem-Alternative Tolls for the Capacitated Flow Problem when $\bar{\gamma}$ is fixed.

<table>
<thead>
<tr>
<th>Link</th>
<th>$\bar{\gamma}$</th>
<th>$\beta_{MSCP}$</th>
<th>$\beta_{D-MINSYS}$</th>
<th>$\beta_{D-MINREV}$</th>
<th>$\beta_{MINMAX}$</th>
<th>$\beta_{MINTB}$</th>
<th>$\beta_{MINTB/MRV}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 5)</td>
<td>0.02</td>
<td>-0.18</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1, 6)</td>
<td>0.02</td>
<td>7.82</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 5)</td>
<td>9.82</td>
<td>10.01</td>
<td>0.18</td>
<td>0.18</td>
<td>0.18</td>
<td>0.18</td>
<td></td>
</tr>
<tr>
<td>(2, 6)</td>
<td>4.35</td>
<td>4.93</td>
<td>0.55</td>
<td>8.38</td>
<td>0.55</td>
<td>0.55</td>
<td></td>
</tr>
<tr>
<td>(5, 6)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5, 7)</td>
<td>8.08</td>
<td>8.00</td>
<td>14.60</td>
<td>8.00</td>
<td>8.36</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5, 9)</td>
<td>0.01</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6, 5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6, 8)</td>
<td>0.48</td>
<td>0.96</td>
<td>-1.10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6, 9)</td>
<td>0.00</td>
<td></td>
<td>2.62</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7, 3)</td>
<td>0.28</td>
<td>0.37</td>
<td>-6.05</td>
<td>0.37</td>
<td>0.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7, 4)</td>
<td>0.27</td>
<td>0.36</td>
<td>-6.07</td>
<td>0.36</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7, 8)</td>
<td></td>
<td>-2.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(8, 3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(8, 4)</td>
<td>4.12</td>
<td>4.29</td>
<td>0.68</td>
<td>-6.04</td>
<td>0.68</td>
<td>0.68</td>
<td></td>
</tr>
<tr>
<td>(8, 7)</td>
<td></td>
<td></td>
<td></td>
<td>0.56</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(9, 7)</td>
<td>0.07</td>
<td>-4.02</td>
<td></td>
<td>0.36</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(9, 8)</td>
<td></td>
<td>-8.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma^*T\bar{v}$</td>
<td>375.50</td>
<td>375.50</td>
<td>375.50</td>
<td>375.50</td>
<td>375.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta^T\bar{v}$</td>
<td>180.23</td>
<td>180.23</td>
<td>180.23</td>
<td>180.23</td>
<td>180.23</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta^T\bar{v}/NUB$</td>
<td>12.00</td>
<td>12.00</td>
<td>12.00</td>
<td>12.00</td>
<td>12.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Toll Booths</td>
<td>4</td>
<td>12</td>
<td>6</td>
<td>13</td>
<td>7</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

In Table 3.6, we present the system and user optimal flows. The net user benefit for both problems differs only by 0.5%. In the system problem, there are four links which have flows at their capacities (in the user equilibrium, there are five links at their capacities). Another interesting fact is the distribution of flow between nodes 5 and 9 in the existence of capacities. The number of users utilizing the path (link) 5-7 in SOPT-ED-C is 11.40% (2.28 users) less than the number of users in the UOPT-ED-C problem. On the contrary, the number of users in
SOPT-ED-C on path 5-9-7 is 2.00 units higher than the UOPT-ED-C flow. A traffic planner might charge on link 5-7 to divert some users to path 5-9-7 in order to have a better utilization of the network. In fact, users are being charged an amount of at least 8.00 units on link 5-7 when MINTB, MINMAX or D-MINSYS tolling schemes are employed.

Table 3.8: Nine Node Problem-Alternative Tolls for the Capacitated Flow Problem when $\gamma_a$ is not fixed.

<table>
<thead>
<tr>
<th>Link</th>
<th>$\beta_{\text{MINSYS}}$</th>
<th>$\gamma_a^1$</th>
<th>$\beta_{\text{MINMAX}}$</th>
<th>$\beta_{\text{MINTB}}$</th>
<th>$\gamma_a^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 5)</td>
<td>0.36</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1, 6)</td>
<td>0.36</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 5)</td>
<td>10.36</td>
<td>10.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 6)</td>
<td>5.26</td>
<td></td>
<td>4.90</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5, 7)</td>
<td>8.00</td>
<td>8.00</td>
<td>8.36</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5, 9)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6, 5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6, 8)</td>
<td>0.90</td>
<td>5.28</td>
<td>5.28</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6, 9)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7, 3)</td>
<td>0.02</td>
<td>0.37</td>
<td>6.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7, 4)</td>
<td></td>
<td>0.36</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7, 8)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(8, 3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(8, 4)</td>
<td></td>
<td>4.03</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(8, 7)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(9, 7)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.36</td>
</tr>
<tr>
<td>(9, 8)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma^*T\bar{v}$</td>
<td></td>
<td>410.79</td>
<td></td>
<td></td>
<td>298.06</td>
</tr>
<tr>
<td>$\beta^T\bar{v}$</td>
<td></td>
<td>144.92</td>
<td>257.67</td>
<td>257.67</td>
<td></td>
</tr>
<tr>
<td>$\beta^T\bar{v}/\text{NUB}$</td>
<td></td>
<td>9.92</td>
<td>17.64</td>
<td>17.64</td>
<td></td>
</tr>
<tr>
<td>Toll Booths</td>
<td></td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 3.7 shows that when $W_{ED-C}(\bar{v}, \bar{t}, \bar{\gamma})$ is used, users not only have to pay 375.50 in terms of the waiting time in the queues but also pay a total toll amount of 180.23. In this case, D-MINREV tolling scheme needs 13 toll booths while MSCP tolls have 12. MINMAX tolling scheme requires seven toll booths, while six toll booths are needed for D-MINSYS and MINTB/MINREV tolls. The total toll cost, $(\beta + \gamma)^T\bar{v}$, which users are paying both in monetary and time units (constraint costs) is 555.73.
The system controller also has the flexibility to control the waiting time in the queues when the toll set, $W_{ED-C}(\bar{v}, \bar{t})$, is utilized. Although the total cost remains the same (555.73), the total waiting time and the total revenue vary for different tolling schemes. In Table 3.8, $\gamma_a^1$ and $\gamma_a^2$ denote the waiting time in queues for the MINSYS tolling scheme\(^2\), and MINMAX and MINTB tolling schemes, respectively. In this example, MINMAX minimizes the maximum of $\gamma_a + \beta_a$ on each link and MINTB minimizes the number of links where users have to pay either in monetary units or in waiting time. In the MINSYS tolling scheme, users have to wait on four links 410.79 in waiting time and paying 144.92 in toll charges, i.e., they are subject to tolls (both in delays and monetary units) on eight links. When the MINTB and MINMAX tolling schemes are utilized, the total waiting time decreases to 298.06 and the total toll revenue increases to 257.67. Moreover, users suffer either in terms of waiting time or in monetary units only on six links. It can also be observed that if users are charged by $\gamma > 0$ then $\beta = 0$, or vice versa. This is because $\gamma = \hat{\beta} + \hat{\gamma}$ and $\beta = 0$ (or vice versa) is also valid constraint cost (or toll charge). When $\gamma$ is allowed to vary, the MINREV tolling scheme is not an interesting toll pricing scheme, since the total MINREV tolls is unbounded in this case.

In the next section, we analyze the case where there are side constraints that are functions of the aggregate flows. We compare the toll set we define with the one that Larsson and Patriksson [66, 67] propose.

### 3.2.9 Elastic Demand TA Models with Side Constraints

Consider the elastic demand model with side constraints (ED-G). This model assumes that the side constraints are functions of the aggregate flows,\(^2\) The reason that we use MINSYS tolling scheme (instead of the D-MINSYS tolling scheme) is the fact that the total toll revenue is no longer constant when $W_{ED-C}(\bar{v}, \bar{t})$ is used.
that is \( g(v) \leq 0 \). Obviously, this is an extension of the ED-C model where side constraints are the capacity constraints. In this section, we define the toll set, \( W_{ED-G}(\bar{v}, \bar{t}) \), for ED-G model and then compare this set with the toll set that Larsson and Patriksson [66, 67] propose, \( W_{ED-G-LP}(\bar{v}, \bar{t}) \). We show that \( W_{ED-G-LP}(\bar{v}, \bar{t}) \) is an instance of \( W_{ED-G}(\bar{v}, \bar{t}) \).

Let \((\bar{v}, \bar{t})\) be the system optimal flows and demands. Based on Lemma 3.3, we define the toll set for the side constrained elastic demand model as follows:

\[
W_{ED-G}(\bar{v}, \bar{t}) \text{ is the polyhedron given by the } \beta \text{ part of the following linear inequality system in variables } (\beta, \gamma, \rho):
\]

\[
s(\bar{v}) + \beta + \gamma^T \nabla g(\bar{v}) \geq A^T \rho^k \quad \forall k \in K
\]

\[
w_k(\bar{t}_k) \leq E_k^T \rho^k \quad \forall k \in K
\]

\[
(s(\bar{v}) + \beta + \gamma^T \nabla g(\bar{v}))^T \bar{v} = w(\bar{t})^T \bar{t} = 0
\]

\[
\gamma^T g(\bar{v}) = 0
\]

\[
\gamma \geq 0
\]

where \((\bar{v}, \bar{t}) \in V\).

As a special case, when \( W_{ED-G}(\bar{v}, \bar{t}, \bar{\gamma}) \) is used (i.e., \( \bar{\gamma} \), the system optimal multiplier for the side constraints, is fixed), the total toll revenue is constant,

\[
\beta^T \bar{v} = w(\bar{t})^T \bar{t} - (s(\bar{v}) + \bar{\gamma}^T \nabla g(\bar{v}))^T \bar{v}.
\]

Larsson and Patriksson address the problem of finding the set of all tolls which will make the unconstrained user problem have the same equilibrium flows with the side constrained user equilibrium problem. This approach can be viewed as a generalization of Hearn’s approach [46] where one set of tolls that allowed converting a problem with capacities to an uncapacitated one were identified (we derive the toll set for this case in Section 3.3). Larsson and Patriksson define
$W_{ED-G-LP}(\bar{v}, \bar{t})$ as:

$$
\begin{align*}
    s(\bar{v}) + \beta_{LP} & \geq A^T \rho^k \quad \forall k \in K \\
    w_k(\bar{t}_k) & \leq E_k^T \rho^k \quad \forall k \in K \\
    (s(\bar{v}) + \beta_{LP})^T \bar{v} &= w(\bar{t})^T \bar{t} \\
    \beta_{LP} &= \gamma^T \nabla g(\bar{v}) \\
    \gamma^T g(\bar{v}) &= 0 \\
    \gamma &\geq 0.
\end{align*}
$$

Larsson and Patriksson comment also on the uniqueness of the total toll revenue,

$$
\beta_{LP}^T \bar{v} = w(\bar{t})^T \bar{t} - s(\bar{v})^T \bar{v}.
$$

As a result, they argue that MSCP tolls are optimal when the objective is minimization of the total toll revenue. They also conjecture that $W_{ED-G-LP}$ is a singleton for large size traffic assignment problems. This analysis is done without ever defining a system problem.

Now we compare our results with the results of Larsson and Patriksson. First of all, $W_{ED-G-LP}(\bar{v}, \bar{t})$ is a subset of $W_{ED-G}(\bar{v}, \bar{t})$. In other words, $W_{ED-G-LP}(\bar{v}, \bar{t}) \subset W_{ED-G}(\bar{v}, \bar{t})$, because when all tolls, $\beta$, are set to zero in $W_{ED-G}(\bar{v}, \bar{t})$, $W_{ED-G-LP}(\bar{v}, \bar{t})$ is obtained.

Based on the uniqueness of toll revenue for $W_{ED-G-LP}(\bar{v}, \bar{t})$, the following theorem proves that one can not conclude that MSCP tolls are optimal.

**Theorem 3.2** $\beta_{MSCP} \notin W_{ED-G-LP}(\bar{v}, \bar{t})$ unless there exists $\gamma$ such that

$$
\gamma^T \nabla g(\bar{v}) = \nabla s(\bar{v}) \bar{v}.
$$

**Proof:** If $\gamma^T \nabla g(\bar{v}) = \nabla s(\bar{v}) \bar{v}$, then MSCP tolls are optimal for the minimal toll revenue problem defined by $W_{ED-G-LP}(\bar{v}, \bar{t})$. On the other hand, consider the case where $g(\bar{v}) < 0$. For example, in the ED-C model, this can happen
when all of the equilibrium link flows are less than the capacity. In this case, the complementary slackness conditions force $\gamma$ to be zero. However, $\beta_{MSCP}$ is greater than zero as long as there is a flow on a given arc unless $s_a(v)$ is a constant. □

Larsson and Patriksson [66, 67] also propose a toll set for the fixed demand side constrained traffic assignment problem

$$s(\bar{v}) + \beta_{LP} \geq A^T \rho^k \quad \forall k \in K$$

$$(s(\bar{v}) + \beta_{LP})^T \bar{v} = \sum_{k \in K} \bar{t}_k E_k^T \rho^k$$

$$\beta_{LP} = \gamma^T \nabla g(\bar{v})$$

$$\gamma^T g(\bar{v}) = 0$$

$$\gamma \geq 0.$$

Similarly, using the same arguments in Theorem 3.2, one can conclude that the MSCP tolls are not always valid for the toll set they propose. The toll set they define is not always feasible. For example, the Nine-Node example in Section 3.5 does not have a solution when $W_{ED-G-LP}(\bar{v}, \bar{t})$ holds. In other words, $W_{ED-G-LP}(\bar{v}, \bar{t})$ is the empty set for this problem. Also, Hearn et al. [52] provide numerical evidence on the Stockholm network, which is a large size toll pricing problem, that the toll set is not a singleton.

To summarize, the toll sets that Larsson and Patriksson propose for the fixed and elastic demand models do not give a complete description of the polyhedron which defines the set of all possible tolls. Furthermore, MSCP tolls might not be a member of those sets. The toll set that they define is a subset of the toll set Hearn and Ramana [49] and we define.

3.2.10 Combined Distribution Assignment Model

In this section, we present the combined distribution assignment model (CDAM) proposed by Evans [30], and Lundgren and Patriksson [75]. We show that
CDAM is a special case of the GVD model. We present the toll set and then give a numerical example.

In the traffic planning process, the trip distribution model (TDM) determines the number of trips per unit time between the OD pairs. The total flow generated at each origin node and the total flow attracted to each destination node are fixed and known. The distribution of travelers between OD pairs follows a gravity model with an exponential decay form of the travel-time function,

\[ t_k = a_p O_p b_q D_q \exp(-\zeta \pi_k), \quad k = (p, q) \in K \]

where \( \pi_k \) is the least route cost, \( \zeta \) is the dispersion parameter (it is also a cost adjustment parameter), \( a_p \) and \( b_q \) are the positive balancing factors and \( \pi_k \) is the estimate of total route cost between an OD-pair \( k \). The TDM minimizes the total entropy, \( \frac{1}{\zeta} \sum_{k \in K} t_k (\ln t_k - 1) \), subject to the trip production and trip attraction constraints. TDM can be defined by the following mathematical program [30]:

\[
\begin{align*}
\min & \quad \frac{1}{\zeta} \sum_{k \in K} t_k (\ln t_k - 1) \\
\text{s.t} & \\
\sum_q t_k &= O_p && \forall p, (p, q) \in K \\
\sum_p t_k &= D_q && \forall q, (p, q) \in K \\
t_k &\geq 0 && \forall k \in K.
\end{align*}
\]

Evans [30] combined the TDM and traffic assignment models to determine simultaneously the distribution of trips between origins and destinations and assignment of trips to links. This model has the aggregate flow, network balance, trip production, trip attraction and nonnegativity constraints. Let \( V \) denote the set of all feasible flows and demands.
The system problem (SOPT-CDAM) is defined as follows:

$$\min \quad s(v)^Tv + \frac{1}{\zeta} \sum_{k \in K} t_k \ln t_k$$

subject to \((v, t) \in V\)

Similarly, the user problem (UOPT-CDAM) can be stated as [30]

$$\min \quad \sum_a \int_0^{v_a} s_a(z)dz + \frac{1}{\zeta} \sum_{k \in K} t_k \ln t_k$$

subject to \((v, t) \in V\)

Note that the only change between the system and user problems is in the first term in the objective function. In the GVD model, when

$$h^p_1 = \sum_q t_k - O_p, \quad h^q_2 = \sum_p t_k - D_q$$

and

$$w_k(t_k) = -\frac{1}{\zeta}(\ln t_k - 1),$$

the CDAM is obtained.

Let \(\theta_p\) and \(\theta_q\) be the KKT multipliers of the trip production and the trip attraction constraints in the SOPT-CDAM. Using Lemma 3.3, \(W_{CDAM}(\bar{v}, \bar{t})\) can be defined as follows:

\(W_{CDAM}(\bar{v}, \bar{t})\) is the polyhedron given by the \(\beta\) part of the following linear inequality system where \((\bar{v}, \bar{t}) \in V\):

\[
A^T \rho^k \leq s(\bar{v}) + \beta \quad \forall k \in K
\]

\[
\theta_p + \theta_q - \frac{1}{\zeta}(\ln \bar{t}_k + 1) \leq E^T_k \rho_k \quad \forall k = (p, q) \in K
\]

\[
\sum_{k \in K} \left(\theta_p + \theta_q - \frac{1}{\zeta}(\ln \bar{t}_k + 1)\right) \bar{t}_k = (s(\bar{v}) + \beta)^T \bar{v}
\]

The toll set inherits the characteristics of the toll set for the GVD model. For example, the total toll revenue is constant when \(\bar{\theta}\) from the system problem is used (see Corollary 3.3). Furthermore, \(\theta_p + \theta_q\) can be interpreted as the benefit gained by a user who decides to take a trip from origin \(p\) to destination \(q\).
Table 3.9: SOPT-CDAM and UOPT-CDAM Demands for the Nine Node Problem

<table>
<thead>
<tr>
<th>OD-PAIR</th>
<th>SOPT-CDAM</th>
<th>UOPT-CDAM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.00</td>
<td>18.00</td>
</tr>
<tr>
<td>2</td>
<td>28.00</td>
<td>42.00</td>
</tr>
</tbody>
</table>

Table 3.10: Nine Node Problem-SOPT-CDAM and UOPT-CDAM Solutions

<table>
<thead>
<tr>
<th>Link</th>
<th>SOPT-CDAM</th>
<th>UOPT-CDAM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$v_a$</td>
<td>$s_a(v_a)$</td>
</tr>
<tr>
<td>(1, 5)</td>
<td>9.41</td>
<td>5.28</td>
</tr>
<tr>
<td>(1, 6)</td>
<td>20.59</td>
<td>7.54</td>
</tr>
<tr>
<td>(2, 5)</td>
<td>38.33</td>
<td>3.65</td>
</tr>
<tr>
<td>(2, 6)</td>
<td>31.67</td>
<td>9.90</td>
</tr>
<tr>
<td>(5, 6)</td>
<td>9.00</td>
<td></td>
</tr>
<tr>
<td>(5, 7)</td>
<td>21.30</td>
<td>6.22</td>
</tr>
<tr>
<td>(5, 9)</td>
<td>26.44</td>
<td>9.28</td>
</tr>
<tr>
<td>(6, 5)</td>
<td>4.00</td>
<td></td>
</tr>
<tr>
<td>(6, 8)</td>
<td>39.47</td>
<td>7.84</td>
</tr>
<tr>
<td>(6, 9)</td>
<td>12.78</td>
<td>7.03</td>
</tr>
<tr>
<td>(7, 3)</td>
<td>29.61</td>
<td>3.89</td>
</tr>
<tr>
<td>(7, 4)</td>
<td>20.76</td>
<td>6.50</td>
</tr>
<tr>
<td>(7, 8)</td>
<td>2.00</td>
<td></td>
</tr>
<tr>
<td>(8, 3)</td>
<td>10.39</td>
<td>8.01</td>
</tr>
<tr>
<td>(8, 4)</td>
<td>39.24</td>
<td>6.62</td>
</tr>
<tr>
<td>(8, 7)</td>
<td>4.00</td>
<td></td>
</tr>
<tr>
<td>(9, 7)</td>
<td>29.06</td>
<td>4.94</td>
</tr>
<tr>
<td>(9, 8)</td>
<td>10.16</td>
<td>8.02</td>
</tr>
<tr>
<td>Total Entropy</td>
<td>46.05</td>
<td></td>
</tr>
<tr>
<td>$s(v)^T v$</td>
<td>2253.92</td>
<td>2517.91</td>
</tr>
<tr>
<td>System Objective($v$)</td>
<td>2207.87</td>
<td>2472.37</td>
</tr>
</tbody>
</table>

To illustrate the toll pricing framework for the CDAM, the Nine Node problem [11, 49, 51] is modified by adding the trip attraction and trip production constraints. $O_1 = 30, O_2 = 70, D_3 = 40, \text{ and } D_4 = 60$ are the total demand in the zones one through four. The travel time functions are those which have been used in Section 3.5. The dispersion parameter, $\zeta$, is 10.

The demand vector for the system and user problems are presented in Table 3.9. The demand vector for both problems differ considerably, but the total entropy for each problem is approximately equal. The entropy term forces both problems to have trips between each OD pair.
Table 3.11: Nine Node CDAM Network-Alternative Tolls

<table>
<thead>
<tr>
<th>Link</th>
<th>$\beta_{MSCP}$</th>
<th>$\beta_{D-MINSYS}$</th>
<th>$\beta_{D-MINREV}$</th>
<th>$\beta_{MINMAX}$</th>
<th>$\beta_{MINTB}$</th>
<th>$\beta_{MINTB/MRV}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 5)</td>
<td>1.13</td>
<td>6.36</td>
<td>15.64</td>
<td>6.36</td>
<td>6.36</td>
<td>6.36</td>
</tr>
<tr>
<td>(1, 6)</td>
<td>6.16</td>
<td>6.36</td>
<td>13.38</td>
<td>5.56</td>
<td>2.54</td>
<td>-2.54</td>
</tr>
<tr>
<td>(2, 5)</td>
<td>2.59</td>
<td>7.81</td>
<td>17.10</td>
<td>7.81</td>
<td>1.45</td>
<td>1.46</td>
</tr>
<tr>
<td>(2, 6)</td>
<td>3.62</td>
<td>3.81</td>
<td>10.84</td>
<td>3.01</td>
<td>-2.54</td>
<td>2.54</td>
</tr>
<tr>
<td>(5, 6)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5, 7)</td>
<td>16.88</td>
<td>8.00</td>
<td>4.39</td>
<td>8.00</td>
<td>20.03</td>
<td>21.56</td>
</tr>
<tr>
<td>(5, 9)</td>
<td>5.13</td>
<td></td>
<td>-9.28</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6, 5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6, 8)</td>
<td>7.37</td>
<td>7.20</td>
<td>0.17</td>
<td>8.00</td>
<td>11.01</td>
<td>16.03</td>
</tr>
<tr>
<td>(6, 9)</td>
<td>0.11</td>
<td></td>
<td>-7.03</td>
<td>0.80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7, 3)</td>
<td>3.54</td>
<td>7.20</td>
<td>1.53</td>
<td>7.20</td>
<td>1.53</td>
<td></td>
</tr>
<tr>
<td>(7, 4)</td>
<td>2.01</td>
<td>5.67</td>
<td>5.67</td>
<td>5.67</td>
<td>-1.52</td>
<td></td>
</tr>
<tr>
<td>(7, 8)</td>
<td></td>
<td>1.08</td>
<td>1.08</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(8, 3)</td>
<td>0.02</td>
<td></td>
<td></td>
<td>-2.47</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(8, 4)</td>
<td>2.50</td>
<td>2.47</td>
<td>2.47</td>
<td>2.47</td>
<td>2.47</td>
<td>2.47</td>
</tr>
<tr>
<td>(8, 7)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(9, 7)</td>
<td>3.75</td>
<td></td>
<td>5.67</td>
<td>9.49</td>
<td>13.56</td>
<td></td>
</tr>
<tr>
<td>(9, 8)</td>
<td>0.06</td>
<td></td>
<td></td>
<td>3.81</td>
<td>8.83</td>
<td></td>
</tr>
<tr>
<td>$\beta^T \bar{v}$</td>
<td>1493.53</td>
<td>1493.53</td>
<td>1493.53</td>
<td>1493.53</td>
<td>1493.53</td>
<td>1493.53</td>
</tr>
<tr>
<td>Toll Booths</td>
<td>14</td>
<td>10</td>
<td>11</td>
<td>11</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>$\frac{\beta^T \bar{v}}{s(\bar{v})^T \bar{v}}$</td>
<td>67.67%</td>
<td>67.67%</td>
<td>67.67%</td>
<td>67.67%</td>
<td>67.67%</td>
<td>67.67%</td>
</tr>
</tbody>
</table>

In Table 3.10, the system and user optimal solutions are given. There is a 12% difference in the utilization of the network when the system objective at the user and system solutions are compared. Thus the traffic planner can utilize the toll pricing framework to make the user equilibrium flows be the same as the system optimal flows by imposing tolls on the transportation network. The system optimal $\bar{\theta}_1 = 37.68$, $\bar{\theta}_2 = 37.50$ and $\bar{\theta}_3 = -1.09$. The $\zeta^{-1}(\ln \bar{t}_k + 1)$ is 0.36 for all OD-pairs. For all utilized paths, the total path cost is equal to generalized user benefit $\bar{\theta}_p + \bar{\theta}_q - \zeta^{-1}(\ln \bar{t}_k + 1)$ for a commodity $k$.

Some alternative toll vectors using the toll set representation $W_{CDAM}(\bar{v}, \bar{t}, \bar{\theta})$ are presented in Table 3.11. The total toll revenue is 1493.53 for each tolling scheme. Network users are charged 67.67% of their actual cost for the externalities they are imposing on others. This is achieved by 14 toll booths with MSCP tolls, 10 with D-MINSYS tolls, 11 with D-MINREV and MINMAX tolls, nine with
MINTB and eight with MINTB/MINREV tolls. Since the toll revenue is constant, the D-MINSYS and D-MINREV tolling schemes do not have meaningful objectives. However, they are used to identify alternative toll vectors.

### 3.3 Another GVD TA Model

In the previous section, we have assumed that users have perfect information about the current conditions of the network. They not only know the conditions on the system but also aware of the constraints (such as capacities) on the network or constraints resulting from a design by the transportation authority.

However one might argue that it is impossible for users to be aware of all such constraints. The users perceive the conditions on the network without any constraints (either physical or those imposed by the traffic planners). In other words, although the system problem is defined by SOPT-GVD, the user problem (UOPT-GVD-2) is defined by the following variational inequality:

Find \((\bar{v}, \bar{t}) \in V'\) such that

\[
s(\bar{v})^T (v - \bar{v}) - w(\bar{t})^T (t - \bar{t}) \geq 0 \quad \forall (v, t) \in V',
\]

where \(V'\) is defined by the aggregate flow and network balance constraints. Note that the user problem is similar to UOPT-ED.

In this section, our goal is to identify the set of all tolls that will make the unconstrained tolled user equilibrium have the same flows and demands as the constrained system optimal (SOPT-GVD) solution.

Let \((\bar{v}, \bar{t}, \bar{\gamma}, \bar{\theta})\) be the system optimal solution. Assume that on each link users are charged by an amount equal to

\[
\lambda_a = \beta_a + \sum_{m \in M} \bar{\gamma}_m \frac{\partial g_m(\bar{v})}{\partial v_a}.
\]
Furthermore, each user is rewarded (gains extra benefit) for an amount equal to
\[ \xi_k = \sum_{n \in \mathcal{N}} \bar{\theta}_n \frac{\partial h_n(\bar{t})}{\partial t_k} \]
for making a trip between OD-pair \( k = (p, q) \).

Suppose \( s_\lambda(v) = s(v) + \lambda \) is the perturbed cost map where \( \beta \) is a toll vector.

Further, let \( w_\xi(t) = w(t) + \xi \) be the perturbed inverse demand function. Then the perturbed user problem can be stated as a variational inequality:

\[ (\bar{v}, \bar{t}) \text{ is tolled user equilibrium flow if and only if} \]
\[ (s(\bar{v}) + \lambda)^T(v - \bar{v}) - (w(\bar{t}) + \xi)^T(t - \bar{t}) \geq 0, \quad \forall (v, t) \in V'. \]

Let \( \hat{U}^*_{(\lambda, \xi)} \) be the set of all \textit{tolled equilibrium solutions}. As it was the case for the GVD models presented in Section 3.2, we identify all toll vectors such that the resulting tolled user equilibrium problem has a system optimal solution, i.e.,

\[ \emptyset \neq \hat{U}^*_{(\lambda, \xi)} \subseteq S^* \]

Any such \( (\lambda, \xi) \) is defined to be a \textit{valid toll vector}. Similarly, let the \textit{toll set} denoted by \( T' \) be the set of all valid toll vectors and for a given vector \( (\bar{v}, \bar{t}) \in V' \), \( W_{GVD-2}(\bar{v}, \bar{t}, \bar{\gamma}, \bar{\theta}) \) is the set of all tolls which ensures that \( (\bar{v}, \bar{t}) \) is a solution of \( U_{OPT}(\lambda, \xi) \cdot \text{-GVD-2} \). In fact, using Lemma 3.1 and Lemma 3.2, we can define \( W_{GVD-2}(\bar{v}, \bar{t}, \bar{\gamma}, \bar{\theta}) \) as:

**Lemma 3.7** Given that \( (\bar{v}, \bar{t}) \in V \), \( W_{GVD-2}(\bar{v}, \bar{t}, \bar{\gamma}, \bar{\theta}) \) is the polyhedron given by the \( \beta \) part of the linear inequality system defined in \( (\beta, \rho) \):

\[
\left( \sum_{m \in M} \bar{\gamma}_m \frac{\partial g_m(\bar{v})}{\partial v_a} \right) + (\rho^k_i - \rho^k_j) \geq 0 \quad \forall k \in \mathcal{K}, \forall a = (i, j) \in \mathcal{A} \\
\left( w_k(\bar{t}_k) + \sum_{n \in \mathcal{N}} \bar{\theta}_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) - (\rho^k_q - \rho^k_p) \leq 0 \quad \forall k \in \mathcal{K} \\
\sum_{a \in \mathcal{A}} \left( s_a(\bar{v}) + \beta_a + \sum_{m \in M} \bar{\gamma}_m \frac{\partial g_m(\bar{v})}{\partial v_a} \right) \bar{v}_a = \sum_{k \in \mathcal{K}} \left( w_k(\bar{t}_k) + \sum_{n \in \mathcal{N}} \bar{\theta}_n \frac{\partial h_n(\bar{t})}{\partial t_k} \right) \bar{t}_k .
\]
Proof: The proof for this lemma is similar to the one for Lemma 3.2. Note that $\xi$ is constant and $\lambda$ is a function of $\beta$ for a given system optimal solution $(\bar{v}, \bar{t}, \bar{\gamma}, \bar{\theta})$. Thus, $\beta$ and $\rho$ are the variables that define the toll set. Furthermore, the system optimal $\bar{\gamma}$ ensures that the complementarity constraint $\bar{\lambda}^T g(\bar{v}) = 0$ holds. □

This result implies that if users are charged by the correct amount, the unconstrained user equilibrium solution will be the same as the constrained system optimal solution. A similar result for the capacitated networks is given by Hearn [46] and Larsson and Pattrikson [63, 62, 65].

In Table 3.7 and 3.8, when users are charged by $\beta + \gamma$, the resulting tolled user equilibrium (without capacity constraints) flows and demands are the same as the ED-C system solution. Similarly, when users are rewarded for taking the trip between OD-pair $k$ by an amount of $\bar{\theta}_p + \bar{\theta}_q$ and tolled by an amount of $\beta$ (see Table 3.11), the tolled user equilibrium solution (for the user problem without trip production and trip attraction constraints) is the same as the CDAM system optimal solution.

3.4 Other Extensions

Now, we show that traffic planners can obtain any basic feasible flow by the toll pricing framework. For notational simplicity, we illustrate this result for the ED model. We then present the notion of toll pricing for the combined travel choice models. Finally, we give alternative descriptions of the tolled user equilibrium solutions.

3.4.1 Toll Pricing for Controlling Flows

We now prove that we can obtain any feasible flow by perturbing the social cost function.
Theorem 3.3 Any feasible flow can be attained as an user equilibrium flow by perturbing the social cost function.

Proof: Assume the flow vector that we want to attain is \((\bar{v}, \bar{t})\). Consider the problem UOPT-ED-M

\[
s(\bar{v})^T(v - \bar{v}) - w(\bar{t})^T(t - \bar{t}) \geq 0 \quad \forall (v, t) \in V'.
\]

where

\[
V' = \{(v, t) | v = \sum_{k \in K} x^k, \quad v = \bar{v}, \quad Ax^k = t_k E_k, x^k \geq 0, t_k \geq 0 \ \forall k \in K\}
\]

is the set of all feasible flows. The vector \((\bar{v}, \bar{t})\) is user optimal if and only if UOPT-ED-M holds. That is,

\[
s(\bar{v})^Tv - w(\bar{t})^Tt \geq s(\bar{v})^T\bar{v} - w(\bar{t})^T\bar{t} \quad \forall (v, t) \in V'.
\]

In other words, \((\bar{v}, \bar{t})\) is user optimal if and only if it solves the following linear program

\[
\min\{s(\bar{v})^Tv - w(\bar{t})^Tt | v = \sum_{k} x^k, v = \bar{v}, \quad Ax^k = t_k E_k, x^k \geq 0, t_k \geq 0 \ \forall k \in K\}.
\]

This linear program has a dual

\[
\max\{\beta^T\bar{v}| A^T \rho^k \leq s(\bar{v}) + \beta, w_k(\bar{t}_k) \leq E_k^T \rho^k \ \forall k \in K\}
\]

where \(\beta\) is the vector of dual variables for \(v - \bar{v} = 0\) constraint.

Since both the primal and dual linear programs have the same optimal objective value, \(\beta^T\bar{v} = s(\bar{v})^T\bar{v} - w(\bar{t})^T\bar{t}\). The optimality conditions guarantee that there exist \(\rho^k\) and \(\beta\) such that

\[
\begin{align*}
s(\bar{v}) + \beta & \geq A^T \rho^k \quad \forall k \in K, \\
w_k(\bar{t}_k) & \leq E_k^T \rho^k \quad \forall k \in K, \\
(s(\bar{v}) + \beta)^T\bar{v} & = w(\bar{t})^T\bar{t}.
\end{align*}
\]
Table 3.12: Nine Node Problem-Alternative Tolls for the Designed Flow Problem.

<table>
<thead>
<tr>
<th>ARC</th>
<th>FLOW</th>
<th>TIME</th>
<th>COST</th>
<th>( \beta_{D-MINREV} )</th>
<th>( \beta_{MINTB/MINREV} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 5)</td>
<td>20.00</td>
<td>5.00</td>
<td>100.00</td>
<td>-30.79</td>
<td></td>
</tr>
<tr>
<td>(1, 6)</td>
<td>20.00</td>
<td>6.08</td>
<td>121.52</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 5)</td>
<td>10.00</td>
<td>3.30</td>
<td>33.03</td>
<td>37.00</td>
<td>67.78</td>
</tr>
<tr>
<td>(2, 6)</td>
<td>30.00</td>
<td>9.06</td>
<td>271.77</td>
<td>17.64</td>
<td>17.64</td>
</tr>
<tr>
<td>(5, 6)</td>
<td>9.00</td>
<td></td>
<td></td>
<td>18.37</td>
<td></td>
</tr>
<tr>
<td>(5, 7)</td>
<td>30.00</td>
<td>4.14</td>
<td>124.21</td>
<td>-22.53</td>
<td>-53.32</td>
</tr>
<tr>
<td>(5, 9)</td>
<td>8.00</td>
<td></td>
<td></td>
<td>12.00</td>
<td></td>
</tr>
<tr>
<td>(6, 5)</td>
<td>4.00</td>
<td></td>
<td></td>
<td>8.63</td>
<td>39.42</td>
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<tr>
<td>(6, 8)</td>
<td>50.00</td>
<td>6.33</td>
<td>316.54</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6, 9)</td>
<td>7.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7, 3)</td>
<td>30.00</td>
<td>3.17</td>
<td>94.97</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7, 4)</td>
<td>6.00</td>
<td></td>
<td></td>
<td>17.93</td>
<td>17.93</td>
</tr>
<tr>
<td>(7, 8)</td>
<td>2.00</td>
<td></td>
<td></td>
<td>21.93</td>
<td>40.95</td>
</tr>
<tr>
<td>(8, 3)</td>
<td>8.00</td>
<td></td>
<td></td>
<td>12.00</td>
<td></td>
</tr>
<tr>
<td>(8, 4)</td>
<td>50.00</td>
<td>6.11</td>
<td>305.74</td>
<td>-7.65</td>
<td>-25.76</td>
</tr>
<tr>
<td>(8, 7)</td>
<td>4.00</td>
<td></td>
<td></td>
<td>12.07</td>
<td></td>
</tr>
<tr>
<td>(9, 7)</td>
<td>4.00</td>
<td></td>
<td></td>
<td>4.70</td>
<td></td>
</tr>
<tr>
<td>(9, 8)</td>
<td>8.00</td>
<td></td>
<td></td>
<td>12.00</td>
<td></td>
</tr>
<tr>
<td>User Benefit</td>
<td>600.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>System Cost</td>
<td>1367.78</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Net User Benefit</td>
<td>-767.78</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total Tolls</td>
<td>-1680.40</td>
<td>-1680.40</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Toll Booths</td>
<td>14.00</td>
<td></td>
<td></td>
<td>7.00</td>
<td></td>
</tr>
</tbody>
</table>

But this is \( W_{ED}(\bar{v}, \bar{t}) \). In other words, these conditions are exactly the optimality conditions for the UOPT\(_\beta\)-ED. So UOPT-ED-M and UOPT\(_\beta\)-ED are equivalent. Thus, the user problem when perturbed by any \( \beta \in W_{ED}(\bar{v}, \bar{t}) \) will produce the planned flow \((\bar{v}, \bar{t})\). Note that this theorem can easily be extended to the general variable demand and fixed demand models. \(\square\)

We now illustrate toll pricing for a controlled flow other than system optimal flow. In the nine node example (Figure 3.2), let \( t_{(1,3)} = t_{(1,4)} = 20, t_{(2,3)} = 10 \) and \( t_{(2,4)} = 30 \). The flows are designed to be on paths \((1-5-7-3), (1-6-8-4), (2-5-7-3) \) and \((2-6-8-4)\). In Table 3.12, link flows obtained from the tolled user equilibrium problem and D-MINREV and MINTB/MINREV tolls are given. For sustaining the given amount of flow on the network, the users should be reimbursed for a total amount of 1680.40 units. There are 16 D-MINREV toll booths compared to seven...
toll booths of MINTB/MINREV. For making links (5,7) and (8,4) more attractive large amount of subsidies are offered. In contrast, the traffic controller should employ large tolls on links (2,5), (6,5) and (7,8) to make these links less attractive. Note that MSCP tolls, D-MINSYS tolls or MINTB tolls do not exist, because the intersection of $W(v, t)$ and $\beta \geq 0$ is empty for the given $(\bar{v}, \bar{t})$.

### 3.4.2 Further Extensions of the GVD Model

The GVD formulation can be extended by adding some further terms to the objective or having other side constraints such as the constraints on commodity flows. To illustrate, we study the combined travel choice model (CTCM) proposed by Balasubramaniam [5] and Boyce et al. [13] which incorporates origin-destination choice, mode choice, and route choice.

In Section 3.6, we analyzed the combined distribution assignment model and assumed that there is a single mode of transportation. However, most of the time, users decide which mode of transportation to use in order to reach a destination. This decision depends on the congestion level, the departure time, and the transportation modes available.

We utilize the notation by Balasubramaniam [5]. Let $m \in \mathcal{M}$ be the set of transportation modes available. Users either can take the transit system or use their automobiles. Let $P_{km}$ be the proportion of person trips between OD-pair $k = (p, q)$ by automobiles ($m = h$) and transit ($m = t$). The additional notation that we employ is presented in Table 3.13. The following inequalities define the feasible region for CTCM:

\[
v = \sum_k x^k \quad \forall k \in \mathcal{K}
\]

\[
A x^k = P_{kh} E_k N_k + T_k \quad \forall k \in \mathcal{K}
\]

\[
\sum_q \sum_m P_{km} = O_p \quad \forall p, k = (p, q) \in \mathcal{K}
\]
Table 3.13: Parameter Definitions for the Combined Travel Choice Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>Mode</td>
</tr>
<tr>
<td>( h )</td>
<td>Automobile</td>
</tr>
<tr>
<td>( t )</td>
<td>Transit</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>Average auto occupancy</td>
</tr>
<tr>
<td>( r_1 )</td>
<td>Operating cost per unit distance</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>Operating cost per unit time</td>
</tr>
<tr>
<td>( \psi_1, \ldots, \psi_7 )</td>
<td>Cost adjustment parameters</td>
</tr>
<tr>
<td>( c_{km} )</td>
<td>In - vehicle travel time for commodity ( k = (p,q) ) by transit</td>
</tr>
<tr>
<td>( l_{kt} )</td>
<td>Transit fare for commodity ( k ) using mode ( m )</td>
</tr>
<tr>
<td>( w_{kt} )</td>
<td>Out of vehicle travel time for commodity ( k ) by transit</td>
</tr>
<tr>
<td>( \omega(v) )</td>
<td>Automobile operation cost</td>
</tr>
<tr>
<td>( N )</td>
<td>Total number of trips</td>
</tr>
<tr>
<td>( O_p )</td>
<td>Proportion of total trip origins from zone ( p )</td>
</tr>
<tr>
<td>( D_q )</td>
<td>Proportion of total trip destinations to zone ( q )</td>
</tr>
<tr>
<td>( T_{pq} )</td>
<td>Number of truck trips in automobile equivalent units</td>
</tr>
</tbody>
</table>

\[
\sum_{p} \sum_{m} P_{km} = D_q \quad \forall q, k = (p,q) \in \mathcal{K}
\]

\[
x^k \geq 0 \quad \forall k \in \mathcal{K}
\]

\[
P_{km} \geq 0 \quad \forall k \in \mathcal{K}, m \in \{h, t\}.
\]

\( P_{kh}E_k \frac{N}{\kappa} + T_k \) is the total number of trips between an origin-destination including the truck traffic. The first two constraints are the aggregate flow and network balance constraints. The next two constraints are the trip generation and trip attraction constraints. The nonnegativity of the variables follow.

The system problem (SOPT-CTCM) aims to minimize the total system cost:

\[
\min \ N \psi_1 s(v)^T v + \frac{1}{N} \psi_2 \omega(v)^T v + \psi_3 \sum_k P_{kh} w_{kh} + \psi_4 \sum_k P_{kt} c_k^f + \psi_5 \sum_k P_{kt} l_{kt} + \psi_6 \sum_k P_{kt} w_{kt} + \psi_7 \sum_k P_{kt} + \frac{1}{\mu} \sum_m \sum_{k=(p,q)} P_{km} \ln \left( \frac{P_{km}}{O_p D_q} \right)
\]

s. t. \((v, P) \in V\).
ψ_1, ψ_2 and ψ_3 correspond to the three auto travel components considered: In vehicle travel time, operating cost and out of vehicle travel time. Similarly, ψ_4, ψ_5 and ψ_6 correspond to the three transit travel components. ψ_7 is the transit bias coefficient, a component of the generalized transit cost.

The total auto cost is given by \( \frac{N}{\kappa} \psi_1 s(v)^T v + \frac{1}{N} \psi_2 \omega(v)^T v \). In reality, there is dispersion of the trips to some higher cost OD-pairs and modes which might be attributed to the imperfect information or some other benefits not accounted in the utilized cost functions. The log term in the objective function is the entropy term which guarantees that trips will not be distributed to only those OD-mode choices with the minimum cost.

Although the GNV model can be reduced to CTCM, for simplicity we continue to utilize the the notation defined for CTCM for the analysis.

Let \( \tilde{s}(\bar{v}) = \frac{N}{\kappa} \psi_1 (s(\bar{v}) + \nabla s(\bar{v}) \bar{v}) \), \( \tilde{\omega}(\bar{v}) = \frac{1}{N} \psi_2 (\omega(\bar{v}) + \nabla \omega(\bar{v}) \bar{v}) \),

\[ c_i^d = \psi_4 c_{kt} + \psi_5 l_{kt} + \psi_6 w_{kt} - \frac{1}{\mu} \left[ \ln(\frac{P_{kt}}{O_p D_q}) + 1 \right], \quad c_i^h = \psi_3 w_{kh} - \frac{1}{\mu} \left[ \ln(\frac{P_{kh}}{O_p D_q}) + 1 \right], \]

and \( \phi_{pq} = \phi_k = \alpha_p + \gamma_q \). The KKT conditions for SOPT-CTCM lead to the following lemma:

**Lemma 3.8** A feasible point \((\bar{v}, \bar{P})\) is a KKT point for SOPT-CTCM if and only if there exists \((\rho^k, \phi_k)\) \(\forall k \in K\) such that the following holds:

\[
A^T \rho^k \leq \tilde{s}(\bar{v}) + \tilde{\omega}(\bar{v}) \quad \forall k \in K
\]
\[
-c_i^h \leq E_k^T \rho^k \frac{N}{\kappa} + \phi_k \quad \forall k \in K
\]
\[
-c_i^d - \psi_7 \leq \phi_k \quad \forall k \in K
\]
\[
(\tilde{s}(\bar{v}) + \tilde{k}(\bar{v}))^T \bar{v} = \sum_k \sum_m (\phi_k + c_i^m) \bar{P}_{km}
\]
The first three inequalities follow from the stationarity of the Lagrangian function. The complementary slackness conditions,

\[ \sum_k \bar{x}^k (\bar{s}(\bar{v}) + \bar{\omega}(\bar{v}) - A^T \rho^k) + \sum_k \bar{P}_{kh} \left( E_k^T \rho^k \frac{N}{K} + \phi_k + c^k \right) + \sum_k \bar{P}_{kt} (\phi_k + c^k + \psi_7) \]

are aggregated to

\[ (\bar{s}(\bar{v}) + \bar{\omega}(\bar{v}))^T \bar{v} = \sum_k \sum_m (\phi_k + c^m) \bar{P}_{km}. \]

We change the system objective function to reflect the user optimum (UOPT-CTCM) objective by replacing the total auto cost over all links with the link cost integrals.

\[
\min \frac{N}{K} \psi_1 \left( \sum_{a \in A} \int_0^{v_a} s(z)dz \right) + \frac{1}{N} \psi_2 \left( \sum_{a \in A} \int_0^{v_a} \omega(z)dz \right) + \psi_3 \sum_k P_{kh} w_{kh} \\
+ \psi_4 \sum_k P_{kt} c^k \bar{P} + \psi_5 \sum_k P_{kt} b_{kt} + \psi_6 \sum_k P_{kt} w_{kt} + \psi_7 \sum_k P_{kt} \\
+ \frac{1}{\mu} \sum_m \sum_{k=(p,q)} P_{km} \ln \left( \frac{P_{km}}{O_{D_q}} \right)
\]

s. t. \((v, P) \in V.\)

Let’s redefine \(\bar{s}(\bar{v}) = \frac{N}{K} \psi_1 s(\bar{v}), \bar{\omega}(\bar{v}) = \frac{1}{N} \psi_2 \omega(\bar{v}).\) Then, we can characterize the optimality conditions for the UOPT-CTCM by the following lemma:

**Lemma 3.9** A feasible point \((\bar{v}, \bar{P})\) is a KKT point for UOPT-CTCM if and only if there exists \((\rho^k, \phi_k) \ \forall k \in K\) such that the following holds:

\[
A^T \rho^k \leq \bar{s}(\bar{v}) + \bar{\omega}(\bar{v}) \quad \forall k \in K \\
-c^k \leq E_k^T \rho^k \frac{N}{K} + \phi_k \quad \forall k \in K \\
-c^k - \psi_7 \leq \phi_k \quad \forall k \in K \\
(\bar{s}(\bar{v}) + \bar{\omega}(\bar{v}))^T \bar{v} = \sum_k \sum_m (\phi_k + c^m) \bar{P}_{km}
\]
Note that the only difference between the optimality conditions of the user and system problems is the term defining the MSCP toll vectors,

$$\beta_{MSCP-CTCM} = \frac{N}{\kappa} \psi_1 \nabla s(\bar{v}) \bar{v} + \frac{1}{N} \psi_2 \nabla \omega(\bar{v}) \bar{v}.$$ 

Thus, the system optimal solution can be obtained when the users are also experiencing the marginal costs of their travel on a link, instead of simply the average cost of travel. This fact for CTCM is also observed by Balasubramaniam [5], Boyce and Balasubramaniam [13], and Boyce and Bar-Gera [14].

Now, assume that the assumptions and definitions in Section 3.2.3 hold. Indeed, the toll set for CTCM, $W_{CTCM}(\bar{v}, \bar{P})$ can be defined as:

**Lemma 3.10** Let $(\bar{v}, \bar{P}) \in V$. Then $W_{CTCM}(\bar{v}, \bar{P})$ is the polyhedron given by the $\beta = (\beta_1, \beta_2)$ part of the following linear inequality system in $\beta$, $\rho$, and $\phi$:

$$A^T \rho^k \leq \tilde{s}(\bar{v}) + \tilde{\omega}(\bar{v}) \quad \forall k \in K$$

$$-c^h_k \leq E^T_k \rho^k \frac{N}{\kappa} + \phi_k \quad \forall k \in K$$

$$-c^l_k - \psi_7 \leq \phi_k \quad \forall k \in K$$

$$(\tilde{s}(\bar{v}) + \tilde{\omega}(\bar{v}))^T \bar{v} = \sum_k \sum_m (\phi_k + c^m_k) \bar{P}_{km}$$

where $\tilde{s}(\bar{v}) = \frac{N}{\kappa} \psi_1 (s(\bar{v}) + \beta_1)$, and $\tilde{\omega}(\bar{v}) = \frac{1}{N} \psi_2 (\omega(\bar{v}) + \beta_2)$

As in the ED model and CDAM, when the multipliers for the side constraints are chosen to be the system optimal multipliers, the total toll revenue is constant:

$$\left(\frac{N}{\kappa} \psi_1 \beta_1 + \frac{1}{N} \psi_2 \beta_2\right)^T \bar{v} = \sum_k \sum_m (\bar{\phi}_k + c^m_k) \bar{P}_{km} - (s(\bar{v}) + \omega(\bar{v}))^T \bar{v}.$$ 

Thus MSCP tolls are optimal when the objective is minimization of the total positive tolls collected.
3.4.3 Alternative Descriptions of Tolled User Equilibrium Solutions

The toll set description presented in Section 3.2.7 is a linked-based characterization of the toll set for elastic demand TA problems. In this section, we present alternative descriptions of the toll set (such as path based, disaggregate link based and path based descriptions) for the elastic demand traffic assignment problems. These descriptions will be utilized in formulating several equivalent second best pricing models in Chapter 4. Note also that, the results given in this section can easily be extended to other GVD models.

First, let’s summarize the result given in Section 3.2.7:

\[ W_{ED}(\bar{v}, \bar{t}) \text{ is the polyhedron given by the } \beta \text{ part of the following linear inequality system in } \beta \text{ and } \rho^k \text{ variables} \]

\[
\begin{align*}
s(\bar{v}) + \beta & \geq A^T \rho^k \quad \forall k \in K \\
w_k(\bar{t}_k) & \leq E_k^T \rho^k \quad \forall k \in K \\
(s(\bar{v}) + \beta)^T \bar{v} & = w(\bar{t})^T \bar{t}
\end{align*}
\]

where \((\bar{v}, \bar{t}) \in V\).

Similarly, one can also define a path based description. Define \(R_k\) to be the set of all possible paths for commodity \(k\). Let \(\chi_{ar_k}\) be 1 if arc \(a\) is on path \(r_k \in R_k\). The following lemma defines the toll set for the ED model as a polyhedron in path costs and user benefits:

**Lemma 3.11** For a given vector \((\bar{v}, \bar{t}) \in V\), \(W'_{ED}(\bar{v}, \bar{t})\) is the polyhedron given by the \(\beta\) part of the following linear inequality system in \(\beta\):

\[
\sum_{a \in A} \chi_{ar_k} (s_a(\bar{v}_a) + \beta_a) \geq w_k(\bar{t}_k) \quad \forall k \in K, \forall r_k \in R_k \\
(s(\bar{v}) + \beta)^T \bar{v} = w(\bar{t})^T \bar{t}.
\]

The theorem below shows the relation between \(W_{ED}(\bar{v}, \bar{t})\) and \(W'_{ED}(\bar{v}, \bar{t})\).
Theorem 3.4 The descriptions of the toll sets in Lemma $W_{ED}(\bar{v}, \bar{t})$ and $W'_{ED}(\bar{v}, \bar{t})$ are equivalent.

Proof: $W'_{ED}(\bar{v}, \bar{t})$ can be obtained from $W_{ED}(\bar{v}, \bar{t})$ by the following procedure:

For all links on path $r_k$, sum the right hand side and left hand side of the inequality $s_a(\bar{v}_a) + \beta_a \geq \rho^k_j - \rho^k_i$ to obtain $\sum_{a \in A} \chi_{ar_k}(s_a(\bar{v}_a) + \beta_a) \geq \rho^k_k - \rho^k_p$. But $\rho^k_q - \rho^k_p \geq w_k(\bar{t}_k)$. Thus the first two constraints in the link based characterization can be combined to yield the first constraint in the path based characterization.

Similarly $W_{ED}(\bar{v}, \bar{t})$ can be obtained from $W'_{ED}(\bar{v}, \bar{t})$ by decomposing $\sum_{a \in A} \chi_{ar_k}(s_a(\bar{v}_a) + \beta_a) \geq w_k(\bar{t}_k)$ into $s_a(\bar{v}_a) + \beta_a \geq \rho^k_j - \rho^k_i$ and $\rho^k_q - \rho^k_p \geq w_k(\bar{t}_k)$ for all $k \in K$ and $r_k \in R_k$ using the procedure above in the reverse order. \(\square\)

Let $\bar{x}^k$ be the optimal commodity flow vector. If the commodity flow vectors and demands are known, the aggregate complementary constraint, $(s(\bar{v}) + \beta)^T \bar{v} = w(\bar{t})^T \bar{t}$, can be dropped from the toll set representation. In this case, the polyhedron that defines the toll set can be modified as:

**Corollary 3.5** If the commodity flow vector is known, then we can define $W_{ED-M}(\bar{v}, \bar{t})$ to be the polyhedron given by the $\beta$ part of the following linear inequality system in $\beta$ and $\rho^k$:

\[
\begin{align*}
  s_a(\bar{v}_a) + \beta_a & \geq \rho^k_j - \rho^k_i & \forall \bar{x}^k_a = 0, a = (i, j) \in A, k \in K \\
  s_a(\bar{v}_a) + \beta_a & = \rho^k_j - \rho^k_i & \forall \bar{x}^k_a > 0, a = (i, j) \in A, k \in K \\
  w_k(\bar{t}_k) & \leq \rho^k_q - \rho^k_p & \forall \bar{t}_k = 0, k = (p, q) \in K \\
  w_k(\bar{t}_k) & = \rho^k_q - \rho^k_p & \forall \bar{t}_k > 0, k = (p, q) \in K.
\end{align*}
\]

Proof: The aggregate complementary constraint is $(s(\bar{v}) + \beta)^T \bar{v} = w(\bar{t})^T \bar{t}$. Add and subtract $\sum_{k \in K} (A\bar{x}^k)^T \rho^k$ from this equation. Then, we obtain

\[
\sum_{a \in A} (s_a(\bar{v}_a) + \beta_a) \bar{v}_a - \sum_{k \in K} w_k(\bar{t}_k) \bar{t}_k - \sum_{k \in K} (A\bar{x}^k)^T \rho^k + \sum_{k \in K} (A\bar{x}^k)^T \rho^k = 0.
\]
But $A\bar{x}^k = E_k\bar{t}_k$. Thus,

$$\sum_{a \in A} (s_a(\bar{v}_a) + \beta_a)\bar{v}_a - \sum_{k \in K} w_k(\bar{t}_k)\bar{t}_k - \sum_{k \in K} (A\bar{x}^k)^T \rho^k + \sum_{k \in K} (E_k\bar{t}_k)^T \rho^k = 0.$$  

This implies

$$\sum_{a \in A} (s_a(\bar{v}_a) + \beta_a)\bar{x}_a^k - \sum_{k \in K} \bar{x}_a^k A^T \rho^k + \sum_{k \in K} \bar{t}_k E_k^T \rho^k - \sum_{k \in K} w_k(\bar{t}_k)\bar{t}_k = 0,$$

or

$$\sum_{k \in K} \sum_{a \in A} (s_a(\bar{v}_a) + \beta_a - \rho_j^k + \rho_i^k)\bar{x}_a^k + \sum_{k \in K} (\rho_q^k - \rho_p^k - w_k(\bar{t}_k))\bar{t}_k = 0.$$

But $s_a(\bar{v}_a) + \beta_a - \rho_j^k + \rho_i^k \geq 0$, $\bar{x}_a^k \geq 0$, $\rho_q^k - \rho_p^k - w_k(\bar{t}_k) \geq 0$ and $\bar{t}_k \geq 0$. This implies if $\bar{x}_a^k > 0$ then $s_a(\bar{v}_a) + \beta_a = \rho_j^k - \rho_i^k$ and if $\bar{x}_a^k = 0$ then $s_a(\bar{v}_a) + \beta_a \geq \rho_j^k - \rho_i^k$. Similarly, if $\bar{t}_k > 0$ then $\rho_q^k - \rho_p^k = w_k(\bar{t}_k)$ and if $\bar{t}_k = 0$ then $\rho_q^k - \rho_p^k \geq w_k(\bar{t}_k)$.

Now, we show that the $W_{ED-M}(\bar{v}, \bar{t})$ formulation implies that the aggregate complementary constraint holds by using the following procedure: Multiply the first two constraints defining $W_{ED-M}(\bar{v}, \bar{t})$ by $\bar{x}_a^k$ and the last two constraints by $\bar{t}_k$.

Then,

$$\sum_{k \in K} \sum_{a \in A} (s_a(\bar{v}_a) + \beta_a - \rho_j^k + \rho_i^k)\bar{x}_a^k + \sum_{k \in K} (\rho_q^k - \rho_p^k - w_k(\bar{t}_k))\bar{t}_k = 0.$$  

Equivalently,

$$\sum_{a \in A} (s_a(\bar{v}_a) + \beta_a)\bar{x}_a^k - \sum_{k \in K} \bar{x}_a^k A^T \rho^k + \sum_{k \in K} \bar{t}_k E_k^T \rho^k - \sum_{k \in K} w_k(\bar{t}_k)\bar{t}_k.$$  

This implies

$$\sum_{a \in A} (s_a(\bar{v}_a) + \beta_a)\bar{v}_a - \sum_{k \in K} w_k(\bar{t}_k)\bar{t}_k - \sum_{k \in K} (A\bar{x}^k)^T \rho^k + \sum_{k \in K} (E_k\bar{t}_k)^T \rho^k.$$  

Thus, $(s(\bar{v}) + \beta)^T \bar{v} = w(\bar{t})^T \bar{t}$. □

Let $\bar{h}_{r_k}$ be the optimal flow on path $r_k$. When utilized paths are known, we can rewrite $W'_{ED}(\bar{v}, \bar{t})$ without having the aggregate complementary constraint.
Corollary 3.6  If the path flow vector is known, then we can define \( W'_{ED-M}(\bar{v}, \bar{t}) \) to be the polyhedron given by the \( \beta \) part of the following linear inequality system in \( \beta \):

\[
\sum_{a \in A} \chi_{ar_k}(s_a(\bar{v}_a) + \beta_a) \geq w_k(\bar{t}_k) \quad \forall \bar{h}_{rk} = 0, k \in \mathcal{K}, r_k \in R_k
\]
\[
\sum_{a \in A} \chi_{ar_k}(s_a(\bar{v}_a) + \beta_a) = w_k(\bar{t}_k) \quad \forall \bar{h}_{rk} > 0, k \in \mathcal{K}, r_k \in R_k.
\]

Proof. We use a procedure similar to the one described in Corollary 3.5.
First, multiply each equation by \( \bar{h}_{rk} \) and sum over all paths for all commodities.
This results in,

\[
\sum_{k \in \mathcal{K}} \sum_{r_k \in R_k} \sum_{a \in A} \chi_{ar_k}(s_a(\bar{v}_a) + \beta_a) \bar{h}_{rk} - \sum_{k \in \mathcal{K}} \sum_{r_k \in R_k} w_k(\bar{t}_k) \bar{h}_{rk} = 0
\]

holds. But this is

\[
\sum_{a \in A} (s_a(\bar{v}_a) + \beta_a) \sum_{k \in \mathcal{K}} \sum_{r_k \in R_k} \chi_{ar_k} \bar{h}_{rk} - \sum_{k \in \mathcal{K}} w_k(\bar{t}_k) \sum_{r_k \in R_k} \bar{h}_{rk} = 0.
\]

This implies

\[
\sum_{a \in A} (s_a(\bar{v}_a) + \beta_a) \sum_{k \in \mathcal{K}} \bar{x}_a^k - \sum_{k \in \mathcal{K}} w_k(\bar{t}_k) \bar{t}_k = 0 \implies (s(\bar{v}) + \beta)^T \bar{v} = w(\bar{t})^T \bar{t}.
\]

The proof for sufficiency is trivial when the following observation is made:

\[
\sum_{k \in \mathcal{K}} \sum_{r_k \in R_k} \left( \sum_{a \in A} \chi_{ar_k}(s_a(\bar{v}_a) + \beta_a) - w_k(\bar{t}_k) \right) \bar{h}_{rk} = 0
\]

implies that either \( \bar{h}_{rk} = 0 \) and

\[
\sum_{a \in A} \chi_{ar_k}(s_a(\bar{v}_a) + \beta_a) - w_k(\bar{t}_k) \geq 0
\]

or \( \bar{h}_{rk} > 0 \) and

\[
\sum_{a \in A} \chi_{ar_k}(s_a(\bar{v}_a) + \beta_a) - w_k(\bar{t}_k) = 0.
\]

One can use the procedure described above in the reverse order to obtain \( W'_{ED-M}(\bar{v}, \bar{t}) \). \( \square \)
Now, assume that all of the commodity flows are positive on the whole transportation network, i.e., $\bar{x}^k > 0$, $\forall k \in \mathcal{K}$. Using Corollary 3.5, the constraints that define the tolled user equilibrium can be stated as:

**Corollary 3.7** If the commodity flows are positive on the whole network, i.e., $\bar{x}^k > 0$, $\forall k \in \mathcal{K}$, then we define $W_{ED-M-R}(\bar{v}, \bar{t})$ to be the polyhedron given by the $\beta$ part of the following linear inequality system in $\beta$ and $\rho$:

$$
\begin{align*}
    s_a(\bar{v}_a) + \beta_a &= \rho_j^k - \rho_i^k \quad \forall a = (i,j) \in \mathcal{A}, k \in \mathcal{K} \\
    w_k(\bar{t}_k) &= \rho_q^k - \rho_p^k \quad \forall k = (p,q) \in \mathcal{K}.
\end{align*}
$$

Now, we identify another description of the toll set when a “relevant path” assumption is made. This assumption requires the utilized paths on the transportation network not to change due to the changes in tolling the network. This leads to the assumption that users are utilizing the same set of the paths in the system optimization and user equilibrium and the difference between the user and system solutions is only in flows and demands. Under this assumption we define the tolled user equilibrium by the following corollary which is a direct result of Corollary 3.6.

**Corollary 3.8** Let $R = \bigcup_{k \in \mathcal{K}} R_k$ be the set of all relevant paths. Assume that the transportation network is defined by the relevant paths, i.e., $h_{r_k} > 0$, $\forall r_k \in R_k, k \in \mathcal{K}$ be the flow on path $r_k$. Then, $W_{ED-M-R}'$ can be defined as

$$
\sum_{a \in \mathcal{A}} \chi_{ar_k}(s_a(\bar{v}_a) + \beta_a) = w_k(\bar{t}_k) \quad \forall k \in \mathcal{K}, r_k \in R_k.
$$

### 3.5 Computational Experimentation

The goal of this section is to present methods by which a traffic planner can solve a moderate scale ED toll pricing problem. We demonstrate by the methods that we present that a moderate size toll pricing problems can be solved by the
mathematical programming solvers like CPLEX [20], OSL [56], RSDNET [47] and MINOS [85] in a reasonable amount of CPU time.

In the last decade, the performance of the mathematical programming solvers have improved by not only using the current literature but also using the state of the art computational techniques. Now, the size of the problem is limited only by the computer memory which allows the solvers keep up with the rapidly increasing capabilities of the today’s computers. These solvers can either be directly run, or called through programming languages like C, C++, and Fortran, or used as solvers by algebraic modeling languages like AMPL [34] and GAMS [37]. The algebraic modeling languages like GAMS and AMPL are powerful and flexible modeling languages that can be utilized to model and then solve (via the available solvers) the linear, nonlinear and integer programming problems. These modeling languages are available on most of the operating systems like UNIX, WINNT, DOS and Sun Solaris. Further, the input files for these mathematical modeling softwares are compatible in all of these operating system platforms. These languages help to handle large sized problems which might require several modifications to construct an accurate model and users can change the formulation quickly and easily that the time available for designing the model and analyzing the results increases considerably.

In this section, we present computational methods for elastic demand toll pricing problems. We give two relaxation procedures and then modify the toll pricing framework in order to handle the moderate sized toll pricing problems.

3.5.1 Implementing the TPF for Elastic Demand TA Problems

First, we generalize the penalty method proposed by Hearn and Ramana [49] for the MINSYS-FD and then we study an $\epsilon-$approximation of the toll set.
Assume that \((\bar{v}, \bar{t}, \bar{x}, \bar{\rho})\) is the system optimal solution. The optimality conditions ensure that complementary slackness conditions hold. That is,
\[
\sum_k \bar{x}^k \left( s(\bar{v}) + \beta_{MSCP} - A^T \bar{\rho}^k \right) - \sum_k \bar{t}^k \left( w_k(\bar{t}_k) - E_k^T \bar{\rho}^k \right) = 0,
\]
\[
\Rightarrow (s(\bar{v}) + \beta_{MSCP})^T \bar{v} - w(\bar{t})^T \bar{t} = 0.
\]

However, when the problem size increases, the complementary slackness conditions might not hold if the nonlinear programming solver does not give an exact optimal solution. This problem might arise when the problem data is ill-conditioned, or the size of the system problem is very large. Also, this will happen when the NLP solver is not run long enough to give the optimal solution. Mathematically,
\[
w(\bar{t})^T \bar{t} - (s(\bar{v}) + \beta_{MSCP})^T \bar{v} = \epsilon \neq 0.
\]

Thus, \(W_{ED}(\bar{v}, \bar{t})\) is not feasible. In this case, it is clear that the toll pricing framework for the elastic demand traffic assignment problems which is proposed in Section 3.2.6 can not be utilized to solve any toll pricing problem. Using the results presented in this section, we propose a new toll pricing framework that can be used when the system optimal solution is approximately optimal.

The first method generalizes the penalty method proposed by Hearn and Ramana [49] so that any toll pricing problem can be solved. The aggregate complementary slackness condition is penalized to obtain the following LP,
\[
\min \ u_1 \bar{Z} - u_2 \left( w(\bar{t})^T \bar{t} - (s(\bar{v}) + \beta)^T \bar{v} \right)
\]
\[
s.t. \quad s(\bar{v}) + \beta \geq A^T \bar{\rho}^k, \quad \forall k \in \mathcal{K}
\]
\[
w_k(\bar{t}_k) \leq E_k^T \bar{\rho}^k, \quad \forall k \in \mathcal{K}
\]
\[
(\beta, \rho, y, z) \in \hat{T}
\]
where \(u_1\) and \(u_2\) are positive scalars and \(y\) and \(z\) are the other variables which are used in toll pricing schemes (see Table 2.1). \(\bar{Z}\) is the objective function and \(\hat{T}\) is the set of additional constraints for the toll pricing problems.
However, this method requires guessing values for $u_1$ and $u_2$ in order to find a good solution which minimizes the total infeasibility in the complementarity constraint and at the same time gives a good approximation of the toll pricing objective.

The second method bounds the aggregate complementarity constraint to obtain an approximation $W_{ED}(\tilde{v}, \tilde{t}, \tilde{\epsilon})$ to the toll set. $W_{ED}(\tilde{v}, \tilde{t}, \tilde{\epsilon})$ is defined by the following set of the inequalities:

\[
\begin{align*}
    s(\tilde{v}) + \beta & \geq A^T \rho^k, \quad \forall k \in K \\
    w_k(\tilde{t}_k) & \leq E^T_k \rho^k, \quad \forall k \in K \\
    w(\tilde{t})^T \tilde{t} - (s(\tilde{v}) + \beta)^T \tilde{v} & \leq \tilde{\epsilon} \\
    -w(\tilde{t})^T \tilde{t} + (s(\tilde{v}) + \beta)^T \tilde{v} & \leq \tilde{\epsilon}.
\end{align*}
\]

Clearly, $W_{ED}(v, \bar{t}, 0)$ is $W_{ED}(v, \bar{t})$. When $\tilde{\epsilon}$ is known, using $W_{ED}(\tilde{v}, \tilde{t}, \tilde{\epsilon})$, the traffic planner avoid guessing particular values of the $u_1$ and $u_2$ for each toll pricing scheme. However, finding good values of $\tilde{\epsilon}$ is essential.

One can find an $\tilde{\epsilon}$ by solving the D-MINSYS-ED problem by penalizing the aggregate complementarity constraint. Then,

\[
\tilde{\epsilon} = (s(\tilde{v}) + \tilde{\beta}_{D-MINSYS})^T \tilde{v} - w(\tilde{t})^T \tilde{t}
\]

is the amount of violation in the aggregate complementarity constraint.

An easier way to determine a suitable $\epsilon$ is by using the MSCP toll vector which is readily available when the system problem is solved. In fact, $\tilde{\beta}_{MSCP}$ is an approximate valid toll vector. Specifically,

\[
\tilde{\epsilon} = w(\tilde{t})^T \tilde{t} - (s(\tilde{v}) + \tilde{\beta}_{MSCP})^T \tilde{v}
\]

\[
\text{can be used and it is not necessary to solve the D-MINSYS-ED linear program since MSCP toll vector is an alternative D-MINSYS-ED solution.}
\]
Then, the toll pricing framework for the elastic demand TA problems can be stated as:

1. Solve the system problem to obtain an approximately optimal solution

\[(\tilde{v}, \tilde{t}), \ (s(\tilde{v}), w(\tilde{t}))\] and \(\tilde{\beta}_{MSCP} = \nabla s(\tilde{v})\tilde{v} \).

2. Let \(\tilde{\epsilon} = w(\tilde{t})^T\tilde{t} - (s(\tilde{v}) + \tilde{\beta}_{MSCP})^T\tilde{v} \) be the amount of violation in the aggregate complementary constraint.

3. Define the toll set approximation \(W_{ED}(\tilde{v}, \tilde{t}, \tilde{\epsilon})\).

4. Define and optimize a toll pricing problem using this approximation.

The first step of the toll pricing framework involves solving a nonlinear programming problem. There exist several specialized solvers for the fixed demand traffic assignment problems. It is also well-known that an elastic demand problem can be transformed to a fixed demand problem by simply adding one link with a specific cost function. In the next section, we present this transformation.

3.5.2 Solving ED Problems with FD Software

In the elastic demand traffic assignment problems, the user and system problems have the user benefit term \(-\int_0^{t_h} w_k(z)dz\) in the objective function.
Assuming that one can identify the upper bound \( t_k^* \) on the demand between an OD-pair \( k \), one can write the user benefit as \([92]\):

\[
\int_0^{t_k} w_k(z)dz = \int_0^{t_k} w_k(z)dz - \int_{t_k}^{t_k^*} w_k(z)dz.
\]

Clearly, the first integral is constant. Thus, it can be dropped from the objective function. The second integral can be expressed in terms of excess demand with a simple change of variable, \( v_k = t_k^* - t_k \):

\[-\int_{t_k}^{t_k^*} w_k(z)dz = -\int_{t_k^* - t_k}^0 w_k(t_k^* - u)(-du) = -\int_0^{v_k} D_k(u)du.\]

Therefore, the elastic demand objective can be written equivalently as

\[
\min \sum_{a \in A} \int_0^{v_a} s(z)dz + \sum_{k \in K} \int_0^{v_k} D_k(u)du.
\]

Note that \( \int_0^{v_k} D_k(u)du \) is convex since \( D(v) \) is a nondecreasing function. Thus the transformed problem is a convex programming problem.

Now, we give a procedure to convert the elastic demand problem to the fixed demand problem:

- Classify the origin and destination nodes into transshipment nodes (those which are on paths used by other OD-pairs) and solely origin and destination nodes.
- For the transshipment-origin node \( p \), add a node \( p' \) and a link between \((p', p)\). Let \( p' \) be the new origin node replacing \( p \) and \( s_{(p', p)}(v_{(p', p)}) = 0 \).
- For the transshipment-destination node \( q \), add a node \( q' \) and a link between \((q, q')\). Let \( q' \) be the new destination node replacing \( q \) and \( s_{(q, q')}(v_{(q, q')}) = 0 \).
- For each OD-pair \( k = (p, q) \), add a link \((p, q)\) with a cost function of \( s_k(v_k) = D_k(v_k) \) for the user problem and \( s_k(v_k) + s'_k(v_k)v_k = D_k(v_k) \) for the system problem (see Figure 3.4). The feasible region of the new network is:
\[ v = \sum_{k \in K} x^k \]
\[ Ax^k = E_k t_k^* \quad \forall k \in K \]
\[ x^k \geq 0 \quad \forall k \in K. \]

- Solve the resulting convex programming problem. Assume that \( \hat{v} \) is the solution. Then \((v, t)\) of the original problem are \( \bar{v}_a = \hat{v}_a, \forall a \neq (p, q) \) and \( \bar{t} = t^* - \hat{v}_k \). Furthermore, \( w_k(\bar{t}_k) = D_k(\hat{v}_k) \).

The excess demand formulation is proposed by Gartner [38, 39]. It has been observed that when the same algorithm is utilized (e.g. a convex combination algorithm), solving the excess demand formulation is computationally more advantageous than solving the elastic demand problem directly (solving the elastic demand formulation directly usually takes 1.25-1.75 times the time needed to solve the fixed demand formulation using the same algorithm [87]). Although in the excess transformation, the size of the network increases, the computational advantages of using a fixed demand algorithm compensates this drawback. One of the main reasons for this lies in the definition of a commodity: in the excess (fixed) demand formulation, the commodities are defined by origins, however, the elastic demand formulation requires the commodities defined by origin-destination pairs. Thus, the resulting elastic demand nonlinear program is usually larger than the fixed demand formulation. Furthermore, in the excess demand formulation, for each origin, one can find the shortest path from the origin to all destinations by solving a single shortest path problem. However, one needs to solve a shortest path problem for each origin-destination pair in the elastic demand formulation.

In the next section, we give computational results on a moderate sized Sioux Falls network.
3.5.3 Numerical Example

In this section, we study the toll pricing framework on the New Sioux Falls Problem [1]. This network is originally proposed for the fixed demand traffic problems and has 24 nodes, 76 arcs and 552 origin-destination pairs. We define an elastic demand problem using the same network having the fixed demand between an OD-pair as the upper bound for the maximum demand. The demand functions are linear and decreasing of the form \( t_{(p,q)}(c_{(p,q)}) = t^*_{(p,q)} - 0.5c_{(p,q)} \). The travel time/cost functions has the following structure:

\[
s_a(v_a) = T_a + 10^{-7}C_a v_a^4.
\]

However, when we tested the ED toll pricing framework on this network, we observed that the resulting toll vectors do not vary a lot, since there is flow almost on all links of the network for all commodities. As a result we defined a new network with fewer number of OD-pairs having the same travel time functions. The elastic demand Sioux Falls problem has 35 nodes and 87 links. In Figure A.1, the nodes one through six are the origin nodes and seven through 11 are the destination nodes. Thus, there is a total of 30 OD-pairs. This results in user and system problems having 2727 variables and 1137 constraints. The data and solution for the results presented for the elastic demand Sioux Falls problem can be found in Appendix A, Tables A.1 through A.4.

The user problem is solved in 9.53 CPU seconds on a 200 MHz IBM SP2 node having 512GB of memory. The user problem has an objective of 11449.86 which results in a NUB of 6695.78. Although the number of OD-pairs is 30, there is traffic only between 15 of them.

---

3 The modified network has the same number of nodes but 117 links. Thus the modified problem has 702 variables and 327 constraints.
The system problem has a NUB of 9416.75 by having demand between 14 OD-pairs. This means, the users are utilizing the transportation network at 71.10% efficiency in the equilibrium. Thus one might consider tolling the transportation system in order to achieve a better utilization. A possible choice is using the MSCP toll vector. The system solution has flows on 46 arcs. As a result there are 46 MSCP toll booths tolling the system with a maximum toll of 35.865. The total first best toll revenue is 5222.45.

When we utilize the toll pricing framework, we observe that there are no feasible toll vectors including MSCP. This is because of \( \bar{\epsilon} = 0.003 \) violation in the aggregate complementarity constraint. As a result we can either use the \( \epsilon \)-approximation of the toll set or penalize the complementarity constraint by guessing \( u_1 \) and \( u_2 \).

When the \( \epsilon \)-approximation of the toll set is used in the toll pricing framework, the following results are obtained: An alternative nonnegative toll vector (D-MINSYS) to MSCP is obtained by charging the users on 58 links with a maximum toll amount of 26.55. The total CPU time for obtaining this solution is 0.29 CPU seconds.

The MINMAX tolling scheme has a maximum toll amount of 11.99 and number of toll booths for this scheme is 43. Thus, one can argue that the MINMAX tolls are more desirable compared to D-MINSYS and MSCP tolling schemes, since the maximum toll amount and number of toll booths needed is less than those needed by both of the tolling schemes. MINMAX is solved in 0.58 CPU seconds.

The best solution that we have obtained in 200000 CPU seconds for the MINTB tolling scheme using CPLEX 6.60 is having 28 toll booths on the network. However, this is not the optimal solution. By ad hoc means, we have determined a
solution with 26 toll booths. The LP relaxation bound for the MINTB problem is 1.25, which is a very weak lower bound. The maximum toll amount is 35.87.

We also compare the $\epsilon$-approximation and penalty methods for solving the Sioux Falls problem. Choosing $u_1 = 1$ and $u_2 = 100$ and solving the relaxed problem yields the following results. The D-MINSYS problem has a total toll revenue of 5222.46 by charging on 40 links. D-MINSYS problem consumes 1.29 CPU seconds. The total violation in the aggregate complementarity constraint is in the order of $10^{-12}$. The maximum MINMAX toll is 12.00 and number of toll booths is 46. The total CPU time required by MINMAX is 1.10 CPU seconds. As we can see, this method requires not only more CPU time for finding the optimal solution but also guessing good values of $u_1$ and $u_2$.

### 3.6 Summary

This chapter extends the first best toll pricing framework for the fixed demand traffic equilibrium model to the variable demand traffic assignment problems. We present the system and user problems for the GVD model and show that there exist toll vectors which can make the tolled user equilibrium problem have $(\bar{v}, \bar{t}) \in V$ as an optimal solution not necessarily restricted to the system optimal flows. We define the toll set for that $(\bar{v}, \bar{t})$ pair to be the set of all such perturbations. For the GVD model, all toll vectors in the toll set produce the same total toll revenue. It is important to emphasize that MSCP tolls are not only an element of the toll set but also optimal when the objective is minimization of the total toll revenue. However, the major disadvantage of using MSCP tolls is that all of the congested links with traffic are being tolled and this is likely to be impractical due to high fixed costs and maintenance costs. By employing the toll pricing framework, a traffic engineer can provide several pricing alternatives to the transportation planning authorities. By using toll pricing, the authorities achieve a better utilization of the roads by spreading the traffic over the system,
and diverting trips to transit. This results in reduced congestion, and thus shorter travel times for auto trips and decrease in the total vehicle miles travelled. We also present methods for solving toll pricing problems when the system solution is not exact.
CHAPTER 4
SECOND BEST PRICING FOR ED NETWORKS

4.1 Introduction

In the foregoing chapter, we assumed that all links of the transportation network can be tolled in order to achieve the most efficient utilization of the network. We defined all such tolls as the first best tolls. We proposed a toll pricing framework by which a traffic planner can obtain alternative toll vectors by imposing additional constraints and objectives on tolling the network and solving a linear or mixed integer program. The traffic planner might have the following constraints: it is usually the case that some portions of the transportation network cannot be tolled due to technological restraints, huge setup costs, political considerations, technical complexities, traffic spill over and the tradition of free access to local streets. Also a regulator might find collecting charges on all links of the network to be inappropriate and inefficient. We define a constrained first best toll pricing (CFB) scheme as one where the optimal usage of the system is still achieved in spite of all of the constraints on tolling some portions of the network.

Assume that the CFB scheme does not produce a feasible toll vector. In other words, the most efficient usage of the system cannot be achieved. In this situation, the transportation authority should consider another type of objective other than having the system optimal flows as the tolled user equilibrium flows. Under such circumstances, a second best toll pricing scheme can be utilized. The second best pricing scheme addresses the following question: Given certain structural facts that prevent prices from being equal to the marginal costs everywhere on the transportation network, what would be the optimal rule to follow in order to
achieve the most efficient utilization of the transportation network? The second best pricing scheme is one where the most efficient utilization of the transportation system can not be achieved because of the constraints on tolling the network.

In the literature, most of the proposed second best toll pricing formulations are path based models. These formulations are utilized for modelling small transportation networks such as those with one, two or three links. In these formulations, it is assumed that the utilized paths are known a priori and do not change whether the users are tolled or not [71, 78, 16, 81, 97, 98]. This assumption is known as the “relevant path” assumption. With this assumption, Marchand [78] shows how the MSCP toll pricing scheme should be modified if only a single route can be tolled on a two parallel routes network. Braid [16] compares several pricing alternatives using a bottleneck-queuing model for the peak load pricing two parallel routes problem. Verhoef et al. [98] analyze the effects of various demand and cost parameters on the relative efficiency of one route tolling using a two link network simulation model.

The two parallel routes problem has been extended in various ways. For example, Liu and McDonald [71] propose a two period, two parallel routes problem. Verhoef [97] proposes a general network second best pricing scheme in which an expression for finding the second best tolls is derived. In this expression, the toll on each arc is a function of MSCP tolls, flows and demands on all other links of the transportation network.

As a consequence of the relevant path assumption, the formulations in the existing literature do not represent the case where there are some unutilized paths. One of the goals of this chapter is to provide appropriate formulations which do not require the relevant path assumption. We propose second best pricing models which are applications of mathematical programs with equilibrium constraints (MPEC). It is important to note that MPECs are difficult to solve even for very
small networks [76]. However, we assume that we can identify stationary points of these formulations by commercial nonlinear solvers like MINOS [85]. Using these stationary points, we identify a second best toll set and then, by defining some additional objectives and constraints, we obtain alternative toll vectors by solving linear or mixed integer programs. However, it is well known that the stationary points of the MPECs are not necessarily local or global optima [76]. As a result, it is important to provide some assessment of the quality of the solution. Thus, another goal is to define some efficiency measures for a given second best stationary point. Using these measures, the traffic planner have a tool to compare several second best pricing alternatives and can decide on the amount of toll revenue users have to pay.

The organization of this chapter is as follows: First, we present the constrained first best pricing scheme. Using the toll set descriptions presented in Section 3.4.3, we present three equivalent SBTP models. Next, we discuss the existence of the solutions to the proposed models. Then, we present the method utilized in the existing second best literature to derive a formula for the second best tolls without making the relevant path assumption. The generalized second best toll pricing framework follows the theory of toll pricing. We illustrate the toll pricing framework by some numerical examples and an extensive computational experimentation. Finally, we introduce another second best toll pricing problem by which we observe the effect of toll revenue on efficiency of the network.

Note that the models in this chapter are second best pricing models with elastic demand. However, using the ideas presented in this chapter, these models can easily be extended to other GVD traffic assignment models.

4.2 Constrained First Best Pricing Model

In the constrained first best pricing scheme, it is assumed that certain links of the network can not be tolled, but the most efficient utilization of the system
can still be achieved even if there are such constraints. In other words, the tolled user equilibrium flows and demands are the same as the system optimal flows and demands despite the fact that there constraints on tolling a certain set of links, \( Y \), on the transportation network. Let \((v^*, t^*)\) denote the system optimal flows and demands. Under the same assumptions as in Section 3.2.7, we define the constrained first best toll set, \( W_{ED}(v^*, t^*, Y) \), as

\[
\begin{align*}
    s(v^*) + \beta & \geq A^T \rho^k \quad \forall k \in K \\
    w_k(t_k^*) & \leq E_k \rho^k \quad \forall k \in K \\
    (s(v^*) + \beta)^T v^* & = w(t^*)^T t^* \\
    \beta_a & = 0 \quad \forall a \in Y
\end{align*}
\]

where \((v^*, t^*) \in V\).

If \( W_{ED}(v^*, t^*, Y) \neq \emptyset \), then \( \beta \in W_{ED}(v^*, t^*, Y) \) is a constrained first best toll and the most efficient utilization of the system is achieved.

### 4.3 Second Best Pricing Models

In this section, we present three different second best toll pricing models using different descriptions of tolled user equilibrium. All of the second best pricing formulations that we propose have the following structure:

\[
\begin{align*}
    \text{max} & \quad \text{The Net User Benefit} \\
    \text{s.t.} & \quad \text{The flows and demands are feasible,} \\
    & \quad \text{The flows and demands are in constrained tolled user equilibrium,} \\
    & \quad \text{Certain links cannot be tolled.}
\end{align*}
\]

The first of the proposed models is the most general one and is an application of mathematical programs with equilibrium constraints (MPEC). The second is an equivalent link-based model (a nonlinear programming model with nonlinear constraints). The third one is a path based model. These models differ only in the
description of the tolled user equilibrium conditions. Further, we show that these models are equivalent.

In this section, after presenting the SBTP models, we present results on equivalency of the proposed models. Finally, we comment on existence of the solutions, and local and global optimality of these models.

4.3.1 MPEC Formulation

In the toll pricing literature, MPECs have been utilized for the maximum toll revenue problem [61, 111]. For example, Labbe et al. [61] assume constant travel time and fixed demand (i.e, constant user benefit) and show that the problem reduces to a mixed integer program. Yang and Lam [111] use the bilevel programming approach to solve toll pricing problems. They also have maximization of the net user benefit as one of their objectives. We propose MPECs for SBTP problems where the objective changes to maximizing the net user benefit instead of maximizing the toll revenue.

In the MPEC formulation, the constraints that define the user equilibrium are represented by a variational inequality in the mathematical program. Let SBTP-VI denote the SBTP model presented below.

\[
\begin{align*}
\max & \quad \sum_{k \in K} \int_{0}^{t_k} w_k(z)dz - s(v)^T v \\
\text{s. t.} & \quad (v, t) \in V \\
& \quad (s(v) + \beta)^T(u - v) - w(t)^T(d - t) \geq 0 \quad \forall(u, d) \in V \\
& \quad \beta_a = 0 \quad \forall a \in Y.
\end{align*}
\]

As we have mentioned before, the MPECs (thus SBTP-VI) are very difficult mathematical programs to solve (Luo, et al. [76]). On the one hand, SBTP-VI is notationally compact: the variational inequality description of the tolled user equilibrium appears as a constraint. On the other hand, one needs a specialized
algorithm for obtaining a solution for this MPEC model. However there is little evidence that the existing algorithms [76] perform better than the standard nonlinear programming methods [69]. This motivates us to propose equivalent nonlinear programming formulations which can be solved by commercial nonlinear solvers like MINOS [85] and SNOPT [40].

4.3.2 Link Based Formulation

The link based formulation for second best pricing models uses a description of the tolled user equilibrium conditions similar to the one presented in Section 3.2.7. The following conditions in \((v, t, \rho, \beta)\) describe the constrained tolled user equilibrium where \((v, t)\) are the tolled user equilibrium flows and demands:

\[
\begin{align*}
    s(v) + \beta & \geq A^T \rho^k \quad \forall k \in \mathcal{K} \\
    w_k(t_k) & \leq E_k \rho^k \quad \forall k \in \mathcal{K} \\
    (s(v) + \beta)^Tv & = w(t)^Tt \\
    \beta_a & = 0 \quad \forall a \in Y.
\end{align*}
\]

This description can equivalently be stated as

\[
\begin{align*}
    s_a(v_a) + \beta_a & \geq \rho^k_j - \rho^k_i \quad a \in \mathcal{A} \setminus Y, \forall k \in \mathcal{K} \\
    s_a(v_a) & \geq \rho^k_j - \rho^k_i \quad a \in Y, \forall k \in \mathcal{K} \\
    w_k(t_k) & \leq E_k \rho^k \quad \forall k \in \mathcal{K} \\
    w(t)^Tt & = \sum_{a \in \mathcal{A} \setminus Y} (s_a(v_a) + \beta_a)v_a + \sum_{a \in Y} s_a(v_a)v_a.
\end{align*}
\]

The link based formulation (SBTP-LINK) uses the descriptions above in the constraint set to ensure that the optimal second best flows and demands are in the constrained tolled user equilibrium. SBTP-LINK is a nonlinear program with a convex objective and a non-convex constraint set. Note that the aggregate complementarity condition, \((s(v) + \beta)^Tv = w(t)^Tt\) makes the feasible region non-convex, even if very restrictive assumptions like constant time and constant
user benefit assumptions are made, because of the term $\beta^Tv$ [61]. The SBTP-LINK formulation is given as

$$
\begin{align*}
\max & \sum_{k \in K} \int_0^{t_k} w_k(z)dz - s(v)^Tv \\
\text{s. t.} & \\
& v = \sum_{k \in K} x^k \\
& Ax^k = t_k E_k \quad \forall k \in K \\
& s_a(v_a) + \beta_a \geq \rho_j^k - \rho_i^k \quad a \in A \setminus Y, \forall k \in K \\
& s_a(v_a) \geq \rho_j^k - \rho_i^k \quad a \in Y, \forall k \in K \\
& w_k(t_k) \leq E_k^T \rho^k \quad \forall k \in K \\
& \sum_{a \in A \setminus Y} (s_a(v_a) + \beta_a)v_a + \sum_{a \in Y} s_a(v_a)v_a = w(t)^T t \\
& x^k \geq 0 \quad \forall k \in K \\
& t_k \geq 0 \quad \forall k \in K.
\end{align*}
$$

This formulation has $1 + |A| + |K| + |K|(|N| + |A|)$ constraints plus nonnegativity constraints and $|A \setminus Y| + |A| + |K| + |K|(|N| + |A|)$ variables.

4.3.3 Path Based Formulation

In this model, the tolled user equilibrium constraints are described by the inequalities given in Lemma 3.11 intersected with $\beta_a = 0$, $\forall a \in Y$

$$
\begin{align*}
\sum_{a \in A} \chi_{ar_k}(s_a(v_a) + \beta_a) & \geq w_k(t_k) \quad \forall k \in K, \forall r_k \in R_k \\
(s(v) + \beta)^Tv & = w(t)^T t \\
\beta_a & = 0 \quad \forall a \in Y,
\end{align*}
$$

or equivalently

$$
\begin{align*}
\sum_{a \in A \setminus Y} \chi_{ar_k}(s_a(v_a) + \beta_a) + \sum_{a \in Y} \chi_{ar_k}s_a(v_a) & \geq w_k(t_k) \quad \forall k \in K, \forall r_k \in R_k \\
\sum_{a \in A \setminus Y} (s_a(v_a) + \beta_a)v_a + \sum_{a \in Y} s_a(v_a)v_a & = w(t)^T t.
\end{align*}
$$
The path based formulation (SBTP-PATH) can be stated as follows:

$$\max \sum_{k \in K} \int_{t_k}^{t_k^*} w_k(z) dz - s(v)^T v$$

s. t.

$$v = \sum_{k \in K} x^k$$

$$Ax^k = E_k t_k, \quad \forall k \in K$$

$$x^k_a = \sum_{r_k \in R_k} \chi_{ar_k} h_{r_k} \quad \forall a \in A, \forall k \in K$$

$$\sum_{a \in A \setminus Y} \chi_{ar_k}(s_a(v_a) + \beta_a) + \sum_{a \in Y} \chi_{ar_k} s_a(v_a) \geq w_k(t_k) \quad \forall k \in K, \forall r_k \in R_k$$

$$\sum_{a \in A \setminus Y} (s_a(v_a) + \beta_a)v_a + \sum_{a \in Y} s_a(v_a)v_a = w(t)^T t$$

$$h_{r_k} \geq 0 \quad \forall r_k \in R_k, \forall k \in K$$

$$t_k \geq 0 \quad \forall k \in K.$$

This formulation has $1 + |A| + |K|(|N| + |A|) + \sum_{k \in K} |R_k|$ constraints and $|A \setminus Y| + |A| + |K| + |A||K| + \sum_{k \in K} |R_k|$ variables. Compared to SBTP-LINK, SBTP-PATH has $\sum_{k \in K} |R_k|$ more constraints and $\sum_{k \in K} |R_k| - |K||N|$ more variables.

In the SBTP-PATH formulation, we need to generate all of the paths in order to have a precise description of a second best pricing scheme. This is a disadvantage because, as the network size increases, the number of available paths increases exponentially.

To summarize, we have presented two equivalent second best toll pricing formulations which have different constraints that define the constrained toll ed user equilibrium. The equivalency between these conditions are presented in Theorem 3.4. The equivalence of the SBTP formulations is also a direct result of Theorem 1.3.5 of Luo et al. [76] because SBTP-LINK and SBTP-PATH are KKT equivalent formulations of SBTP-VI, the MPEC model.
4.3.4 Local and Global Optimality of SBTP Models

The second best models presented in this chapter generalize the formulations given in the literature. These models do not assume relevant paths and also do not need *a priori* knowledge of the utilized paths. Now, we address whether these models have feasible solutions or not and then comment on the local and global optimality of the stationary points. We give bounds on where the global optimal solution of the SBTP model lies, which might help in devising algorithms to solve these problems.

The following theorem shows that the SBTP-VI always has a solution given that the user and system problem have solutions. In other words, the feasible region, which is defined by constraints in \((v, t, x, \rho, \beta)\), is not empty when the user and system problems have solutions.

**Theorem 4.1** Assume that the user and system problems have solutions \((v^U, t^U)\) and \((v^*, t^*)\). Further, assume that \(s(v)\) and \(-w(t)\) are monotonic and continuous and

\[
(\nabla s(v), \nabla w(t))
\]

exists and is continuous. Then, the SBTP-VI has a global optimal solution objective value in the interval \([\text{NUB}(v^U, t^U), \text{NUB}(v^*, t^*)]\).

**Proof.** The objective function is bounded from above because the SBTP-VI is the system problem having the constrained user equilibrium problem as a constraint. Thus,

\[
\text{NUB}_{SBTP-VI}(\bar{v}, \bar{t}) \leq \text{NUB}(v^*, t^*)
\]

where \((\bar{v}, \bar{t})\) is the global optimal solution for the SBTP-VI problem. In addition, \((v^U, t^U, \beta = 0)\) is a feasible solution to the MPEC. This implies

\[
\text{NUB}(v^U, t^U) \leq \text{NUB}_{SBTP-VI}(\bar{v}, \bar{t}).
\]
Thus, the objective function value of SBTP-VI is bounded. Given the assumptions stated above and using Theorem 1.4.1 of Luo et al. [76], we can conclude that the SBTP-VI problem has a global optimal solution. □

The following result gives conditions on the cost and inverse demand functions where stationary points of the SBTP problem are local minima.

**Theorem 4.2** Suppose that the conditions in Theorem 4.1 hold. Further, assume that \( s_a(v_a) \) and \( w_k(t_k) \) are linear functions. Then any stationary point to the SBTP-VI problem is a local minimum.

**Proof:** This is a direct result of Theorem 4.1.4 and Theorem 5.2.4 of Luo et al. [76]. □

We use this result to design some numerical examples and illustrate the toll pricing framework.

### 4.4 Deriving the Second Best Tolls Analytically

The goal of this section is to derive a second best toll vector analytically for a general network. The models in the existing literature are of the form

\[
\begin{align*}
\max & \quad \text{Net User Benefit} \\
\text{s.t.} & \quad \text{The flows and demands are feasible,} \\
& \quad \text{Conditions on constrained user equilibrium,} \\
& \quad \text{All of the paths are relevant,} \\
& \quad \text{Certain links cannot be tolled.}
\end{align*}
\]

Verhoef [97, 96] determines second best toll vector from the first order conditions of this problem. We extend this approach to the SBTP models without making the relevant path assumption and show that the toll formula derived for the general model reduces to the one proposed by Verhoef.
Below is the path based model SBTP-PATH in the standard form with corresponding multipliers for each constraint (the $z_{rk}$ are the surplus variables for the total path cost constraints):

$$\max \sum_{k \in K} \int_{0}^{t_k} w_k(z) dz - s(v)^T v$$

s. t.

$$v = \sum_{k \in K} x^k : \mu$$

$$Ax^k = E_k t_k, \quad \forall k \in K : \rho^k$$

$$x^k_a = \sum_{r_k \in R_k} \chi_{ar_k} h_{rk} \quad \forall a \in A, \forall k \in K : \varsigma^k_a$$

$$\sum_{a \in A \setminus Y} \chi_{ar_k} (s_a(v_a) + \beta_a) + \sum_{a \in Y} \chi_{ar_k} s_a(v_a) - z_{rk} = w_k(t_k) \quad \forall k \in K, \forall r_k \in R_k : \lambda_{rk}$$

$$\sum_{a \in A \setminus Y} (s_a(v_a) + \beta_a)v_a + \sum_{a \in Y} s_a(v_a)v_a = w(t)^T t : \phi$$

$$h_{rk} \geq 0 \quad \forall r_k \in R_k, \forall k \in K : u_{rk}$$

$$t_k \geq 0 \quad \forall k \in K : \zeta_k$$

$$z_{rk} \geq 0 \quad \forall r_k \in R_k, \forall k \in K : \nu_{rk}$$

The following necessary first order conditions can now be derived:

$$s_a(v_a) + s'_a(v_a)v_a - \sum_{k \in K} \sum_{r_k \in R_k} \lambda_{rk} \chi_{ar_k} s'_a(v_a) + \phi(s_a(v_a) + s'_a(v_a)v_a) - \mu_a = 0, \quad \forall a \in A$$

$$\mu_a - \rho^k_j + \rho^k_i - \varsigma^k_a = 0, \quad \forall a \in A, \forall k \in K$$

$$-w_k(t_k) + \rho^k_q - \rho^k_p + \sum_{r_k \in R_k} \lambda_{rk} w'_k(t_k) + \phi(w_k(t_k) + w'_k(t_k)t_k) - \zeta_k = 0, \quad \forall k \in K$$

$$\sum_{a \in A \setminus Y} \chi_{ar_k} \lambda_{rk} + \phi v_a = 0, \quad \forall a \in A \setminus Y$$

$$\sum_{a \in A} \chi_{ar_k} \zeta^k_a - u_{rk} = 0, \quad r_k \in R_k, \forall k \in K$$

$$\lambda_{rk} - u_{rk} = 0, \quad \forall k \in K, r_k \in R_k$$

$$\zeta_k t_k = 0, \quad \zeta_k \geq 0, \quad u_{rk} z_{rk} = 0, \quad u_{rk} \geq 0, \quad u_{rk} h_{rk} = 0, \quad u_{rk} \geq 0.$$

In these equations, $a = (i, j)$ denotes an arc between nodes $i$ and $j$ and $k = (p, q)$ denotes an OD-pair.
Let \( MSs_a(v_a) = s_a(v_a) + s'_a(v_a)v_a \), \( MSw_k(t_k) = w_k(t_k) + w'_k(t_k)t_k \), \( MSW_k = \sum_{r_{k'} \in R_k} \lambda_{r_{k'}}w'_k(t_k) \) and \( MSP_a = \sum_{k' \in K} \sum_{r_{k'} \in R_{k'}} \lambda_{r_{k'}}s'_a(v_a) \). Then, the following holds:

\[
MSs_a(v_a) - MSP_a + \phi MSs_a(v_a) - \varsigma_a^k = \rho^k_i - \rho^k_j, \quad \forall a = (i, j)
\]

Sum this equation over all arcs on path \( r_k \in R_k \). Then,

\[
\sum_{a \in A} \chi_{ar_k} \left( MSs_a(v_a) - MSP_a + \phi MSs_a(v_a) - \varsigma_a^k \right) = w_k(t_k) - MSW_k - \phi MSw_k(t_k) + \zeta_k, \quad \forall k \in K
\]

is obtained. But

\[
\rho^k_q - \rho^k_p = w_k(t_k) - MSW_k - \phi MSw_k(t_k) + \zeta_k, \quad \forall k \in K.
\]

Thus \( \forall r_k \in R_k \),

\[
\sum_{a \in A} \chi_{ar_k} \left( MSs_a(v_a) - MSP_a + \phi MSs_a(v_a) - \varsigma_a^k \right) - w_k(t_k) + MSW_k + \phi MSw_k(t_k) - \zeta_k = 0
\]

holds. Equivalently,

\[
\sum_{a \in A} \chi_{ar_k} \left( MSs_a(v_a) - MSP_a + \phi MSs_a(v_a) - \varsigma_a^k \right) - \sum_{a \in A \setminus Y} \chi_{ar_k} \beta_a + \zeta_{r_k} = 0
\]

which implies

\[
\sum_{a \in A} \chi_{ar_k} \left( s'_a(v_a) - MSP_a + \phi MSs_a(v_a) - \varsigma_a^k \right) - \sum_{a \in A \setminus Y} \chi_{ar_k} \beta_a + \zeta_{r_k} + MSW_k + \phi MSw_k(t_k) - \zeta_k = 0.
\]

Then, solving for \( \lambda_{r'_{k}} \) yields the following result:

\[
\lambda_{r'_{k}} = \frac{S_{r'_{k}}(v) - \sum_{a \in A \setminus Y} \chi_{ar'_{k}} \beta_a + \zeta_{r'_{k}} + W_{r'_{k}}(t_k)}{\sum_{a \in A} \chi_{ar'_{k}} s'_a(v_a) - w'_k(t_k)}
\]
where for $r'_k \in R_k$,

$$S_{r'_k}(v) = \sum_{a \in A} \chi_{ar'_k} \left( s'(v_a)v_a - \sum_{k' \in K, o_{a'} \in R_{k'}, o_{a'} \neq r'_k} \sum_{k'' \in K} \lambda_{k''} \lambda_{a''} s''(v_a) + \phi M S a(v_a) - \zeta_k \right)$$

and

$$W_{r'_k}(t_k) = \sum_{o_k \in R_k, o_k \neq r'_k} \lambda_{o_k} w^1_k(t_k) + \phi M S w_k(t_k) - \zeta_k.$$ 

Using $\sum \sum \chi_{ar_k} \lambda_r + \phi v_a = 0$, $\forall a \in A \setminus Y$, the second best tolls can be obtained by using the following formula:

$$\beta_{a'} = \frac{\phi v_a' + \sum \sum \chi_{ar_k} - \sum_{a \in A \setminus Y, a \neq a'} \sum_{a \in A} \chi_{ar_k} s'(v_a) - w'_k(t_k)}{\sum \sum \sum \sum \chi_{ar_k} s'(v_a) - w'_k(t_k)}, \quad \forall a' \in A \setminus Y.$$

Although the expressions for $\lambda$ and $\beta$ are linear in $\lambda$ and $\beta$, we still have to satisfy the complementarity conditions,

$$\zeta_k t_k = 0, \quad \zeta_k \geq 0, \quad \nu_r z_r = 0, \quad \nu_r \geq 0, \quad u_r h_r = 0, \quad u_r \geq 0$$

in order to find the correct second best tolls.

Now, consider a special case where the cost and inverse demand functions are linear, i.e., assume that $s_a(v_a) = s_a v_a + c_a$ and $w_k(t_k) = -w_k t_k + d_k$. In this case, the formula for $\beta$ can be further simplified:

$$\beta_{a'} = \frac{\phi v_a' + \sum \sum \chi_{ar_k} \sum_{a \in A \setminus Y, a \neq a'} \sum_{a \in A} \chi_{ar_k} s_a + w_k}{\sum \sum \sum \sum \chi_{ar_k} s_a + w_k}, \quad \forall a' \in A \setminus Y,$$

where

$$S_{r^1}(v) = \sum_{a \in A} \chi_{ar^1} \left( s_a v_a (1 + 2 \phi) + \phi c_a - \sum_{k' \in K, o_{a'} \in R_{k'}, o_{a'} \neq r'_k} \sum_{k'' \in K} \lambda_{k''} \lambda_{a''} s''(v_a) - \zeta_k \right).$$
and

\[ W_{r_k}(t_k) = -\sum_{o_k \in R_k, o_k \neq r_k'} \lambda_{o_k} w_k + \phi(d_k - 2w_k t_k) - \zeta_k. \]

Now, consider a special case of SBTP-PATH where the relevant path assumption is made as in [97, 96]. The relevant path model (SBTP-PATH-VERHOEF) for elastic demand networks can be written as

\[
\begin{align*}
\max \quad & \sum_{k \in K} \int_{0}^{t_k} w_k(z)dz - s(v)^T v \\
\text{s. t.} \quad & v = \sum_{k \in K} x^k \\
& Ax^k = E_k t_k, \quad \forall k \in K \\
& x_a^k = \sum_{r_k \in R_k} \chi_{ar} h_{rk} \quad \forall k \in K, \forall a \in A \\
& \sum_{a \in A \setminus Y} \chi_{ar}(s_a(v_a) + \beta_a) + \sum_{a \in Y} \chi_{ar} s_a(v_a) = w_k(t_k) \quad \forall k \in K, r_k \in R_k \\
& h_{rk} > 0 \quad \forall k \in K, r_k \in R_k \\
& t_k > 0 \quad \forall k \in K.
\end{align*}
\]

Using the first order conditions, the second best toll on each arc as derived in [97], is

\[
\beta_{a'} = \frac{\sum_{k \in K, r_k \in R_k} \sum_{a \in A} \chi_{ar} s_a(v_a) - W_{r_k}(t_k)}{\sum_{k \in K} \sum_{a \in A} \chi_{ar} s_a(v_a) - W_{r_k}(t_k)}, \quad \forall a' \in A \setminus Y
\]

where

\[
S_{r_k}(v) = \sum_{a \in A} \chi_{ar_k} \left( s'_a(v_a) v_a - \sum_{k' \in K, o_{a'} \in R_k, o_{a'} \neq r_{k'}} \sum_{k' \in K} \lambda_{o_{a'}} \chi_{ao_{a'}} s'_a(v_a) \right)
\]

and

\[
W'_{r_k}(t_k) = \sum_{o_k \in R_k, o_k \neq r_k'} \lambda_{o_k} w'_k(t_k), \quad \forall r_k' \in R_k.
\]
Now, we show that the formula that we derived for SBTP-PATH can be reduced to SBTP-PATH-VERHOEF:

- The relevant path assumption implies $t_k > 0$, $\forall k \in K$ and $h_{rk} > 0$, $\forall r_k \in R_k$, $\forall k \in K$, thus $\zeta_k$ and $u_{rk}$ are zero. As a result, $\sum_{a \in A} \chi_{ar_k}s_a^k = 0$.

- The relevant path assumption makes

$$w_k(t_k) - \sum_{a \in A \setminus Y} \chi_{ar_k}(s_a(v_a) + \beta_a) - \sum_{a \in Y} \chi_{ar_k}s_a(v_a) = 0$$

hold on all paths, thus $\nu_{rk} \geq 0$ can be relaxed since $z_{rk} = 0$.

- Since all of the utilized paths are known, the

$$\sum_{a \in A \setminus Y} (s_a(v_a) + \beta_a)v_a + \sum_{a \in Y} s_a(v_a)v_a = w(t)^T t$$

constraint becomes redundant (see Corollary 3.8) and can be dropped from this formulation (thus $\phi = 0$).

Using all of these facts, the definitions of $S_{r_k}'(v)$ and $W_{r_k}'(t_k)$ for SBTP-PATH simplify to the ones for Verhoef’s model. Furthermore, since $\phi = 0$ and $z = 0$, the $\beta$ formulas of both models are the same.

To summarize, we have presented two MPEC models and a system of nonlinear equations to obtain the second best flows, demands and tolls. However, our intention is not to give solution algorithms for these equations. The goal of this chapter is to provide an extension to the first best toll pricing framework that Hearn and Ramana [49] proposed and the one that we have extended for general networks with variable demand in Chapter 3.

Before doing so we comment on the algorithm proposed by Verhoef [97] to solve the system of equations above. Verhoef’s Algorithm can be summarized as

1. Start with an initial toll vector $\beta_a = 0$

2. Solve the tolled user problem to obtain the equilibrium flows and demands.
3. Solve the first order conditions to calculate \((\mu, \rho, \zeta, \lambda)\) and \(\beta\) for SBTP-PATH-VERHOEF given the flows and demands.

4. Check if there is convergence. If not, go to step 2.

Using the algorithm above, SBTP toll vectors can be found numerically. However, it is not clear how effective the method might be. We have tested Verhoef’s Algorithm on a two parallel routes network where only one route can be tolled. The data and the results are presented in the Appendix B Tables B.1 and B.2. Verhoef’s algorithm terminates in 38 iteration when the cost function is nonlinear and 54 iterations when it is linear. However, we could not verify his results for larger networks. When we tested this algorithm on the 10 node example given in [96], we observed that the algorithm converges to other stationary points not necessarily having a better solution than the user solution (i.e., no tolling strategy yields a higher user benefit).

Although, Verhoef’s method can be extended to solve SBTP-PATH, we cannot guarantee the existence of the multipliers in step three of the algorithm for SBTP-PATH and SBTP-PATH-VERHOEF at each iteration. Furthermore, this method is rather complex and requires solving a system of nonlinear equalities (for SBTP-PATH there are also the complementarity slackness constraints). If one would like to generate alternative toll vectors, different starting points and alternative multipliers can be used. However, this method could be as difficult as solving the SBPT problem itself.

In the next section, we extend the first best toll set idea to second best pricing given that a second best solution is known. Then we present a toll pricing framework to find alternative toll vectors.
4.5 General Toll Pricing Framework for Elastic Demand Networks

After the transportation planning authority decides about the restrictions on allowing toll booths on certain parts of the network, the following framework can be utilized to find alternative toll vectors for different pricing schemes:

1. Define the links of the network that are not to be tolled, i.e., define the set \( Y \).
2. Solve SOPT-ED to obtain the system optimal solution \((v^*, t^*)\).
3. Check if \( W_{ED}(v^*, t^*, Y) \neq \emptyset \), i.e., if the polyhedron defining the first best toll set is feasible or not. If feasible, go to Step 4.
4. Solve the second best toll pricing formulation to obtain a stationary point \((\bar{v}, \bar{t}, \bar{\beta})\) (GAMS/MINOS [37, 85] can be used to solve the KKT second best pricing formulations). Define the toll set as \( W(\bar{v}, \bar{t}, Y) \).
5. Define and optimize an objective function over the toll set, possibly intersected with other constraints (See Table 2.1 for examples).

As we noted before solving the second best toll pricing problem is very difficult. Thus, given the restrictions on tolling the network, Step 2 first determines if a constrained first best pricing scheme is feasible or not. If not, then Step 3 solves the SBTP toll pricing problem and define the second best toll set to obtain alternative toll vectors. In Theorem 4.1 we prove that the NUB of the SBTP problem is in the interval \([\text{NUB}(v^U, t^U), \text{NUB}(v^*, t^*)]\). In the next section, we use this fact to define how good a second best solution is.

As shown in Table 2.1, let \( \hat{T} \) be the set of additional (side) constraints for alternative toll pricing problems and \( \hat{Z} \) be the objective function for each problem. The feasible region for each constrained first best or the second best toll pricing problem is the intersection of \( \hat{T} \) with \( W_{ED}(v^*, t^*, Y) \) or \( W(\bar{v}, \bar{t}, Y) \), respectively, where \( Y = \emptyset \) for the first best pricing scheme. The objective is linear in each case, thus each toll pricing problem is either a linear program or a linear integer program.
4.6 Examples

In this section, first, we verify the solutions to the three second best pricing examples from the literature by the models that we have presented. Then, a Nine Node Network example illustrate the toll pricing framework to obtain alternative first best, constrained first best and second best toll vectors.

Before presenting these examples, we define several indexes to measure the efficiency of the pricing scheme. The first index measure the distance of the second best solution from the first best solution, i.e., the system optimal one.

\[ \Upsilon = \text{Relative distance from the User Optimal Solution} = \frac{NUB(\bar{v}, \bar{t}) - NUB(v^u, t^u)}{NUB(v^*, t^*) - NUB(v^u, t^u)}, \]

where \((v^u, t^u)\), \((v^*, t^*)\), and \((\bar{v}, \bar{t})\) denote user, system and second best flows and demands respectively. \(1 - \Upsilon\) is the relative distance from first best optimal solution.

The second index \(\Omega\) compares the second best toll revenue with the first best one:

\[ \Omega = \text{Tolling Index} = \frac{\bar{\beta}^T \bar{v}}{\beta^* T v^*}. \]

The last index \(\Psi\) is the index of relative system efficiency when a second best toll pricing scheme is applied:

\[ \text{Efficiency} = \text{Relative Distance from the User Optimal Solution}(1 - \text{Tolling Index}), \]

or mathematically,

\[ \Psi = \Upsilon(1 - \Omega). \]

Clearly, \(0 \leq \Upsilon < 1\) and \(0 \leq \Omega < 1\). As a result, \(0 \leq \Psi < 1\).

4.6.1 Two Parallel Routes Example

The two parallel routes example [81, 97] is a network with two nodes and two links having flow on each of the arcs. The total demand between the origin-destination pair is \(t\) with an inverse demand function of \(w(t)\). On path one there
is a flow of $v_1$ and there is a flow of $v_2$ on path two. The travel time functions are $s_1(v_1)$ and $s_2(v_2)$, respectively. Let the toll on path one be $\beta$ and assume that path two can not be tolled. Since there is flow on all paths, we use the tolled user equilibrium conditions in Corollary 3.8. This results in the following nonlinear program:

$$
\begin{align*}
\max & \int_0^t w(z)dz - s_1(v_1)v_1 - s_2(v_2)v_2 \\
\text{s. t.} & \\
& t = v_1 + v_2 \\
& w(t) = s_1(v_1) + \beta \\
& w(t) = s_2(v_2) \\
& v_1 > 0 \\
& v_2 > 0
\end{align*}
$$

We use the formula derived in Section 4.4 to solve for the toll on path one. First of all, observe that the SBTP formulation does not have the aggregate complementarity constraint since we assumed that all paths are carrying flows, thus $\phi = 0$. $\zeta = \nu = u = 0$ since $(h, t, z) > 0$. This implies $\sum\chi_{ar_k}x_{ar_k} = 0$. Using

$$
\sum_{k \in K} \sum_{r_k \in R_k} \chi_{ar_k} \lambda_{r_k} + \phi v_a = 0, \lambda_1 = 0.
$$

As a result, $\lambda_2 = \frac{s_2'(v_2)v_2}{s_2'(v_2) - w'(t)}$. Then applying the formula for $\beta$, we obtain the optimal second best toll as:

$$
\beta_1 = s_1'(v_1)v_1 + \frac{w'(t)}{s_2'(v_2) - w'(t)}s_2'(v_2)v_2.
$$
Figure 4.2: Parking Pricing Network Example: There is a Charge for Using the Private Parking Lot.

This result is consistent with McDonald’s [81] and Verhoef’s [97] results. The second best optimum toll is composed of the MSCP toll on the tollway plus an adjustment term which is a function of the MSCP toll on the untolled link. Clearly, \( \beta \) does not need to be a positive toll. It can be zero or negative based on the sign of \( s'_2(v_2) - w'(t) \). However, note that this expression is just rephrasing of the optimality conditions of the given model in order to yield an expression which is a function of the MSCP tolls. Note that \( \beta_1 = w(t) - s_1(v_1) = s_2(v_2) - s_1(v_1) \) are also valid tolls.

4.6.2 Parking Pricing Example

Assume that users are willing to park in a major shopping area where there are two parking areas. The first one is a private parking area and the second one is a free parking lot which is located a little bit farther away. The demand for the private parking lot is \( v_1 > 0 \) and for the open parking lot is \( v_2 > 0 \). The user benefits for these parking lots are \( w_1(v_1) \) and \( w_2(v_2) \) respectively. It is assumed that the travel time in the parking lots is negligible and congestion occurs only on the link that carries traffic to the parking lots. These parking lots are connected to
the traffic by a road having a travel cost of $s(v)$. In this problem, we would like to calculate the parking charges for the private parking lot. This example is due to Glazer and Niskanen [41] and appears in Verhoef [97].

The network representation of this problem is given in Figure 4.2. Assuming that there is demand for both of the parking lots, using the first order conditions of the following nonlinear program, one can estimate the parking charges:

$$\begin{align*}
\max & \quad \int_0^{v_1} w_1(z)dz + \int_0^{v_2} w_2(z)dz - s(v)v \\
\text{s. t.} & \\
& v = v_1 + v_2 \\
& w_1(v_1) = s(v) + \beta_1 \\
& w_2(v_2) = s(v) \\
& v_1 > 0 \\
& v_2 > 0
\end{align*}$$

Note that the formulation above has the tolled user equilibrium conditions presented in Corollary 3.8 as the conditions that describe the constrained user equilibrium (thus $\phi = 0$). Let’s assume that path one consists of nodes 1-2-3 and path two follows through nodes 1-2-4. In addition, let $\lambda_1$ and $\lambda_2$ denote the multipliers related to these paths. $\zeta = \nu = u = 0$ since $(h, t, z) > 0$. As a result, $\sum_{a \in A} \chi_{ar_k} s_a^k = 0$. Using $\sum_{k \in K} \sum_{r_k \in R_k} \chi_{ar_k} \lambda r_k + \phi v_a = 0$, $\lambda_1 = 0$. Thus, $\lambda_2 = \frac{s'(v)v}{s'(v) - w'_2(v_2)}$. Then, we can estimate the parking charge as follows:

$$\beta_1 = s'(v)v - s'(v) \frac{s'(v)v}{s'(v) - w'_2(v_2)} = s'(v)v \frac{-w_2(v_2)}{s'(v) - w'_2(v_2)}.$$  

This result is similar to Verhoef’s [97] result. The parking charge is a fraction of the MSCP toll on the road access, where the denominator is a function of $s'(v)$ and $w'_2(v_2)$. However, note that there is an alternative way of deriving the formula for
the second best toll, which is,

$$\beta_1 = w_1(v_1) - s(v) = w_1(v_1) - w_2(v_2).$$

It is clear that this formula is a simpler expression compared to the one in the economics literature and is easier to analyze. The toll is equal to the difference between the benefit of using the private parking lot and cost of using the congested road access, or is equal to the difference between the benefit of using the private and free parking lots.

4.6.3 Liu and McDonald’s Example

Liu and McDonald [71] consider a two-period two parallel routes example by which they simulate the effect of rush hour traffic on the two parallel routes where one of the routes can not be tolled. The network representation of this problem is the time expanded network of the two parallel routes network: it has three nodes, two origins, one destination and four links. In this network, the links between node one and node three represent the peak period traffic and links between nodes two and node three represent the prepeak period. The main difference in the Liu and McDonald model is that the inverse demand in a given period is a function of the demand in both periods. However, it is assumed that the effect of the price change in the peak period on the prepeak period is the same as the effect of price change in the prepeak period on the peak period, i.e.,

$$\frac{\partial w_1(t_1, t_2)}{\partial t_1} = \frac{\partial w_2(t_1, t_2)}{\partial t_2}.$$ 

This effect is called the substitution effect. The substitution effect enables the line integral to be uniquely defined and to be independent of the path taken. Mathematically, the two time period two parallel route example has the user
<table>
<thead>
<tr>
<th>LINK</th>
<th>$T_a$</th>
<th>$C_a$</th>
<th>$B_a$</th>
<th>Tollable</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 3)</td>
<td>1</td>
<td>2000</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(1, 3)</td>
<td>2</td>
<td>2000</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>1</td>
<td>2000</td>
<td>32.5</td>
<td>1</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>2</td>
<td>2000</td>
<td>32.5</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) Problem Data

(b) User Equilibrium Flows, $\text{NUB}= 38813.16$

(c) System Optimal Flows, $\text{NUB} = 39530.13$

(d) MSCP Toll Vector, Toll Revenue = 4371.19

Figure 4.3: Liu and McDonalds’ Network-First Best Solution.
benefit function as,

\[
\int_{(0,0)}^{(t_1,t_2)} w_1(z_1, z_2) dz_1 + w_2(z_1, z_2) dz_2 = \begin{cases} 
\int_0^{t_1} w_1(z,0) dz_1 + \int_0^{t_2} w_2(t_1, z_2) dz_2 \\
\int_0^{t_2} w_2(0, z_2) dz_2 + \int_0^{t_1} w_1(z_1, t_2) dz_1.
\end{cases}
\]

The demand functions for peak and prepeak functions are \( w_1(t_1, t_2) = 247500 - 25t_1 - 15t_2 \) and \( w_2(t_1, t_2) = 196500 - 15t_1 - 21t_2 \) respectively. The cost functions are of the form \( s_a(v) = s_a(v_a) = T_a(1 + 0.15(v_a/C_a)^4) + B_a \) where \( B_a \) represents an adjustment parameter (See Liu and McDonald [71] for how these parameters are set). The data for this example is given in Figure 4.3.a.

From Figure 4.3.b and 4.3.c it is evident that the user equilibrium solution and system optimal solution are considerably different. The system optimal demand is 3.3% less than the user equilibrium demand. Furthermore, the net user benefit (NUB) for the system problem is 39530.13 which is 1.81% higher than the NUB of the user problem. Thus, the traffic planner might toll the system in order to make the user equilibrium solution be the same as the system optimal one. Note that there is flow on all of the links, thus the inequalities describing the toll set hold as equalities. It happens that the MSCP tolls (Figure 4.3.d) are the same as the MINTB tolls. The total tolls collected for making the users behave in the system optimal way is 4371.19.

Now consider the case where the traffic planner cannot toll all of the routes, i.e., a second best toll pricing scheme should be employed to have a better utilization of the network. In Figure 4.4, the bold arcs represent the tollable arcs. Since a second best toll pricing scheme is utilized, it is expected to have

\[
NUB(v^u, t^u) \leq NUB(\bar{v}, \bar{t}) \leq NUB(v^*, t^*)
\]

where \((\bar{v}, \bar{t})\) is the second best solution.
Figure 4.4: Liu and McDonalds’ Network: Second Best Solution: Flows and Demands and Alternative Toll Vectors
For this example, we present the results for two stationary points. We cannot guarantee if these points are local minimums or not, since the cost functions are nonlinear. For both stationary points, the total second best demand is less than the user equilibrium demand but higher than the system optimal demand. An interesting fact is the relationship between the amount that users have to pay in toll revenue and the efficiency level of the transportation network. For example, charging the users by 890.25 which is 20.37% of the first best toll revenue, i.e., $\Omega = 0.20$, we are 0.98% away from the first best optimal objective function value for the first stationary point. The relative distance from the user optimal solution, $\Upsilon$, is 0.46. This leads to an efficiency of $\Psi = 0.3644$. In the second example, the NUB differs by 1.08% even though we just toll the system by 558.01, i.e., $\Omega = 0.13$, $\Upsilon = 0.40$ and $\Psi = 0.3531$. Based on the efficiency measure, for a traffic planner the second stationary point might be more desirable, since both points have approximately the same efficiency and in the second one, users have to pay less tolls. We also observe that users have to pay more when transportation authority tries to get closer to the first best optimal flows and demands.

4.6.4 Nine Node Network

In this section, we have modified the nine node example from [51], which is depicted in Figure 4.5, in order to illustrate the toll pricing schemes presented in this chapter. This network has a cost data similar to large-scale traffic assignment problems. Emphasis here will be on comparison of the alternative tolling schemes such as first best, constrained first best and second best toll pricing schemes.

The nine node network has 18 links with cost functions of the same structure: $s_a(v) = s_a(v_a) = T_a(1 + 0.15(v_a/C_a))$ where $T_a$ is a measure of travel time when there is zero flow and $C_a$ is the practical capacity of link $a$ ($T_a$ and $C_a$ values for all arcs are presented in Table 4.1). In fact $s(v)^T v$ is strictly convex and separable, and $w(t)$ is strictly monotonic. Therefore, the user and system problems have
Table 4.1: Problem Data for the Nine Node Problem: $T_a$, $C_a$ and whether the Arc is Tollable or not.

<table>
<thead>
<tr>
<th>LINK</th>
<th>(1, 5)</th>
<th>(1, 6)</th>
<th>(2, 5)</th>
<th>(2, 6)</th>
<th>(5, 6)</th>
<th>(5, 7)</th>
<th>(5, 9)</th>
<th>(6, 5)</th>
<th>(6, 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_a$</td>
<td>6</td>
<td>7</td>
<td>2</td>
<td>8</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>$C_a$</td>
<td>11</td>
<td>9</td>
<td>2</td>
<td>35</td>
<td>20</td>
<td>5</td>
<td>44</td>
<td>6</td>
<td>48</td>
</tr>
<tr>
<td>Tollable</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.2: Demand between OD-Pairs for the User, System and Second Best Pricing Problems.

<table>
<thead>
<tr>
<th>LINK</th>
<th>(6, 9)</th>
<th>(7, 3)</th>
<th>(7, 4)</th>
<th>(7, 8)</th>
<th>(8, 3)</th>
<th>(8, 4)</th>
<th>(8, 7)</th>
<th>(9, 7)</th>
<th>(9, 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_a$</td>
<td>7</td>
<td>2</td>
<td>7</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$C_a$</td>
<td>29</td>
<td>34</td>
<td>9</td>
<td>49</td>
<td>13</td>
<td>5</td>
<td>47</td>
<td>42</td>
<td>5</td>
</tr>
<tr>
<td>Tollable</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

unique optimal solutions. There are four OD-pairs. The demand functions between those OD-pairs are, $t_{(1,3)}(c_{(1,3)}) = 10 - 0.5c_{(1,3)}$, $t_{(1,4)}(c_{(1,4)}) = 20 - 0.5c_{(1,4)}$, $t_{(2,3)}(c_{(2,3)}) = 30 - 0.5c_{(2,3)}$, and $t_{(2,4)}(c_{(2,4)}) = 40 - 0.5c_{(2,4)}$ where $c_{pq}$ is the generalized cost between OD-pair $(p, q)$.

As seen in Table 4.2, SOPT-ED and UOPT-ED have different demands. The total demand is 55.34 in the system problem and 61.66 in the user problem which results in a 11.4% difference.

Figure 4.5 provides the optimal flows for the SOPT-ED and UOPT-ED problems. When the individual link flow values for SOPT-ED and UOPT-ED are compared, it is clear that all of the links used by both problems differ in flows at least by 17.7%. Another difference is in the net user benefit: the system problem has 7.97% more net user benefit than the user problem.

In Figure 4.5, we also give the MSCP and MINTB tolls. Recall that by using these tolls, the transportation authority can make the tolled user equilibrium flows
Figure 4.5: First Best Solution: User and System Optimal Flows, and Alternative Toll Vectors

(a) User Equilibrium Flows, NUB = 1351.68

(b) System Optimal Flows, NUB = 1468.11

(c) MSCP Tolls, Total Tolls Collected = 332.24

(d) MINTB Tolls, Total Tolls Collected = 332.24
and demands be the same as the system optimal demands and flows. The total revenue is constant for the elastic demand traffic assignment problems which is 332.24 (22.6% of the net user benefit) for the first best pricing models.

The system problem has a total flow of 15.50 on path 2-6-8-4. In contrast, there is no flow on this path in the user equilibrium solution. As a consequence, there is a relatively high toll on link 2-5 in order to shift some traffic flow to that path in both tolling schemes. In the MSCP tolling scheme, nine arcs are being tolled. Note that the MSCP tolls occur on any arc with positive flow. The MINTB tolling scheme achieves the same system utilization by just tolling five arcs. In this scheme, the largest toll is on link 2-5 with a value of 3.79 (which is less than 4.79 of the MSCP tolling scheme). Thus, it could be argued that the MINTB solution is the best implementable tolling scheme if there is a high cost of building and maintaining the toll booths.

Now, assume that the transportation authority is not willing to toll the incoming and outgoing arcs of node nine (the downtown of the nine node network). In this case, the goal is to obtain the best system efficiency given the tolling restrictions on the traffic network. We use the toll pricing framework presented in Section 4.5 to obtain alternative toll vectors for this scenario. The first step is checking if the constrained first best toll set is empty or not. It is clear that the MSCP and MINTB tolling schemes presented in Figure 4.5 do not impose tolls on the incoming and outgoing arcs of node nine. Thus, we can conclude that the net user benefit obtained by imposing these tolls is equal to 1468.11 which is the same as the system optimal one.

Consider another scenario where the transportation authority is forced to allow the users from the origin nodes one and two to destination nodes three and four without being tolled if they choose to. Thus, the transportation authority designs the scheme presented in Figure 4.6 to allow the users choose certain paths
Figure 4.6: Second Best Solution: Flows, Demands and Alternative Toll Vectors

(a) Second Best Equilibrium Flows, $\text{NUB} = 1436.89$

(b) MINREV Tolls, Total Tolls Collected = 75.92

(c) MINTB Tolls, Total Tolls Collected = 75.92
to make their trips without being tolled. In these schemes the bold arcs represents
the tollable arcs. Given these conditions, first, the traffic planner should check if
the most efficient utilization of the transportation network can be achieved. For
this scenario, unfortunately, it turns out that $W(v^*, t^*, Y)$ is an empty set, i.e., the
system optimal flows and demands can not be achieved. In other words, a first
best pricing scheme can not be applied. As a result, we have to apply a second
best toll pricing scheme in order to achieve a better system utilization. We utilize
SBTP-LINK model to find the second best flows, demands and tolls, $(\bar{\bar{v}}, \bar{\bar{t}}, \bar{\bar{\beta}})$. We
use $W(\bar{\bar{v}}, \bar{\bar{t}}, Y)$ to identify the alternative second best toll vectors.

The second best flows and alternative toll vectors are presented in Figure 4.6.
The solution presented in terms of the utilized arcs is similar to the system optimal
one with three exceptions: link 1-6, 5-9 and 9-8. The total demand in the second
best pricing model is higher than the system and user optimal demands (see Table
4.2) and the utilization of the network is very close to the best utilization possible.
The net user benefit in the second best problem is 2.12% less than the system
optimal one. For this problem the relative distance from the user optimal solution,
$\Upsilon$, is 0.7319 which means that by charging 75.92, i.e., $\Omega = 0.23$, the transportation
planning agency can achieve 73.19% relative improvement compared to the untolled
case. The efficiency of this solution, $\Psi$, is 0.5647.

In Figure 4.6, we also present two alternative second best tolling schemes.
Both of the schemes are essentially same. As long as the total toll on links 5-9 and
9-8 is 2.22, i.e., $\beta(5,9) = \alpha$ and $\beta(9,8) = 2.22 - \alpha$, these tolling scheme will produce
the same toll revenue and the same second best flows and demands. Note that we
can also toll the arcs with zero flows. However, this scheme will result in having
“spurious” tolls.

To summarize, we can argue that it might be advantageous to implement a
second best toll pricing policy because it might be easier to toll less number of arcs
and still obtain a relatively high efficiency level which might help to decrease the resistance against first best pricing in political circles.

4.7 Effect of Tolling on Total Demand

One would expect to have a higher NUB when the users are tolled efficiently (e.g., by first best tolls). Intuitively, tolling the system should result in a smaller number of users since the costs of travel on utilized links (also on the utilized paths) increases. For example in Table 4.2, when users are tolled with a first best toll pricing scheme, there is a decrease in demand for all OD-pairs, and as a result the total demand decreases from 61.66 to 55.34. Quite the contrary for the same network, this is not the case when a SBTP scheme is applied. Although the NUB increases by 85.12 as a result of tolling efficiently with an amount of 75.92, the total demand also increases by 0.19. One of the reasons why this abnormality occurs might be because of not having the effect of tolling in the objective function (i.e., the NUB is a function of $v$ and $t$). Another reason might be that tolling diverts traffic to the paths with smaller travel costs, and the net decrease in the total system costs compensates the decrease in the user benefit. In this section, we explain in detail why tolling the network by a SBTP scheme results in an increase in demand for the Nine Node network.

Table 4.3: Total Benefit for the User, System and Second Best Pricing Problems.

<table>
<thead>
<tr>
<th></th>
<th>User</th>
<th>System</th>
<th>SBTP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_{(1,3)}(t_{(1,3)})$</td>
<td>14.90</td>
<td>16.71</td>
<td>13.55</td>
</tr>
<tr>
<td>$w_{(1,4)}(t_{(1,4)})$</td>
<td>20.98</td>
<td>24.37</td>
<td>23.61</td>
</tr>
<tr>
<td>$w_{(2,3)}(t_{(2,3)})$</td>
<td>17.35</td>
<td>20.28</td>
<td>14.55</td>
</tr>
<tr>
<td>$w_{(2,4)}(t_{(2,4)})$</td>
<td>23.43</td>
<td>27.44</td>
<td>24.61</td>
</tr>
</tbody>
</table>

In Figure 4.7, we give the cost of travel and the toll amount on each link of the network where the first number indicates $s_a(v_a)$ and the second number below $s_a(v_a)$ is $\beta_a$. Note that in the SBTP scheme, links (7, 4), (7, 8) and (5, 9) are tolled.
Now, let's examine why tolling the system results in a higher number of users. In the SBTP problem, the cost of travel between OD pairs (1, 3) and (2, 3) is lower compared to the user problem. However, the cost of travel between OD-pairs (1, 4) and (2, 4) increases when a SBTP scheme is utilized. This can be verified by checking the $w_k(t_k)$ values in Table 4.3. Note that the cost of travel on any used path is equal to the total benefit gained on that path. On all other paths, the cost of travel is greater than or equal to the total benefit. For the Nine Node problem, the net change in the total user benefit is a decrease of 14.03 when a SBTP scheme is utilized (i.e., 2621.97 versus 2607.94).

Now consider the total cost of travel $(s(v) + \beta)^T v$ on the Nine Node Network ($\beta = 0$ for the user problem). Tolling links (7, 4), (7, 8) and (9, 8) enables the traffic planner to reduce some of the flow which goes through link (5, 7) and has the final destination as node four. As a result, some of the flow is diverted to paths 1-6-8-4 and 2-6-8-4. Analyzing in detail, for example, we can observe that, tolling
Table 4.4: Simulation Results for First and Second Best Pricing Test Problems.

<table>
<thead>
<tr>
<th></th>
<th>First Best Pricing</th>
<th>Second Best Pricing</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>NUB</strong></td>
<td><strong>FB</strong></td>
</tr>
<tr>
<td><strong>Linear</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td>6.79</td>
<td>6.88</td>
</tr>
<tr>
<td><strong>σ</strong></td>
<td>27.17</td>
<td>27.38</td>
</tr>
<tr>
<td><strong>Min</strong></td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td><strong>Max</strong></td>
<td>553.60</td>
<td>557.60</td>
</tr>
<tr>
<td><strong>Nonlinear</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td>51.42</td>
<td>54.34</td>
</tr>
<tr>
<td><strong>σ</strong></td>
<td>160.06</td>
<td>163.08</td>
</tr>
<tr>
<td><strong>Min</strong></td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td><strong>Max</strong></td>
<td>4166.30</td>
<td>4180.50</td>
</tr>
</tbody>
</table>

the network decreases the flow from upper part of the network to node four by an amount of 18.51 (i.e., from 37.79 to 19.28). Furthermore, a total flow of 16.98 is diverted to the paths 1-5-9-8-4, 1-6-8-4 and 2-6-8-4. As a result, the total system cost decreases to 1173.55 (in the user problem the total system cost is 1270.39). Although the number of users on the network increases, the total system cost decreases. This is because alternative paths are being utilized and the flows on each link is below the capacity except (5, 7) and (7, 4). Furthermore, the total amount of decrease in the system cost is higher than the total amount of decrease in total user benefit (due to increase in demand). As a result, the net user benefit of the SBTP problem is higher than the net user benefit of the user problem.

4.8 Computational Experimentation with the TPF

The toll pricing framework (TPF) requires solving a nonlinear program and system of linear equations prior to solving the SBTP problem which is an MPEC instead of solving the SBTP directly. In this section, we compare both approaches by studying randomly generated nine node test problems. We focus on statistics on the expected running time of both methods and likelihood of occurrence of the SBTP scheme. Furthermore, we compare the experimental results for the test problems where the cost functions are assumed to be linear and nonlinear.
We have tested the TPF on 100000 randomly generated Nine Node problems with linear and nonlinear travel time functions by varying the cost and demand function parameters and the part of the network which can not be tolled. Before presenting the results, we give the experimental setup used in this section: The nine node network has 18 links with cost functions of the same structure either of the form

\[ s_a(v) = s_a(v_a) = T_a (1 + 0.15(v_a/C_a)) \]

(linear cost functions (LCF)) or of the form

\[ s_a(v) = s_a(v_a) = T_a (1 + 0.15(v_a/C_a)^4) \]

(nonlinear cost functions (NCF)) where \( T_a \) is Uniform[1, 5] and \( C_a \) is Uniform[1, 30]. There are four OD-pairs (1, 3), (1, 4), (2, 3) and (2, 4). The demand functions between those OD-pairs are of the form, \( t_{(p,q)}(c_{(p,q)}) = d_{(p,q)} - m_{(p,q)}c_{(p,q)} \) where \( d \) is Uniform[10, 40] and \( m \) is Uniform[1, 5]. The set \( Y \), which defines the untollable parts of the network, is also generated randomly. The test problems are solved using GAMS/ MINOS [85] and GAMS/CPLEX [20].

When we utilize the TPF, we observe that 37393 (out of 50000) of the linear test problems (approximately 74.77%) and 31256 (out of 50000) of the nonlinear
test problems (approximately 62.50%) do not require solving SBTP-LINK (i.e.,
they are constrained first best pricing problems). The remaining problems are
SBTP problems. We also note that GAMS/MINOS and GAMS/SNOPT could
not solve 501 of the SBTP problems with LCFs and 609 of the SBTP problems
with NCFs. In these cases, even though \((v^U, t^U, \beta = 0)\) is a feasible solution to
the second best toll pricing problems, MINOS (or SNOPT) fails to find a feasible
solution using the default set parameters before the reduced gradient is less than a
certain tolerance value.

We present the experimental results in Tables 4.4, 4.5 and 4.6. In these tables,
\((v^U, t^U)\) denotes the user equilibrium flows and demands, \((v^*, t^*)\) is the system
optimal solution, \(\beta^*\) is a FB toll and \((\bar{v}, \bar{t}, \bar{\beta})\) is the second best solution. \(|\beta|\)
denotes number of links for which \(\beta_a \neq 0\).

As can be seen in Table 4.4, there is a huge discrepancy in the scale of data
between the results for linear and nonlinear cases. For example, the NUB of the
user problem increases from 6.79 to 51.42 when linear and nonlinear test problems
are compared. There is almost 800% difference in the scale of data for the test
problems where a first best pricing scheme is applied. This is approximately
300% when a SBTP scheme is needed. Also the range of data for both cases is
considerably high which results in large standard deviations. The distribution of
the data is usually skewed to right. It is also clear that the discrepancy between
the user, second best and system solution is higher in nonlinear test cases, i.e., the
spread in the data is higher. Because of the high variability in the data, we have
scaled the data in Tables 4.5 and 4.6 in order to reduce the variability in the data
range for different problems.

Analyzing Table 4.4, we observe that the user problem is approximately 99%
efficient for the LCF test problems. This might lead to a conclusion that the tolling
almost does not achieve any improvement on the traffic system when the LCF
Table 4.6: Simulation Results for Second Best Pricing: Linear and Nonlinear Cost Functions.

| Linear | \( \frac{\text{NUB}(v,U,t)}{\text{NUB}(v^*,U,t)} \) | \( |\beta^*_\text{MSCP}| \) | \( |\beta^*_\text{MINTB}| \) | \( \frac{\beta^Tv}{\text{NUB}(v^*,U,t)} \) | \( \frac{\text{NUB}_{\text{SBTP}}(v,t)}{\text{NUB}(v^*,t)} \) | \( |A\setminus Y| \) | \( |\tilde{\beta}_{\text{SBTP}}| \) | \( \frac{\beta^Tv}{\text{NUB}(v^*,U,t)} \) | \( \Omega \) | \( \Upsilon \) | \( \Psi \) |
|--------|---------------------------------|----------------|----------------|---------------------------------|---------------------------------|----------------|----------------|---------------------------------|----------------|----------------|----------------|
| Average | 0.98                           | 5.82           | 2.31           | 0.21                           | 0.99                           | 8.69           | 2.43           | 0.15                           | 0.71           | 0.57           | 0.18           |
| \( \sigma \) | 0.03                           | 1.61           | 0.87           | 0.16                           | 0.02                           | 2.12           | 1.68           | 0.17                           | 0.42           | 0.48           | 0.16           |
| Min | 0.66                           | 3              | 1              | 0.00                           | 0.66                           | 1              | 0              | 0.00                           | 0.00           | 0.00           | 0.00           |
| Max | 1.00                           | 11             | 6              | 0.53                           | 1.00                           | 15             | 9              | 1.00                           | 1.00           | 1.00           | 1.00           |
| Nonlinear | \( \frac{\text{NUB}(v^*,U,t)}{\text{NUB}(v,t)} \) | \( |\beta^*_\text{MSCP}| \) | \( |\beta^*_\text{MINTB}| \) | \( \frac{\beta^Tv}{\text{NUB}(v,t)} \) | \( \frac{\text{NUB}_{\text{SBTP}}(v,t)}{\text{NUB}(v,t)} \) | \( |A\setminus Y| \) | \( |\tilde{\beta}_{\text{SBTP}}| \) | \( \frac{\beta^Tv}{\text{NUB}(v,t)} \) | \( \Omega \) | \( \Upsilon \) | \( \Psi \) |
| Average | 0.94                           | 6.81           | 2.74           | 0.19                           | 0.98                           | 8.47           | 3.01           | 0.13                           | 0.34           | 0.57           | 0.45           |
| \( \sigma \) | 0.07                           | 2.02           | 1.24           | 0.14                           | 0.04                           | 2.08           | 1.76           | 0.14                           | 0.37           | 0.45           | 0.39           |
| Min | 0.61                           | 3              | 0              | 0.00                           | 0.61                           | 1              | 0              | 0.00                           | 0.00           | 0.00           | 0.00           |
| Max | 1.00                           | 14             | 8              | 0.52                           | 1.00                           | 15             | 11             | 1.00                           | 1.00           | 1.00           | 1.00           |
assumption is made. However, when we analyze the data in detail we observe that 10% of the test problems have a relative efficiency less than 97% (i.e., the users are utilizing the system at most 97% efficiency). On the average 1.19 MSCP toll booths (a maximum of 10 toll booths) or 0.52 MINTB toll booths (a maximum of six) can be used to make the users utilize the transportation system in the most efficient way. For the NCF case, users are utilizing the network at most 97% efficiency in 31% of the test problems. A traffic engineer should have 4.23 MSCP toll booths (a maximum of 14) or 1.30 MINTB toll booths (a maximum of six).

When we compare the FB MINTB scheme on all test problems, we observe that the MINTB tolling scheme produces smaller number of links to be tolled in the LCF case compared to the NCF case. As can be seen from Tables 4.5 and 4.6, this observation actually holds for all tolling schemes including CFB and SBTP MINTB tolling schemes.

As we can observe from Tables 4.5 and 4.6, on the average, when the number of links allowed to be tolled is higher than 50% of the links, a constrained first best pricing scheme is applied. When this value is less than 50% a SBTP scheme should be applied. Comparing the LCF and NCF test problems, we also observe that the relative toll revenue in the linear case is higher than the one for the nonlinear case. Although the first best pricing schemes charge more compared to second best pricing schemes (see Tables 4.5 and 4.6), if a traffic planner carefully chooses the links to be tolled (e.g. by a MINTB tolling scheme), users have to pay tolls on relatively small number of links which is usually much less than the number of links allowed to be tolled. This observation also holds for the second best toll pricing schemes. In the SBTP problems, the MINTB tolling scheme needs two to three FB toll booths to achieve the most efficient utilization of the system while MSCP tolling scheme requires six to seven toll booths. The maximum number of toll booths required for the MINTB scheme is six to eight toll booths while at
most 11-14 MSCP toll booths are needed. However, these schemes do not satisfy the constraints on tolling the network. Although the system planner allows eight to nine links to be tolled, the SBTP MINTB tolling scheme requires two to four links to be tolled in order to achieve the SB efficiency. Another observation is that MINSYS tolling scheme is usually a good upper bound for the number of links to be tolled.

In the SBTP problems, when LCF assumption is made, 21% of the test problems results having a user equilibrium which is at most 97% efficient. This is 55% when the NCFs are used. It seems a SBTP scheme is needed when users are utilizing the system significantly inefficient, i.e., the relative distance between the user and system problems is relatively high.

The efficiency index has a range between zero and one as expected. For all of the problems, the relative distance to the user optimal solution is approximately the same. However, the linear case achieves this by charging 208% relatively more tolls (34% versus 71%) compared to the NCF case. As a result the efficiency index in the linear test problems is less than the one for the NCF test problems. For the NCF case, by tolling $\Omega = 34\%$ of the FB toll revenue, one achieves the relative distance to user problem as $\Upsilon = 57\%$. This results in an efficiency index of $\Psi = 0.45$. Note that the efficiency index has higher values when the toll revenue is small and the resulting NUB is high.

Now we compare the expected CPU times for finding the alternative toll vectors by utilizing the TPF and directly by solving the SBTP models. The linear test problems has CPU times of 0.014, 0.001 and 0.05 CPU seconds for the system, MINTB and SBTP problems on a 200Mhz node of IBM SP2 respectively. This results in an expected TPF running time of 0.024 which is 48.43% of average running time of the SBTP problem. Similarly, the nonlinear case has the average CPU time for the system, MINTB and SBTP-LINK problems as 0.011, 0.008
and 0.082 CPU seconds respectively. Thus, the expected running time for TPF is 0.049 which is 39.33% less than the average CPU time needed to solve the SBTP-LINK formulation directly. As a result, we can conclude that for the Nine Node Problem, the TPF is computationally more advantageous than solving the MPEC. Furthermore, we also note that the size of the MPECs which are solvable is limited (for example, currently the Sioux Falls problem can not be solved by any MPEC solver).

To summarize, first of all, we observe that the LCF test problems are not as interesting as the NCF test problems, since for most of the cases, the user problem utilizes the transportation network quite efficiently. Thus, one might question the validity of the LCF assumption in the transportation networks. Second, if a traffic planner chooses the tollable links carefully (e.g. by using the MINTB tolling scheme), the most efficient system utilization of the system can be achieved by relatively low number of toll booths. A SBTP scheme usually occurs when the distance between the user and system problems is relatively high and the number of tollable links is relatively low. We also observe that on the average the TPF is consuming less amount of time comparing to solving the SBTP problem directly.

In the next section, we explain the relationship between the NUB and the maximum allowable toll revenue.

4.9 Effect of Toll Revenue on NUB

Consider the following SBTP problem (SBTP-2-VI) where the total toll revenue is restricted to be below a certain threshold:

$$\max \sum_{k \in K} \int_{0}^{t_k} w_k(z)dz - s(v)^T v$$

s. t.

$$(v,t) \in V$$
It is clear that

\[(s(v) + \beta)^T(u - v) - w(t)^T(d - t) \geq 0 \quad \forall (u, d) \in V\]

\[\beta^Tv \leq K\]

\[\beta_a = 0 \quad \forall a \in Y.\]

It is clear that

\[NUB(v^U, t^U) \leq NUB_{SBTP-2-VI}(\hat{v}, \hat{t}) \leq NUB_{SBTP-VI}(\bar{v}, \bar{t}) \leq NUB(v^*, t^*)\]

where \((\hat{v}, \hat{t})\) is the global optimal solution for the SBTP-2-VI problem. Furthermore,

\[\tilde{\beta}^T\hat{v} \leq \tilde{\beta}^T\hat{v} \leq \beta^Tv^*\]

holds. Thus, when \(K \leq \beta^T\tilde{v}\), the toll revenue constraint might be binding. Note that when \(K > \beta^T\tilde{v}\), SBTP-VI and SBTP-2-VI are exactly the same formulations.

In this section, we investigate the relationship between \(K\) and \(NUB_{SBTP-2-VI}\). By analyzing this relation, a transportation planner might decide to implement a tolling scheme where users are charged well below the first and second best toll revenue and obtain a reasonable efficiency in the system. The experimental design
in this section is as follows: We choose the upper bound on the toll revenue as $K_l = l\delta, \quad l = 0, \ldots, n$ where $\delta = \frac{\bar{\beta}^T \bar{v}}{n}$ and $n$ denotes the number of intervals. We define the “marginal returns” as

$$NUB_{SBTP-2-VI}(\hat{v}^{l+1}, \hat{t}^{l+1}, K_{l+1}) - NUB_{SBTP-2-VI}(\hat{v}^l, \hat{t}^l, K_l)$$

i.e., when $\delta \to 0$, marginal returns is the derivative of the NUB function over the feasible region of SBTP-2-VI. Denote the feasible region of SBTP-2-VI as $\mathcal{V}_l$. We observe that

$$NUB_{SBTP-2-VI}(\hat{v}^{l+1}, \hat{t}^{l+1}, K_{l+1}) - NUB_{SBTP-2-VI}(\hat{v}^l, \hat{t}^l, K_l) \geq 0.$$ This is because $\mathcal{V}_l \subseteq \mathcal{V}_{l+1}$. As a result $SBTP - 2 - VI(v^l, t^l, K_l)$ is monotonically nondecreasing as a function of $K$.

If SBTP-2-VI were a convex programming problem, one would expect a decrease in marginal returns as $l$ increases. Since, SBTP problems are highly non-convex (unless very simplistic assumptions are made), one can not conclude the shape of the marginal returns curve. We give two examples on the nine node
work to illustrate the relation between the NUB and the maximum allowable toll revenue.

We first consider the Nine Node network given in Section 4.6.4. Using the experimental setup described above, we have observed the principle of “decreasing marginal returns” as can be seen in Figure 4.8. When toll revenue is less than 40, the toll revenue has almost a linear relationship with the NUB. After that point, an increase in the toll revenue returns smaller increases in the NUB. We analyze this effect in Figure 4.9 (which compares the marginal benefit and maximum toll revenue) in detail. As can be seen, when the system is tolled even for a very small amount, there is a huge improvement in the system performance\footnote{In this figure, we have not included the first data point, (where the maximum allowable toll revenue is 3.32 and NUB is 400.71), which has a marginal benefit of 49.03, solely for the scaling purposes.}. But this improvement decreases as the tolling amount increases.

Table 4.7: Problem Data for the Second Nine Node Problem: $T_a$, $C_a$ and whether the Arc is Tollable or not.

<table>
<thead>
<tr>
<th>LINK</th>
<th>(1, 5)</th>
<th>(1, 6)</th>
<th>(2, 5)</th>
<th>(2, 6)</th>
<th>(5, 6)</th>
<th>(5, 7)</th>
<th>(5, 9)</th>
<th>(6, 5)</th>
<th>(6, 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_a$</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>$C_a$</td>
<td>17</td>
<td>13</td>
<td>9</td>
<td>19</td>
<td>17</td>
<td>24</td>
<td>16</td>
<td>22</td>
<td>13</td>
</tr>
<tr>
<td>Tollable</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>LINK</th>
<th>(6, 9)</th>
<th>(7, 3)</th>
<th>(7, 4)</th>
<th>(7, 8)</th>
<th>(8, 3)</th>
<th>(8, 4)</th>
<th>(8, 7)</th>
<th>(9, 7)</th>
<th>(9, 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_a$</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$C_a$</td>
<td>15</td>
<td>11</td>
<td>16</td>
<td>23</td>
<td>13</td>
<td>22</td>
<td>18</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>Tollable</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

However, decreasing marginal returns does not hold for all SBTP problems. For example, consider the Nine Node network given in Table 4.7. The cost functions for this network are of the form

$$s_a(v) = s_a(v_a) = T_a(1 + 0.15(v_a/C_a)^4)$$
Figure 4.10: Comparison of the NUB and Maximum Toll Revenue (DMR Property does not Hold)

Figure 4.11: Comparison of the Marginal Benefit and Maximum Toll Revenue
and the demand functions are \( t_{(1,3)}(c_{(1,3)}) = 19 - 2.612c_{(1,3)} \), \( t_{(1,4)}(c_{(1,4)}) = 24 - 0.1817c_{(1,4)} \), \( t_{(2,3)}(c_{(2,3)}) = 21 - 3.514c_{(2,3)} \) and \( t_{(2,4)}(c_{(2,4)}) = 26 - 1.103c_{(2,4)} \).

In this problem, number of links allowed to be tolled is 11. The user, system and SBTP problems have NUBs of 63.2, 70.4 and 69.7, respectively. The total first best toll revenue is 8.98 and the second best toll revenue is 7.30. This results in an efficiency of 0.21.

As can be seen in Figure 4.10, the NUB is a non-decreasing function of the maximum allowable toll revenue. However, in Figure 4.11, it can clearly be seen that the DMR principle does not hold since the increase in marginal returns increases for the first five intervals and then starts to decrease. This might be due to highly non-convexity of the feasible region.

Since the DMR principle does not always hold, we have done a simulation study to check the likelihood of DMR occurrence. We run the toll pricing framework and obtain 6808 SBTP problems for which

\[
\frac{\text{NUB}_{\text{SBTP}}(v, t) - \text{NUB}(v', t')}{\text{NUB}_{\text{SBTP}}(v, t)} \geq 0.02.
\]

The average NUB for the user, system and SBTP problems are 28.51, 31.18 and 31.07 respectively. On the average, 9.05 links are allowed to be tolled. The total first best toll revenue is 6.44 while the second best toll revenue is 6.03.

Among 296 that we have tested, 124 (approximately 42%) of SBTP problems do not have the DMR property. This indicates that DMR property is almost random. We conjecture (and numerical evidence suggests) that because of the highly non-convex feasible region, the DMR principle might not always hold\(^2\).

\(^2\) Labbe et al. [61] show that for the case where the demand is fixed and the travel time on a link is constant, the aggregate complementary constraint for the tolled user equilibrium can be replaced by a set of mixed integer constraints (thus
We also test in how many of SBTP-2-VI problems, the DMR principle is violated. 296 SBTP-VI problems corresponds to 6808 cases (we have \( n = 22 \), thus a total of 23 intervals for each SBTP problem). We observe that 264 cases (3.88\%) of SBTP-2-VI violate the DMR property. As can be seen in Table 4.8, most of these cases occur in the first five or last five intervals (69.69\% of the cases). The average marginal return decreases for the first 11 intervals, and then increases for the next interval, decreases for seven intervals but it starts to increase in the next two intervals. We also present the average and 3\( \sigma \) confidence interval (Figure 4.9) for the marginal increase in NUB (thus data is in this interval with a probability

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the feasible region is non-convex and note that there is not a through sensitivity analysis for mixed integer problems).
of 0.9973). We also present the maximum increase in the NUB for a given interval. For example, in the first interval, the maximum increase in the NUB is 99.1% of the distance between $NUB(v^U, t^U)$ and $NUB_{SBTP}(v, t)$.

To summarize, the traffic planning authority can limit the total toll revenue and still achieve a reasonable utilization of the traffic network. Although the decreasing marginal returns principle is intuitive, numerical evidence verifies that this principle is almost “random.” However, it still worth investigating the relation between the maximum toll revenue and the net user benefit for a particular problem instance.

### 4.10 Further Extensions: Fixed Demand SBTP Problems

The second best models are readily extendable to the fixed demand (FD) traffic assignment problems. In this section, we present the variational inequality, the path based and link based formulations of the second best pricing models for fixed demand traffic networks. First, we give the fixed demand MPEC formulation.
(SBTP-VI-FD):

$$\begin{align*}
\text{min} & \quad s(v)^T v \\
\text{s. t.} & \quad v = \sum_{k \in K} x^k \\
& \quad A x^k = E_k \bar{t}_k \quad \forall k \in K \\
& \quad (s(v) + \beta)^T (u - v) \geq 0 \quad \forall u \in V \\
& \quad \beta_a = 0 \quad \forall a \in Y \\
& \quad x^k \geq 0 \quad \forall k \in K.
\end{align*}$$

Similarly, the link based formulation for FD second best pricing models uses a description of the tolled user equilibrium conditions similar to the one presented in Lemma 2.1. The following conditions in $(v, \rho, \beta)$ describe the constrained tolled user equilibrium:

$$\begin{align*}
s_a(v_a) + \beta_a & \geq \rho^k_j - \rho^k_i \quad a \in A \setminus Y, \forall k \in K \\
s_a(v_a) & \geq \rho^k_j - \rho^k_i \quad a \in Y, \forall k \in K \\
\sum_{k \in K} \bar{t}_k E_k^T \rho^k & = \sum_{a \in A \setminus Y} (s_a(v_a) + \beta_a) v_a + \sum_{a \in Y} s_a(v_a) v_a
\end{align*}$$

where $v$ is the equilibrium flows. The link based formulation (SBTP-LINK-FD) uses the description above in the constraint set to ensure that the optimal second best flows and demands are in the constrained tolled user equilibrium. SBTP-LINK-FD is a nonlinear program with a convex objective and a non-convex constraint set:
min \ s(v)^Tv

s. t.

\[ v = \sum_{k \in K} x^k \]

\[ Ax^k = E_k \bar{t}_k, \quad \forall k \in K \]

\[ s_a(v_a) + \beta_a \geq \rho^k_j - \rho^k_i; \quad a \in A \setminus Y, \forall k \in K \]

\[ s_a(v_a) \geq \rho^k_j - \rho^k_i; \quad a \in Y, \forall k \in K \]

\[ \sum_{k \in K} \bar{t}_k E_k^T \rho^k = \sum_{a \in A \setminus Y} (s_a(v_a) + \beta_a)v_a + \sum_{a \in Y} s_a(v_a)v_a \]

\[ x^k \geq 0, \quad \forall k \in K. \]

Finally, the path based formulation uses the following constrained tolled user equilibrium conditions as constraints in the mathematical model:

\[ \sum_{a \in A \setminus Y} \chi_{ar_a}(s_a(v_a) + \beta_a) + \sum_{a \in Y} \chi_{ar_a} s_a(v_a) \geq \rho^k_q - \rho^k_p \quad \forall k \in K, \forall r_k \in R_k \]

\[ \sum_{a \in A \setminus Y} (s_a(v_a) + \beta_a)v_a + \sum_{a \in Y} s_a(v_a)v_a = \sum_{k \in K} \bar{t}_k E_k^T \rho^k \]

The path based formulation (SBTP-PATH-FD) can be stated as follows:

min \ s(v)^Tv

s. t.

\[ v = \sum_{k \in K} x^k \]

\[ Ax^k = E_k \bar{t}_k \quad \forall k \in K \]

\[ \sum_{a \in A \setminus Y} \chi_{ar_a}(s_a(v_a) + \beta_a) + \sum_{a \in Y} \chi_{ar_a} s_a(v_a) \geq \rho^k_q - \rho^k_p, \quad \forall k \in K, r_k \in R_k \]

\[ \sum_{a \in A \setminus Y} (s_a(v_a) + \beta_a)v_a + \sum_{a \in Y} s_a(v_a)v_a = \sum_{k \in K} \bar{t}_k E_k^T \rho^k \]

\[ x^k \geq 0, \quad \forall k \in K. \]

These three formulations are KKT equivalent. Furthermore, the fixed demand SBTP problem is bounded above by the system optimal solution and below by the
system objective at the user solution. One can conclude that if the user and system problems can be solved, the MPECs above also have solutions.

The fixed demand SBTP problem has interesting properties: For example, consider the two parallel routes problem, parking pricing problem and Liu and McDonalds’ examples. In all of these examples, there are two routes where only one route can be tolled. In the elastic demand case, when only one route is tolled, a SBTP scheme should be utilized. However, this is not the case when the demand is fixed. In this case, a constrained first best pricing scheme achieves the most efficient system utilization when one of the routes is constrained not to be tolled. For example, consider the two parallel route problem having $v_1^*$ and $v_2^*$ as the system optimal flows. The transportation planner can not toll link two. Then a $\beta^* = s_1(v_1^*) - s_2(v_2^*)$, which is a CFB toll vector, would satisfy the constraints on tolling the network and still achieves the most efficient utilization of the transportation network. This was not the case in the elastic demand case because of the constant toll revenue property (i.e., the aggregate complementarity constraint).

We also note that for the models that we have presented if we assume that $s(v)$ is constant and the demand is fixed (Labbe et al. [61]), the optimal solution is “not tolling the system.” In other words, the $\beta = 0$ strategy achieves the most efficient utilization of the transportation network since the user and the system problems have the same solution.

4.11 Concluding Remarks

In this chapter, we define the constrained first best and the second best toll pricing schemes, and show that there exist a toll set for each of these pricing schemes.

In summary, the first best and constrained first best pricing schemes achieve the optimal/most efficient utilization of the network. The only difference between
these schemes is that the constrained first best pricing scheme has constraints on
tolling certain parts of the network a priori. In the second best pricing scheme,
the goal is to optimize the efficiency of the transportation network given that
there are constraints on tolling certain parts of the network and the most efficient
utilization of the network cannot be achieved. The reason that we distinguish
between the constrained first best and second best pricing schemes is that the
constrained first best tolls can be found by using linear or mixed integer programs
but a mathematical program with equilibrium constraints has to be solved in order
to obtain the optimal second best tolls.

After introducing the first best pricing concepts, we present several second
best toll pricing formulations. These models have a solution given that the
 corresponding user and system problems have solutions. Further, we show that
the resulting second best toll set can be characterized as a polyhedron. Thus one
can define additional constraints and objectives on tolling the network and obtain
alternative toll vectors by solving a linear or mixed integer program. Using this
idea, we propose a novel approach to obtain alternative toll vectors using a general
toll pricing framework.

We also give examples from the literature to validate the second best
formulations presented in this chapter. Also on a nine node 18 arcs example,
we illustrate the toll pricing framework. Based on the numerical experience, we
observe that a second best pricing scheme is desirable in some circumstances where
users might pay a small fraction of the first best toll revenue and have a relatively
high service level (net user benefit). This concept is also known as the decreasing
rate of marginal returns. Unfortunately, decreasing marginal returns property is a
random property due to the non-convex feasible region of the SBTP problems. We
also present computational results on simulating how the toll pricing framework
performs, and conclude that on average it takes less time to run the TPF then solving the SBTP-LINK directly.

In this chapter, we mainly focus on the modelling aspects of second best toll pricing models. However, solving these models is NP-Hard even if very simplistic assumptions like constant travel times and user benefits are made [61]. Unfortunately, the MPEC literature reports results just for small sized networks when more complicated assumptions are made. Thus, it is important to devise efficient algorithms to solve the models we have presented. Another way to solve these models might be solving the formulas we have derived for the path based model.

Heuristics to find the effect of tolling/building toll roads on the net user benefit of the transportation is a necessity. For example, given a transportation network, finding some greedy rules to determine the most profitable (or finding the $n$ most profitable) road segment(s) to toll will help to devise efficient algorithms for solving the second best toll pricing problems.
CHAPTER 5
FURTHER RESEARCH AND CONCLUDING REMARKS

5.1 Introduction

In this chapter, we first summarize the findings in the first four chapters and then give some future research directions. Mainly, we discuss how we can extend the ideas presented in this dissertation both theoretically and computationally including extending the toll pricing framework for the first best and second best pricing schemes to other models including the stochastic and dynamic traffic assignment models.

5.2 Summary of Findings

In this thesis, one of my goals is to synthesize congestion pricing approaches by the economists and the operations researchers by clarifying the first best pricing, constrained first best pricing and second best pricing schemes in order to include toll vectors other than the MSCP toll vector.

In Chapter 1, we present the road pricing concept, some successful implementations and failures and the road pricing technology. We also define the first best pricing scheme as a toll pricing scheme in which the tolled user equilibrium flows and demands utilize the network in the most efficient way. We define the set of all such toll vectors (including the MSCP toll) as the first best toll set.

Chapter 2 formally introduces the traffic planning process. In addition, we give a brief survey of mathematical models in the literature and point out the fact that all of these models employ the MSCP toll pricing scheme to obtain the system optimal solution as the tolled user equilibrium flow. We then present the first best the toll pricing theory and framework for the fixed demand models.
In Chapter 3, we develop the first best pricing theory for general variable demand networks. For the GVD model, we first show that the MSCP toll vector is valid, i.e., the resulting tolled user equilibrium problem have the system flows and demands in the equilibrium. Next, we identify the set of all such vectors for the GVD model. Toll set can either be utilized by the transportation planner to simulate the system optimal flows and demands in the equilibrium or in addition to simulate the effect of side constraints exactly as in the system problem.

In Chapter 3 we also give the alternative descriptions of the first best toll set. These include link based and path based formulations when link flows and path flows are known, and when the relevant path assumption is made. Finally, we give an approximate toll set and toll pricing framework for the case where the aggregate complementary constraint is infeasible.

Chapter 4 addresses the second best toll pricing theory and formulations. When a constrained first best pricing scheme is not feasible, a second best pricing scheme should be utilized to obtain the best system utilization given the constraints on tolling certain portions of the network. We propose a mathematical program with equilibrium constraints formulation for the elastic demand networks. We also provide two equivalent nonlinear formulations in link flows and path flows where the resulting formulation is a nonlinear program with nonlinear constraints. Given that the user and system problems have solutions, we prove that the SBTP formulations have a global optimal solution. Furthermore, when the demand and cost functions are linear, we observe that every stationary point is a local optima. We also derive the second best tolls for general networks analytically. In contrast to the approaches in the literature, we do not assume that all paths on the network are relevant. We show that the second best toll on a link can be expressed as a function of MSCP toll vector and all other toll vectors on the transportation network. We further show that when the relevant path assumption is made, the
derived formula reduces to those ones proposed in the literature. Given a second best solution, we identify the toll set and extend the toll pricing framework to second best pricing problems. We define some efficiency measures and demonstrate the toll pricing framework on small networks.

By testing the toll pricing framework on randomly generated nine node networks, we observe that on average, the toll pricing framework consumes less time compared to solving the SBTP problem directly. We also show that the decreasing marginal returns property is random. Finally, we extend the SBTP formulations to fixed demand traffic assignment problems. We show that for the two parallel routes example, a constrained first best toll always exists.

5.3 Proposed Research

5.3.1 Extensions to Other Traffic Assignment Problems

In the foregoing chapters, we have assumed that each individual has perfect information about the network and the link travel times for a given flow rate. However, the perfect information assumption about the conditions on the network is not very realistic. The imperfect information and the variability in perceptions results in a stochastic user equilibrium (SUE) [22, 77]. Tatineni et al. [95], Hazelton [45] and Damberg et al. [23] define the Wardrop’s first principle for the SUE as:

IN SUE, no user believes he or she can improve his or her travel time by unilaterally changing routes. In other words, users will select the route which he or she perceives to have the minimum cost.

In the toll pricing perspective, Bell [9], Damberg et al. [23], Dial [26], Johansson and Mattson [57], Prashker and Bekhor [86], Smith et al. [93] and Yang [108] show that the MSCP tolling scheme is a valid tolling first best tolling strategy.
The second class of congestion pricing models that can be investigated is the dynamic traffic assignment (DTA) models. In these models, demand and costs vary over time and thus congestion tolls need to be time varying. As we have mentioned in Chapter 2, the MSCP tolling scheme is valid for the DTA problems. One should investigate how the toll pricing framework idea can be extended to stochastic and dynamic traffic assignment models.

Other extensions include extending the GVD model to include other costs and side constraints and investigating the toll pricing theory for queuing models. For example, the GVD model does not have either a term in the objective or a side constraint to capture the effect of individual commodity flows. In addition, a case where the side constraints are not linear has not been considered.

5.3.2 Second Best Pricing Models

In Chapter 4, we have presented SBTP formulations for the elastic demand traffic assignment and fixed demand traffic assignment problems. One can use a similar approach to generalize these formulations to GVD models including capacitated traffic assignment models, combined trip choice models and combined distribution assignment models.

Unfortunately, neither the MPEC solvers nor the nonlinear solvers in the literature can handle large sized problems. Other than the bilevel programming approach, to our best knowledge, Verhoef’s algorithm [96] is the only method for solving specifically the SBTP problems using the problem structure. It is essential to design algorithms specifically tailored for solving the SBTP problems in order to handle larger sized real life toll pricing problems. In addition, computationally efficient heuristics might help to find a good feasible solution in reasonable amount of time. One should also address the existence of the KKT multipliers for the SBTP problems. A conjecture is that the relevant path assumption implies the existence of these multipliers.
In Chapter 4, we have illustrated that the decreasing marginal returns (DMR) property for SBTP problems is almost random. It might be interesting to find some special cases for which the DMR property holds (or does not hold). This might be important in especially finding the SBTP solutions which have high efficiency indexes.

5.3.3 Extensions to Computational Methods

The toll pricing schemes such as MINSYS, MINREV and MINMAX tolling schemes are linear programs, which have well-defined duals. For example, the problem of minimization of the total toll revenue (MINSYS-FD) for the fixed demand traffic assignment models has a dual which is a multicommodity maximum cost network flow problem and, when there is a single commodity, the problem has a polynomial time solution algorithm (Dial [25, 27], Hagstrom [42] and Ahuja (in an unpublished work)). We would also like to investigate if the dual formulation for other linear toll pricing problems for both fixed and variable demand models will lead to promising solution algorithms or not.

In Section 3.5, we described methods for solving moderate sized toll pricing problems by using commercial mathematical modeling languages and solvers. However when these solvers are utilized for solving the MINTB problem even for a moderate sized network, it takes a considerable amount of CPU time to find the optimal solution. This shows the difficulty of solving the MINTB problem even for a moderate sized toll pricing problem. Therefore, developing specialized algorithms for the toll pricing schemes with mixed integer formulations is also a priority. Therefore, we plan to develop heuristics that will give a “good” solution in reasonable amount of time.
APPENDIX A
SIOUX FALLS ED NETWORK

Table A.1: Upper Bounds on Total Demand Between Origin Destination Pairs.

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Table A.3: Total System Optimal Demand Between Origin Destination Pairs.

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Figure A.1: Sioux Falls Network
Table A.4: Problem Data and Solution for the Elastic Demand Sioux Falls Problem.

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APPENDIX B
VERHOEF’S ALGORITHM

We have tested Verhoef’s Algorithm on a two parallel routes example with linear demand functions and linear and nonlinear cost functions. The demand function is of the forms

\[ t = 10 - 0.5c \]

where \( c \) is the generalized cost on a given path. We assume that the \( s(v) \) is linear, the travel time functions are \( s_1(v_1) = v_1 \) and \( s_2(v_2) = 2v_2 \). In the nonlinear case, \( s_1(v_1) = v_1 \) and \( s_2(v_2) = 2v_2^4 \). Note that, we allow tolling only on link one.

In Table B.1 and Table B.2 we present the computational results including the user, system and second best optimal flows, NUB, the toll amount and total toll revenue.

Table B.1: NUB and Flows on the Two Parallel Routes Example when the Cost Functions are Linear.

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When the linear cost function case is compared with the nonlinear case, one can see that the total number of iterations required for the method to converge is smaller for the nonlinear case (54 versus 38). The stopping criteria that has been utilized is checking if \( v_l - v_{l-1} \leq 10^{-5} \) where \( l \) is the iteration counter.
Table B.2: NUB and Flows on the Two Parallel Routes Example when the Cost Functions are Nonlinear.

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The NUB of the linear travel time cost function problem is higher than the nonlinear travel time cost function problem for the system, user and second best models. Quite the contrary, the total toll charge is higher for the nonlinear case.
REFERENCES


Mehmet Bayram Yıldırım was born on January 15, 1973 in Kurtalan, Turkey. He received his high school education in Diyarbakır Anadolu Lisesi. In 1994, he was awarded a bachelor’s degree in industrial engineering from Boğaziçi University in Istanbul, Turkey. He had his master’s degree from Bilkent University in 1996. After his master’s studies, he began doctoral studies at the University of Florida, and received his Ph.D. in December 2001.