INSTRUMENTATION OF THE NEXT GENERATION GRAVITATIONAL WAVE DETECTOR: TRIPLE PENDULUM SUSPENSION AND ELECTRO-OPTIC MODULATOR

By
WAN WU

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To people who are persistent with their dreams
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LIST OF ABBREVIATIONS

AOM Acoustic optic modulator
ASC Alignment sensing and control
BS Beam splitter
BH Black hole
EOM Electro-optic modulator
ETM End test mass
FI Faraday Isolator
FSR Free spectral range
GW Gravitational Wave
IFO Interferometer
ITM Input test mass
IO Input optics
LIGO Laser interferometer gravitational wave observatory
LSC Length sensing and control
MMT Mode matching telescope
MSPRC Marginally stable power recycling cavity
NS Neutron Star
SPRC Stable power recycling cavity
PD Photo detector
PDH Pounder-Drever-Hall
PM Phase modulation
PRM Power recycling mirror
RAM Residual amplitude modulation
RF Radio frequency
RSE Resonant-sideband-extraction
SAW Surface acoustic wave
**SQL**  Standard quantum limit

**SRM**  Signal recycling mirror

**TT**  Transverse-traceless
The Laser Interferometer Gravitational Wave Observatory (LIGO) is now operating at its design sensitivity. To extend the ability to sense gravitational wave signals in a broader range with reasonable rate of detections, the Advanced LIGO has been planned and R&D for it is underway. Advanced LIGO is designed to have an order of magnitude better sensitivity then the current detector. This will be achieved by improving several technical features, including the use of triple pendulum suspension systems and an order of magnitude higher laser power.

Going from LIGO to Advanced LIGO requires new designs for some key components or subsystems. As part of the LIGO group at the University of Florida, I was involved in the research work related to two components of Advanced LIGO. These include the triple pendulum suspension system which has been designed and installed in the JIF lab at Glasgow University. My research provides a complete mechanical analysis for the triple pendulum and describe the local control system with feedback forces applied on the top mass via six electro-magnetic actuators. Also presented is the characterization measurement of the transfer function of this suspension system.

Another research project I have worked on focuses on the development and test of a novel electro-optic phase modulator (EOM), which is a key component of the input optics subsystem. The experiments include measurements of the physical properties of
the EOM crystals (e.g. piezo-resonance and optical absorption at 1064 nm wavelength), characterization of the residual amplitude modulation and investigation of the additional laser amplitude and phase noise imposed by the EOM on a 1064nm laser beam.
CHAPTER 1
INTRODUCTION

1.1 Gravitational Waves

1.1.1 Introduction

Albert Einstein described how matter and energy change the geometry of space time using his famous Einstein field equation (EFE).

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]  

(1-1)

The curved geometry, being interpreted as the gravitational field of the matter source, is characterized by the Ricci tensor \( R_{\mu\nu} \), the Ricci scalar \( R \) and the metric tensor \( g_{\mu\nu} \) in the left part of the equation. \( T_{\mu\nu} \) is the stress-energy tensor, and the constant is given in terms of \( c \) (the speed of light) and \( G \) (the gravitational constant). The EFE is used to determine the metric tensor \( g_{\mu\nu} \).

The general definition of the space-time interval in General Relativity is:

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]  

(1-2)

Because gravitational waves (GWs) are from astronomical objects very far away from us and thus very weak, they can be seen as small disturbances to the flat metric (Minkowski space), defined as

\[ \hat{g} = \hat{\eta} + \hat{h}, \]  

(1-3)

where

\[ \hat{\eta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  

(1-4)

The small disturbance \( h_{\mu\nu} \) can be simplified into a useful and simple form when the transverse-traceless (TT) gauge is applied to solve for GWs in vacuum,
General relativity (GR) predicts that GWs travel with the speed of light $c$ and can be treated as plane waves, being seen to be non-dispersive in the theoretical treatment.

The quantity $h_{\mu\nu}$ is the waveform which can be interpreted as the superposition of two orthogonal polarization states for GWs,

$$\hat{h} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx} & h_{xy} & 0 \\ 0 & h_{yx} & -h_{yy} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (1-5)$$

where $h_+ = h_{xx} = -h_{yy}$ and $h_\times = h_{xy} = h_{yx}$ are the strain amplitude of GWs with two distinct polarizations. A passing GW will impose a compressing or stretching effect on ‘free falling’ test masses depending on the phase and polarization of the wave.

For example, the effect on test particles arranged in a circular ring by GWs of two polarizations coming into the plane is shown in Figure 1-1. If $L$ is the original diameter of the ring and two GWs have equal strain amplitude $h_+ = h_\times = h$, the change of the diameter as indicated in Figure 1-1 can be given as

$$\Delta L = hL. \quad (1-7)$$

In other word, the strain amplitude of a GW can be characterized as

$$h = \frac{\Delta L}{L}. \quad (1-8)$$

Intuitively speaking, GWs are emitted when the mass distribution of a object or system changes in an oscillatory way. This is similar to the way that a change in charge
distribution creates an electrical dipole moment and thus radiates electro-magnetic waves. However, the lowest order radiation multipole that the mass distribution variation can produce is quadrupole radiation. Gravitational monopoles exist and appear as Newtonian gravity, but oscillatory changes in the monopole moment of mass distribution are forbidden due to mass conservation. And changes in the gravitational dipole moment are forbidden due to conservation of momentum. Specifically, GWs are emitted by physical objects with a changing quadrupole moment. The strain of the quadrupole radiation $h$ can be evaluated as

$$h_{\mu\nu} \approx \frac{G}{R c^4} \frac{d^2}{dt^2} \hat{Q}_{\mu\nu}. \quad (1-9)$$

Here $R$ is the distance to the source and $\hat{Q}$ is the reduced quadrupole moment tensor of the source mass, defined as

$$Q_{\mu\nu} = \int \rho(\vec{r}) \left( x_\mu x_\nu - \frac{1}{3} \delta_{\mu\nu} r^2 \right) dV, \quad (1-10)$$

where $\rho(r)$ is the mass density, the $r$ is the distance from the center of the mass, and the integral is over the volume of the source. The amplitude of the strain is very small, which makes the direct detection of GWs a very challenging task. Take a typical coalescing
binary neutron star system to be about 20 Mpc away and have a mass on the order of that of the sun. The resulting amplitude strain on earth is of the order $10^{-21}$.

1.1.2 Gravitational Wave Sources

GW sources can be classified as four types according to the frequencies of the radiation [1]. The first type of sources emit waves in the ‘extremely low’ frequency band ($10^{-15}$ to $10^{-18}$ Hz), and it is believed that initial density perturbations associated with inflation of the universe introduce gravitational-field perturbation in this frequency band. The second type of sources are ‘very low’ frequency ones ($10^{-7}$ to $10^{-9}$ Hz). The expected sources include super-massive binary black hole (BH) systems. The third type sources, the ‘low’ frequency ones ($10^{-6}$ to 1 Hz), include the inspiral of the stellar-mass binary objects, short period stellar-mass binaries, the merger of super-massive binaries, the gravitational collapse of super-massive stars, and a cosmic gravitational wave background. And sources of the fourth type, the ‘high’ frequency sources (1 Hz to 10 KHz), consist of the merger of stellar-mass binaries, supernovae, a stochastic background and non-axisymmetric spinning neutron stars (NSs).

In general, heavier sources correspond to lower frequencies. And the same physical objects may generate GW radiations at different frequencies as they evolve. For example, the inspiral of a stellar-mass binary at early stage radiates in the ‘low’ frequency band. The radiation period decreases as two components of the binary approaches each other. Hence the GW radiation frequency increases gradually. In the last few seconds of the binary’s inspiral, the frequency has already been in the ‘high’ frequency band. The frequency of the GW radiation keeps on increasing as the binary merges and rings down.

1.1.3 Detection of Gravitational Waves

Although the existence of gravitational waves has been predicted since the early twentieth century, there were no experimental efforts to prove it until in the 1960’s when Weber suggested to use resonant bars to sense the passing GWs and carried out subsequent experimental work [2]. In 1974, Russell Hulse and Joseph Taylor provide
substantial evidence for the existence of GWs when they discovered a binary pulsar (known as PSR 1913+16) with a decreasing orbit. The observed orbital decay was in agreement with the prediction by the theory of GR that the change of the orbit is due to the energy loss through the emission of gravitational waves \([3, 4]\). However, this indirect detection of the GWs generated by a binary pulsar did not provide us the information carried by the waves themselves. Hence the direct measurement of the GW strain \(h(t)\) is needed. Direct detection of GWs will open up a way for examining the theoretical prediction of \(h(t)\), testing the astrophysical models, and characterizing the astronomical systems that generate GWs.

Several techniques for measuring GWs directly, besides the resonant bar detectors, have been proposed or implemented. These techniques include spacecraft Doppler-tracking \([5]\), pulsar timing searches \([6]\), ground-based Michelson laser interferometric detectors \([7–10]\), and the laser interferometer space antenna (LISA) \([11]\). The ground-based laser interferometric detectors, as the most promising GW detectors that are currently in operation, are going to be discussed in Section 1.2.

1.2 Gravitational Wave Detection Using Michelson Interferometric Detectors

A simple laser Michelson interferometer is shown in Figure 1-2. An ideal laser interferometric gravitational wave detector is built up by mirrors which are free of forces. The laser beam is split into two beams at the beam splitter (BS). The distance between the BS and the end mirrors (M), the arm length, can be altered by a passing GW. The laser interferometer defines its own coordinates which are different from the TT coordinates. In this coordinate system, the \(x\) and \(y\) axes are determined by the directions of the two arms with the \(z\) axis pointing perpendicularly to them (see Figure 1-2).

It is the light that is used as the ruler to measure the change of the arm length, which is the signature by the passing GWs on the interferometer. We know that the \(proper\)
length between two space-time events linked by a light is zero, which can be expressed as

\[ ds = \int \sqrt{\left| g'_{\mu \nu} dx^\mu dx^\nu \right|} = 0, \]  

(1–11)

where \( g'_{\mu \nu} \) is the metric tensor defined in the coordinates associated with the detector. For light in one of the arms (e.g. the arm along the x-axis),

\[ -c^2 dt^2 + (1 + h'_{xx}) dx^2 = 0, \]  

(1–12)

where \( h'_{xx} \) is the strain along the x-arm of the interferometer. The distance that light travels from the BS to the end of the x-arm and back can be calculated as

\[ 2L_x = \int_0^{L_x} dx - \int_{L_x}^0 dx = \int_{t-\tau_x}^t \frac{c}{\sqrt{1 + h'_{xx}(t)}} dt' \approx \int_{t-\tau_x}^t c \left( 1 - \frac{1}{2} h'_{xx}(t) \right) dt', \]  

(1–13)
where $L_x$ is the length of the x-arm and $\tau_x$ is the round-trip travel time. If we can approximate the waveform $h'_{xx}$ as

$$h'_{xx}(t) = h_0 \cos(\omega_g t),$$  \hspace{1cm} (1-14)$$

where $\omega_g$ is the frequency of a GW signal, Equation 1–13 can then be written as

$$\frac{2L_x}{c} = \int_{t-\tau_x}^{t} \left(1 - \frac{1}{2}h_0 \cos(\omega_g t)\right) dt' = \tau_x - h_0 \frac{\sin(\omega_g t) - \sin(\omega_g (t - \tau_x))}{2\omega_g}. \hspace{1cm} (1-15)$$

Equation 1–15 can be rewritten as

$$\tau_x = \frac{2L_x}{c} + h_0 \frac{\sin(\omega_g t) - \sin(\omega_g (t + \tau_x))}{2\omega_g} = \frac{2L_x}{c} + h_0 \frac{\sin(\frac{1}{2}\omega_g \tau_x)}{\omega_g} \cos(\omega_g t - \frac{1}{2}\omega_g \tau_x). \hspace{1cm} (1-16)$$

Therefore, the phase that light accumulates during a round trip in the x-arm is

$$\phi_x(t) = \omega_0 \tau_x = \omega_0 \frac{2L_x}{c} + \omega_0 h_0 \frac{\sin\left(\frac{1}{2}\omega_g \tau_x\right)}{\omega_g} \cos\left(\omega_g t - \frac{1}{2}\omega_g \tau_x\right). \hspace{1cm} (1-17)$$

A similar expression can be written for the light travels in the y-arm. This is given as

$$\phi_y(t) = \omega_0 \tau_y = \omega_0 \frac{2L_y}{c} - \omega_0 h_0 \frac{\sin\left(\frac{1}{2}\omega_g \tau_y\right)}{\omega_g} \cos\left(\omega_g t + \frac{1}{2}\omega_g \tau_y\right). \hspace{1cm} (1-18)$$

The motion of the end mirrors in each arms introduced by GWs will be in a differential mode. This introduces variations of the phase difference $\Delta \phi = \phi_x - \phi_y$ between the two beams when they are recombined at the BS, changing the interference pattern at the dark port. This change can be precisely sensed by a photodetector (PD). Thus the information of GWs is encoded in the readout signal from the PD.

When the round trip time is short compared to the GW period such that $\omega_g \tau_x \ll 1$ and $\omega_g \tau_y \ll 1$, the phase difference $\Delta \phi$ becomes

$$\Delta \phi \approx 2\omega_0 \frac{L_x - L_y}{c} + \frac{1}{2} \omega_0 h_0 (\tau_x + \tau_y) \cos \omega_g t \approx 2\omega_0 \frac{L_0}{c} (L_x - L_y) + \frac{\omega_0}{c} (L_x + L_y) h_0 \cos \omega_g t.$$ \hspace{1cm} (1–19)
In this case, the signal simply scales with the length of the arms. However, if the length of the arms become too long such that

$$\omega_g \tau_i = \omega_g \frac{2L_i}{c} \geq \frac{\pi}{2},$$  \hspace{1cm} (1–20)

where \( i = x, y \), \( \phi_x \) will decrease as \( L_x \) increases while \( \phi_y \) will increase with \( L_y \), as can be seen from Equation 1–17 and Equation 1–18. Hence the signal \( \Delta \phi \) will degrade. So there is a limit on the arm length, which is

$$L_i = \frac{\pi c}{4\omega_g}.$$  \hspace{1cm} (1–21)

Once the length of the arms become longer than the limit, the GW strain starts to oscillate as the light propagates, shifting the round trip phase back towards zero. This deteriorates the response of the interferometer to GWs.

The coordinate defined by the interferometer and the TT coordinate can be related by a rotational transformation,

$$\vec{x} = R^T \vec{x}_T,$$  \hspace{1cm} (1–22)

where \( \vec{x}_T \) represents the coordinates associated with the TT gauge and \( \vec{x} \) represents the coordinates associated with the detector. The rotational transformation induces the transformation of the metric. The strain tensor in the detector frame \( h' \), can be found from the TT tensor \( h \) by means of the induced transformation,

$$h' = R^T h R.$$  \hspace{1cm} (1–23)

Euler angles are used to specify the orientation of the gravitational-wave frame with respect to the detector frame,

$$R^T = R_z(\varphi)R_y(\theta)R_z(\phi),$$  \hspace{1cm} (1–24)

where \( \varphi, \theta, \phi \) are the angles defined as shown in Figure 1-2. The rotations around the \( x \), \( y \), and \( z \) axes are thus given by
These rotations are the transformations within 3-dimensional space. The strain tensor $h'$ can thus be represented by a $3 \times 3$ matrix. Hence we can describe $\Delta \phi$ as the sum of two parts each originating from an independent polarization,

$$\Delta \phi = F_+ h_+ + F_x h_x.$$  (1–28)

The information of the strain amplitude $h_+, h_x$ is thus encoded in $\Delta \phi$. $F_+$ and $F_x$ are called the ‘beam pattern.’ Their explicit expressions are

$$F_+ = \frac{1}{2} (1 + \cos^2 \theta) \cos 2\phi \cos 2\varphi - \cos \theta \sin 2\phi \sin 2\varphi,$$  (1–29)

$$F_x = \frac{1}{2} (1 + \cos^2 \theta) \cos 2\phi \sin 2\varphi + \cos \theta \sin 2\phi \cos 2\varphi.$$  (1–30)

$\varphi$ is the polarization angle of the wave. $\theta$ and $\phi$ keep on changing as the earth orbits around the sun and rotates daily. This motion also generates a Doppler effect of the gravitational wave signals. The beam patterns are nearly omnidirectional except for certain orientations. Hence the gravitational wave interferometers are sensitive to a very large area of the sky at all times.
1.3 Signals For The Ground Based Interferometric Detectors

Ground based laser interferometric gravitational wave detectors are expected to detect GWs in the frequency range of a few tens of Hz to several KHz. These GW signals can be summarized into four categories.

1.3.1 Chirp Signals

Chirp signals are emitted by coalescing binaries. A compact binary system can consist of two compact objects (NS/NS, NS/BH, BH/BH) orbiting around their common center of mass. Two binary components may be widely separated and move along an eccentric orbit when the binary system is formed. A coalescing binary system can be characterized by the masses radial separation, the inclination of the orbit relative to the plane of the sky, the starting orbital phase, and eccentricity of the two orbiting bodies. Gravitational radiation causes the system to lose energy so that the stars will draw closer to each other and the orbit circularizes eventually. This effect of drawing closer is called an inspiral. The orbiting period decreases as the orbit shrinks. The GW frequency, which is twice the orbiting frequency, moves from the low frequency region to the high frequency region, sweeping through the detection bandwidth of the ground based interferometric detectors eventually. The amplitude of gravitational radiation also increases as the orbit shrinks, as can be seen from Equation 1–9.

The waveforms of chirp signals from coalescing binaries have been known to a high degree of precision. Hence the classical matched filtering technique can be applied on the output data of the detectors to enable a reliable detection. However, only a few Coalescing NS/NS systems have been observed while no NS/BH or BH/BH have yet been discovered, which implies that the distribution density of coalescing binaries in the universe could be small. Hence the detection rate of those signals using GW detectors is very uncertain.

1.3.2 Periodic Signals

The most important sources of continuous periodic gravitational waves are pulsars. Pulsars are highly magnetized NSs which rapidly rotates in a non-axisymmetric way. A
pulsar emits a beam of electromagnetic radiation along its magnetic axis. If the magnetic axis is offset from the rotational axis, the electromagnetic beam will sweep out circular paths as the star rotates. The radiation can only be detected by the ground based radio telescopes when the beam points towards the Earth. The rotation period of pulsars are very stable. The regularity of the time interval between observed electromagnetic radiation from some pulsars could be as precise as an atomic clock.

The characteristic amplitude of gravitational waves from pulsars can be given as

$$h_c = \frac{16\pi^2 G f^2}{c^4 r} I \varepsilon,$$

where $I$ is the moment of inertia of the pulsar around the rotation axis, $f$ is the GW frequency, $\varepsilon$, the equatorial ellipticity, is a measure of the asymmetry of the pulsar, and $r$ is the distance between the observer and the source. For example, The characteristic frequency of gravitational wave generated by the Crab pulsar is 59.6 Hz, an upper limit of strain amplitude of the GW signal from the Crab pulsar is calculated to be $h \approx 10^{-24}$ with $\varepsilon \approx 7 \times 10^{-4}$ and $r = 1.8 \text{kpc}$. Weak periodic signals like this will be buried in the noise of the output signals of the GW detectors. However, the signal to noise ratio can be increased by integrating the output signal over a long observation time.

The detection of periodic GW signals will be realized by applying fast Fourier transform (FFT) analysis on the detector output and construct the power spectrum. If we could keep on searching the sky with long enough observation time $T$, the signals with nearly fixed frequencies peaks will show up in the power spectrum since the signal to noise ratio grows as $T$. The statistically important peaks in the spectrum could be associated with these periodic GW signals. However, such data analysis algorithm just imposes a big computational burden on the signal processing procedure. This is because that the integration required time in order to achieve a reasonable signal to noise ratio will likely be so long that a huge number of data points need to be handled in the FFT
analysis. The required processing speed goes beyond the limits of even the most powerful computers.

Pulsars can lose rotational energy by electromagnetic radiation, the emission of particles, and gravitational wave dissipation. That is why the rotational frequency varies over a time scale in the order of the pulsar age, which is called the ‘spindown’. This leads to the frequency drift of the GW signals. The other mechanism that causes the frequency change of the observed GWs from a pulsar is the Doppler shift. Since the ground detectors are carried by the spinning Earth orbiting the Sun, a monochromatic signal in the source reference will be Doppler modulated by the motion of the detectors. The frequency modulation due to the Earth’s orbital and rotational motion are expected to be seen in the pulsar signals. These frequency characteristics can be utilized to extract pulsar signals from the data obtained using GW detectors. Moreover, the changing orientation of the detectors with respect to the pulsar sources due to the motion of the Earth will change the ‘beam pattern’ mentioned in Section 1.2, imposing amplitude and phase modulation on the GW signals. The phase and frequency modulation of the signals broaden the spectral lines in the power spectrum and spread the power into the frequency bins around the signal frequencies. This just attenuates the amplitude of the main signal components and thus imposed an additional demand for increasing the integration time. A solution to this problem is to build a new time coordinate so that frequency of the signals remains fixed. Unfortunately, the construction of the new time coordinates means more data analysis work. The situation becomes extremely worse in the case of searching for sources whose position and frequencies are not well known since the all-sky and all-frequency search will be needed.

Hence searching periodic signals is believed to be far more complicated than that for chirp signals.
1.3.3 Burst Signals

GW bursts originate from mergers of stellar mass binaries, supernovae, the gamma-ray burst engines and other unknown sources.

A supernova is an explosion of a star which creates an extremely luminous object. This results in a burst of radiation in the electromagnetic domain which emits as much energy as the Sun would emit over about 10 billion years. Although only a small fraction of energy is converted into GWs, it could be big enough to make supernovae important sources for ground based interferometric detectors. Supernovae is believed to be triggered in two ways. In the first case, a white dwarf star in a binary system may accumulate sufficient material from a stellar companion. As the white dwarf approaches the Chandrasekhar limit of roughly 1.4 times the mass of the Sun, it will collapse, triggering a stellar explosion. In the second case, the iron core of a massive star ceases to create nuclear fusion energy to resist the gravity. The resulting sudden collapse of the star will generate a NS or BH, releasing gravitational potential energy that heats and expels the star’s outer layers. The collapse must be non-axial to produce the change in the quadruple momentum in order to generate GWs. The waveform of supernovae signals are not clearly understood. People currently believed that the current sensitivity of ground based interferometric detectors can only allow us to detect GWs from galactic supernovae and the signals from the extra-galactic sources would be too weak.

Gamma-ray bursts (GRBs) are intense flashes of gamma-rays which last from milliseconds to many minutes, coming from random locations in the sky. GRBs can be classified based on their duration as two types: The ‘short’ GRBs have a life time less than 2 seconds while the ‘long’ ones last longer than 2 seconds. The present consensus is that GRB emission is associated with BH formation processes such as hypernovae, collapsars and compact binary mergers. Since collapsars and compact binary mergers are both good GW sources detectable by ground based detectors, we have good reasons to take GRBs as important targets of the ground based detectors as well.
Since cosmic gamma-ray bursts were first discovered in the late 1960s by the US Vela nuclear test detection satellites [12], large number of GRB events have been observed. The detectable event rate could be as high as once per day. This is very useful in data analysis using statistical methods.

The waveforms of gravitational waves from burst sources are poorly known. The cross-correlation analysis using data from more than one GW detectors is needed to extract signals from instrumental noise, assuming there are burst signals with enough strength in the sensitive band of detectors. And benefiting from the non-GW observations of GRBs, we can use externally trigger methods to reduce the detection threshold for GRB signals.

1.3.4 Stochastic Signals

The stochastic background of GWs are expected to be either generated from the very early universe or a result of the superposition of signals from many unresolved astrophysical sources. If stochastic signals can be detected using GW observatories, we might be able to extend our view of the early universe to a time as early as \(10^{-22}\) seconds after the big bang. In comparison, the measurement of the cosmic microwave background radiation can only provide us the information of the universe about \(10^5\) years after the big bang.

If the stochastic GW background is isotropic, stationary and Gaussian, its spectrum can then be well characterized by a quantity \(\Omega_{GW}(f)\) which is defined as

\[
\Omega_{GW}(f) = \frac{1}{\rho_c} \frac{d\rho_{GW}}{d\ln f}, \tag{1-32}
\]

where \(\rho_c\) is the critical energy density to close the universe, \(\rho_{GW}\) is the GW energy density.

Like what we did to acquire burst signals from the outputs of more than one detectors, the cross-correlation analysis is also demanded to search for stochastic signals. The basic idea is to multiply together the outputs from two detectors and integrate them. Although the waveforms of stochastic signals are not clear, their event rate might be much
higher than that for GW signals of other types. Hence the data analysis for a stochastic background could possibly be done with much shorter integration time.

1.4 Noise In The Ground based Interferometric Detectors

The sensitivity of an interferometric detector is limited by several noise sources. Principal noise sources can be divided into two types: the displacement noise and the sensing noise. Displacement noise could prevent us from building free-falling test masses and sensing noise could limit our ability to measure the signature of GWs on the interferometer, the differential arm length. Principal displacement noise include seismic noise, thermal noise, radiation pressure noise, residual gas noise and gravity gradient noise. Sensing noise include shot noise and optical noise.

Ground based interferometers will be inevitably affected by seismically driven vibrations. Suspension isolation systems reduce the seismic noise in the measurement bandwidth. An ideal isolation system is expected to suppress the seismic noise while longitudinal motions of mirrors in the GW bandwidth are free of damping. However, the unavoidable seismic wall at low frequencies prevents current ground based interferometers from sensing GW signals from sources such as the mergers of supermassive black holes. LISA will be a good complementarity in this regard.

Thermal noise originates through two mechanisms. First, the thermal motion of the suspension system causes the mirrors to move. Second, the thermally excited vibration of the mirror deforms the mirror surfaces, i.e., changes the arm length of the interferometer. The thermal noise can be characterized through the indirect measurement of the mechanical loss of the system. The loss can be reduced by improving several aspects of the system such as providing better coatings for the test mass, utilizing fiber suspension wires and implementing cryogenic cooling devices.

The temperature gradient in the chamber housing the interferometer could produce differential pressure on the suspended masses via the residual gas molecules. Thus the thermally introduced motion of the residual gas molecules leads to the displacement of test
masses. Moreover, the residual gas molecules in the beam path of the laser could introduce optical path length fluctuations in the interferometer [13]. So the residual molecule noise could also be viewed as a sensing noise. Since the ground based interferometric GW detectors are all placed in vacuum chambers, the effect due to residual molecules is considered to be trivial.

The ultimate limit on displacement noise is the gravity gradient noise – noise due to fluctuating Newtonian gravitational forces that induce motions in the test masses. An important gravity gradient noise source is the fluctuating density of the earth near each of the interferometers test masses. Atmospheric fluctuations could also change the gravity gradient, but the effect is much weaker than that from earth motions. Gravity gradient noise due to moving bodies or objects can be more serious than seismic gravity-gradient noise if such bodies or objects are not kept at an adequate distance from the test masses. The gravity gradient noise is negligible in the current existing ground based interferometric detectors. However, advanced interferometers with higher sensitivities could be limited by gravity gradient noise at frequencies below 10 Hz.

Light inside the interferometer exerts radiation pressure forces on the test masses. Variations of the light intensity introduces fluctuations in the forces, causing vibrations of the masses. This is known as the radiation pressure noise. The anti-correlated motion of the masses can be calibrated in strain sensitivity $h$ as [15]

$$h_{rp} (f) = \frac{1}{m f^2 L} \sqrt{\frac{\hbar P_{in}}{2\pi^3 c \lambda}},$$

where $P_{in}$ is the optical power in the arm cavity, $L$ is the length of the arm cavity, $\lambda$ is the wavelength of the laser, and $\hbar$ is the Planck constant. $h_{rp}$ increases with the optical power.

Shot noise, also known as the ‘photon-counting error’, is the fundamental limit to the measurement of the optical power. It is the statistical fluctuation in the number of the photons measured by a photo detector causes fluctuations in the amplitude of the signal. The signal to shot noise ratio increases with the root of the power. The contribution of
shot noise in the output signal, calibrated in the strain sensitivity $h$, can be calculated as

$$ h_{\text{shot}}(f) = \frac{1}{L} \sqrt{\frac{hc\lambda}{2\pi P_{in}}} \quad (1-34) $$

Radiation pressure noise and shot noise are also called Quantum noise [14]. The total quantum noise is the quadrature sum of the shot and radiation pressure noise. Minimizing the quantum noise with respect to the power yields a minimum noise level which is close to the standard quantum limit (SQL) [16],

$$ h_{\text{SQL}}(f) = \sqrt{\frac{8\hbar}{(2\pi f)^2 mL^2}} \quad (1-35) $$

Here, $m$ is the mass of each mirror, and $L$ is the length of the interferometer arms.

It is noteworthy to mention that the radiation pressure noise we described above is due to the uncorrelated power fluctuation in two arms that is generated by the quantum uncertainty. Technical radiation pressure noise which originates from the fluctuation of the laser power is not related to the SQL.

Amplitude and phase change of the laser field are common noise sources in a perfectly balanced interferometer. However, when the two arms of the interferometer are not perfectly identical in terms of having different losses and mirror reflectivities, then the amplitude and phase changes of the field in each arm are no longer identical. This shows up as differential displacement noise and can not be distinguished from a GW signal.

1.5 LIGO & Advanced LIGO

1.5.1 Optical Configuration

The laser interferometer gravitational wave observatory (LIGO) operates three ground based Michelson interferometric detectors. The two interferometers in Hanford, WA have 2 km and 4 km arms respectively and the one in Livingston, LA has arms of 4 km in length. These interferometers use Fabry-Perot cavities to increase the effective length of the arms and a power recycling mirror to enhance the optical power circulating inside the interferometer (see Figure 1-3). The optical configuration of LIGO is plotted
Figure 1-3. Optical configuration of the LIGO detector. BS - beam splitter, PBS - polarizing beam splitter, EOM - electro-optic modulator, ITM - input test mass, ETM - end test mass, PRM - power recycling mirror, AOM - acoustic optical modulator.
in Figure 1-3. A small fraction of the light from a Nd:YAG laser is split away from the main beam, transmits a polarizing beam splitter (PBS) and passes an acoustic-optic modulator (AOM) and a $\lambda/4$ plate. The light is reflected back by a mirror placed after the $\lambda/4$ plate, passes through the $\lambda/4$ plate and the AOM again. The double pass of light through the $\lambda/4$ plate rotates the polarization of the light to be orthogonal to the incoming light’s polarization. Hence the light is reflected by the PBS, passes through an EOM and goes to a reference cavity. A $\lambda/4$ plate and a PBS are used again to send the light reflected from the reference cavity to a photodetector. The signal from the photodetector is demodulated to obtain the error signal that is used to adjust the frequency of the laser. This is a typical application of using the Pound-Drever-Hall technique \[17\] to stabilize the frequency of a laser. The AOM serves as a frequency offset between the laser frequency and the resonant frequency defined by the reference cavity. This way, the laser frequency can be tuned/controlled by changing the frequency offset, while the high frequency stability is ensured. The EOM after the beam splitter (BS) is used to lock the pre-mode cleaner. The main laser beam is spatially filtered by the pre-mode cleaner first. The transmitted light is phase modulated by two EOMs placed in series to generate two pairs of RF sidebands around the carrier field for the length sensing and control of the mode cleaner and the interferometer. Both carrier and RF sidebands are spatially filtered again by the mode cleaner before being sent to the interferometer.

The carrier field is kept resonant inside the power recycling cavity formed by the power recycling mirror (PRM) and the input test masses (ITMs). In addition, the carrier light must be held resonant inside the two Fabry-Perot arm cavities while RF sidebands are reflected by the ITMs.

To meet the resonance conditions for the cavities and the dark fringe condition for the Michelson interferometer, the lengths of this optical configuration should be controlled. It is convenient to describe the two arm cavities by use of the common length, $L_+ = L_1 + L_2$, and the differential length, $L_- = L_1 - L_2$. These two lengths, plus the
length of the power recycling cavity, \( l_+ = l_p + \frac{1}{2}(l_1 + l_2) \) and the path length difference in the interferometer, \( l_- = l_1 - l_2 \) define the four longitudinal degrees of freedom in a power recycling interferometer such as LIGO. It is the arm length difference \( L_- \) that needs to be sensed with the most stringent requirement since the variation of \( L_- \) is directly sensed as the GW signals.

The whole interferometer is controlled via the length sensing and control (LSC) subsystem. The LSC of the interferometer evolves from the Pound-Drever-Hall feedback control technique. The modulated laser field reflects at and transmits through the interferometer. The amplitude and the phase relationship between various frequency components of the field change as a function of positions of the mirrors that the interferometer is comprised of. The change is sensed by photo detectors placed at various locations in the interferometer. The signals from these photo detectors are demodulated at specified frequencies to create the error signals used in the feedback control loops.

LIGO has achieved its design goal with peak strain sensitivity of \( h(f) \approx 3 \times 10^{-23}/\sqrt{Hz} \) at 150 Hz and \( h_{rms} \approx 10^{-21} \) in a 100 Hz band. Advanced LIGO is proposed to have an order of magnitude better strain sensitivity which increases the observational range by a factor of 10. This means an increment of the event rates by 3 orders of magnitude since the volume of space that the detector can see grows as the cube of the distance.

The research work towards an upgrade from LIGO to Advanced LIGO has been carried out. Besides the development of better materials and suspension systems to reduce the thermal and seismic noise, the application of a high power laser to increase the signal to shot noise ratio and the modification of the optical configuration have been proposed.

Compared with the power-recycled scheme in LIGO, the resonant-sideband-extraction, or RSE topology in Advanced LIGO (see Figure 1-4) improves the performance of the laser interferometer by adding a signal extraction mirror. The new configuration allows very high power to build up in its arms with relatively low power at BS. RSE, which was
first proposed by Mizuno [18], is more practical in Advanced LIGO as compared with the dual-recycled topology (power recycling and signal recycling) proposed by Meers [19], which has the advantage of recycling the signal due to the differential motion of the arm cavities. As an analog to the way Fabry-Perot cavities optimize the time of interaction of light with the gravitational wave signal by enlarging the light storage time, the signal recycling cavity increases the ‘signal storage time’ by keeping the carrier resonant inside the cavity formed by the front mirrors of the two arm cavities and the signal-recycling mirror. Although the signal-recycling configuration was demonstrated to considerably improve the performance of laser interferometric GW detectors [20], the storage time for the signal sidebands must be kept short enough to achieve a desired detection bandwidth. The storage time limit prevents us from using high-finesse arm cavities in the interferometer. Thus a high power laser incident on the ITMs becomes necessary in order to increase the shot noise limited signal to noise ratio. This results in
serious thermal lensing in the BS which is exceedingly difficult to deal with. The signal extraction mirror is positioned to keep the light anti-resonant inside the signal extraction cavity. It serves to decrease the storage time for the signal sidebands so that high-finessee arm cavities can be used without deteriorating the detection bandwidth. The signal extraction cavity can be detuned to achieve an optimized sensitivity at various frequencies. This is useful in searching for gravitational wave signals from known narrow band sources.

The introduction of the signal extraction mirror in Advanced LIGO greatly enhances the complexity of the length sensing and control system. The signal recycling mirror and two ITMs form the signal recycling cavity whose length, $l_s = l_{sr} + \frac{1}{2}(l_1 + l_2)$, has to be controlled as well as the length in the other five degrees of freedom.

Since the carrier light will not be transferred to the asymmetric port effectively, the control of the length of the signal recycling cavity requires an additional sideband so that beat signals between two pairs of sidebands can be used to monitor the variation of the signal recycling cavity. Length sensing and control in dual-recycling laser interferometer has been studied through a number of table-top and prototype experiments [21–24]. However, the final control scheme for Advanced LIGO is still to be determined.

It has been proposed to place an output mode cleaner at the asymmetric port of the interferometer in Advanced LIGO. The output mode cleaner serves to filter out the light in high order modes which increases the shot noise on the photo detectors does not contribute to the detection of GW signals.

1.5.2 Readout Scheme

There are two sensing schemes, DC and RF, to extract GW signals encoded in the optical signal detected by the photo detector at the detection port. A GW signal with certain frequency $f$ can be seen as a signal sideband around the carrier light. The amplitude and phase information of the GW signal can be obtained by demodulating beat notes between the signal sideband and RF sidebands using RF oscillators. RF readout scheme is currently used in LIGO. The problem associated with this heterodyne detection
scheme is the coupling of the phase noise of RF oscillators to the readout signal. The coupling will be enhanced when RF sidebands at the detection port of the interferometer become unbalanced, which is the case in an interferometric gravitational detector with RSE configuration such as Advanced LIGO.

Advanced LIGO will use the DC sensing scheme, in which the carrier light emerging from the detection port is used as the local oscillator. As compared with the RF readout scheme, the DC readout scheme also has several practical advantages: the optical noise couplings are smaller with a DC readout; the readout electronics will be much simpler and no RF sidebands emerge from the detection port so that the photo detector will not be saturated when the laser power is scaled to high levels [25]. The disadvantage of DC sensing is its subjection to many low frequency noise sources.

1.6 Overview Of The Thesis

As part of the Input Optics group of LIGO at the University of Florida, I has been involved in instrumentation work for Advanced LIGO. I have mainly worked on two projects. The first project is about the characterization of a triple pendulum suspension system. The details of the mechanical analysis of the triple pendulum, which was designed by the group at Glasgow University, are given in Chapter 2. Also described in Chapter 2 are the local control scheme of the pendulum system and some results from the closed-loop transfer function measurement that was done in the JIF lab at Glasgow University. The second one is the project of developing a novel RTP crystal electro-optic modulator to meet the requirements of Advanced LIGO. The design and characterization of RTP crystal based EOMs are presented in Chapter 3. The summary and the discussion of the future work will be given in Chapter 4.
CHAPTER 2
TRIPLE PENDULUM SUSPENSION SYSTEM

2.1 Seismic Isolation Suspension Systems

The seismic isolation systems in gravitational-wave detectors isolate the test masses from the seismic vibrations of the ground. This ensures that test masses are as much as possible in ‘free fall’.

The simplest idea of the suspension isolation is to put the test masses on the ‘stacks’. Figure 2-1 shows a simple stack consists of a supporting plate which is held with springs. The relationship between the ground motion, \( x_0 \), and the displacement of the mass, \( x_1 \), can be expressed as:

\[
m \ddot{x}_1 = -k (x_1 - x_0),
\]

where \( m \) represents the mass and \( k \) is the spring constant. The Laplace transform gives the transfer function between \( x_1 \) and \( x_0 \):

\[
\frac{x_1}{x_0} = \frac{\omega_0^2}{s^2 + \omega_0^2}
\]

where \( s = j\omega \), \( \omega \) is the angular frequency and \( \omega_0 = 2\pi f_0 = 2\pi \sqrt{\frac{k}{m}} \) is the resonance frequency of the spring.

If a viscous damping force is introduced with a damping constant \( b \), the transfer function can be rewritten as:

\[
\frac{x_1}{x_0} = \frac{\omega_0^2 + \left( \frac{b}{m} \right) s}{\omega_0^2 + s^2 + \left( \frac{b}{m} \right) s}.
\]

The transfer function of the isolation system (see Figure 2-2) is similar to a low-pass filter so that the seismic noise above the resonance frequency will be effectively filtered out. The dissipation of the system can be defined by the quality factor \( Q \), which is given by \( Q = \frac{\omega_0 m}{b} \). The fall off of the transfer function is proportional to \( 1/f^2 \) from \( f_0 \) to \( Qf_0 \) and becomes \( 1/f \) at higher frequencies.
Figure 2-1. A simple spring stack and a pendulum.

Figure 2-2. Transfer function of a simple spring stack.
The other way to improve the seismic isolation is to increase the slope of the transfer function by implementing multiple suspension layers. The isolation will fall off as $1/f^{2n}$ where $n$ is the number of layers. In principle, the attenuation of the seismic noise increases with the number of layers. However, the complexity of the system prevents us from increasing the layers without limit. The cross coupling of the horizontal motion to the vertical motion due to the imperfect symmetry of the mechanical system will become worse as more layers are implemented.

A spring stack can provide good isolations along the vertical direction. Another example of the suspension isolation is a pendulum (also see Figure 2-1). Pendulum suspension systems can be used to isolate the test mass from being affected by the horizontal seismic noise. A simple model for the pendulum is made of a mass which is suspended on a wire of length $l$. Supposing the mass of the wire is negligible and the damping constant is the same constant $b$ that we mentioned above, the same transfer function can be derived as that for a simple spring stack except that the resonant frequency is now given by $\omega_0 = 2\pi \sqrt{g/l}$ in the horizontal direction.

### 2.2 Introduction to the Triple Pendulum Suspension

The suspension system of Advanced LIGO is based on the triple pendulum designed for GEO600 - the German/UK detector. As shown in Figure 2-3, this suspension system consists of a two-layer isolation stack, two cantilever springs which are attached to the top of the stack, and a triple pendulum. The upper mass of the pendulum is suspended from the springs with two wires and the double pendulum consisting of the intermediate mass and the lower mass are suspended from extra four cantilever springs with four wires. The intermediate mass and the lower mass are also connected with four wires. The intermediate mass and the lower mass are equal, in order to ensure a monolithic response to disturbances. The test mass needs to be controlled in order to lock the interferometer. A reaction pendulum is hence installed parallel to the pendulum which holds the test mass. Integrated Optical position Sensor/ElectroMagnetic drivers (OSEMs) are mounted
Figure 2-3. Three dimensional view of the triple pendulum suspension system.
Figure 2-4. Schematic view of the triple pendulum. The stacks and the damping arm have been omitted for clarity. This a modified drawing which originates from [26].
on the pendulum with the flag magnet on the test mass and the coil on the reaction mass.
The motion of the test mass leads to the change in the amount of light which shines on the photo detector, which introduces the variation of electric current in the coil (see Figure 2-4). This, in turn, adjusts the feedback force on the magnet.

Details of the triple pendulum suspension system such as the design of the two-layer isolation stack, the design of the cantilever spring, and the active isolation which includes both the feedback and the feed-forward control techniques have been given in the theses of Torrie [26] and Husman [27]. They each built a mechanical model to give a complete characterization of the mode frequencies and the dynamic response of the single pendulum, using standard vector analysis and the Lagrangian analysis respectively. The mechanical analysis is extended to the triple pendulum. However, the details about the triple pendulum model for all degrees of freedom (DOF) are not completely outlined. In the following sections, a complete analysis of the triple pendula which are installed in the JIF lab at Glasgow University is carried out. In addition, a Matlab model is constructed with the active control on the top mass being included. This model considers some differences between the pendulum in the JIF lab and the pendulum installed for GEO 600. It will be a good supplement to the two triple pendulum models that are mentioned above.

2.3 Mechanical Analysis of the Triple Pendulum

2.3.1 Variables and Parameters

Rigid bodies can move with six DOF in three dimensional space. We can name these DOF as:

(i) Longitudinal motion $x$, along the $\hat{x}$-axis,
(ii) Sideways motion $y$, along $\hat{y}$-axis,
(iii) Vertical motion $z$, along the $\hat{z}$-axis
(iv) Roll $\psi$, about the $\hat{x}$-axis,
(v) Pitch $\theta$, about the $\hat{z}$-axis,
(vi) Yaw $\phi$, about the $\hat{y}$-axis.
Hence the triple pendulum has 18 degrees of freedom and therefore 18 mode frequencies. The differential equations of motion of the triple pendulum for all degrees of freedom are derived here. The control forces on the top mass are applied via six magnets, which are included in the equations. The global control on the intermediate mass and the lower test mass has been planned, but is not considered here. The parameters as specified in Figure 2-7 used are listed below:

(i) Separation of wires in the $\hat{x}$ direction

$s_1$ - the half separation of the wires on the intermediate mass,

$s_2$ - the half separation of the low wires on the lower mass.

(ii) Separation of wires in the $\hat{y}$ direction

$n_0$ - the half separation of the upper wires at the suspension point,

$n_1$ - the half separation of the upper wires at the suspension point,

$n_2$ - the half separation of the upper wires on the upper mass,

$n_3$ - the half separation of the intermediate wires on the upper mass,

$n_4$ - the half separation of the intermediate wires on the intermediate mass,

$n_5$ - the half separation of the lower wires on the intermediate mass.
Figure 2-6. Photograph of the top mass.
(iii) Height of the suspension points

\( d_0 \) - height of upper wire break-off (above the center of mass of the upper mass),
\( d_1 \) height of intermediate wire break-off (below the center of mass of the upper mass),
\( d_2 \) - height of intermediate wire break-off (above the center of mass of the intermediate mass),
\( d_3 \) - height of lower wire break-off (below the center of mass of the intermediate mass),
\( d_4 \) - height of lower wire break-off (above the center of mass of the lower mass).

(iv) Separation of magnets on the upper mass

\( l_p \) - the half spacing of magnets (between 3 and 4) acting on pitch,
\( l_y \) - the half spacing of magnets (between 1 and 2) acting on yaw,
\( l_r \) - the half spacing of magnets (between the line along 3,4 and 5) acting on roll.

(v) Height of magnets 1, 2 and 6 above the center of mass, \( l_s \).

(vi) Length of wires and their projection in the vertical directions

\( l_1, l_2 \) and \( l_3 \) represents the length of upper wires, intermediate wires and lower wires respectively. Their projection on the vertical directions are \( l_{t1}, l_{t2} \) and \( l_{t3} \), which are calculated as

\[
\begin{align*}
l_{t1} &= \sqrt{l_1^2 - (n_1 - n_0)^2}, \\
l_{t2} &= \sqrt{l_2^2 - (n_3 - n_2)^2}, \\
l_{t3} &= \sqrt{l_3^2 - (n_5 - n_4)^2}.
\end{align*}
\]

(vi) Momenta of inertia \( I_{i,j} \) and masses \( m_i, i = 1, 2, 3 \) and \( j = x, y, z \)

1, 2, 3 - the upper mass, the intermediate mass and the lower mass and \( x, y, z \) - directions of the angular motion.

(vii) Spring constants of the upper cantilever blades, intermediate blades and lower wires, \( k_1, k_2, k_3 \).
Figure 2-7. Parameters of a triple pendulum.
Figure 2-8. Photograph of a triple pendulum.
2.3.2 Vertical Response

The vertical motion of the pendulum remains uncoupled from the other five degrees of freedom motion. The force applied on the lower test mass is \(-4k_3 \Delta l_3 \cos \Omega\), \(\Delta l_3\) is the extension in the wire from the static equilibrium point, \(\Omega\) is the angle of the wires with respect to the vertical direction, \(\Delta l_3 \cos \Omega\) is the displacement along the \(z\) direction. Thus the equation for the vertical motion of the lower test mass is

\[ m_3 \ddot{z}_3 = -4k_3 (z_3 - z_2). \]  \(2-7\)

where \(z_3, z_2\) represents the vertical displacement of the lower test mass and the intermediate mass. Similarly, the equation for the intermediate mass is

\[ m_2 \ddot{z}_2 = -4k_2 (z_2 - z_1) + 4k_3 (z_3 - z_2). \]  \(2-8\)

Considering a control force \(F_v\) applied vertically on the upper mass via OSEMs, the equation for the upper mass is

\[ m_1 \ddot{z}_1 = -2k_1 z_1 + 4k_2 (z_2 - z_1) + F_v. \]  \(2-9\)

To derive the transfer function, the above equations are rewritten in the Fourier domain, utilizing the transformation \(z(t) = e^{i\omega t} Z(\omega)\). These equations become

\[
\begin{align*}
-m_1 z_1 \omega^2 &= -2k_1 z_1 + 4k_2 (z_2 - z_1) + F_v, \\
-m_2 z_2 \omega^2 &= -2k_2 (z_2 - z_1) + 2k_3 (z_3 - z_2), \\
-m_3 z_3 \omega^2 &= -2k_3 (z_3 - z_2). 
\end{align*}
\]  \(2-10\)

The relationship between the control force and the vertical motion of the upper mass can be described in the transfer function given below:

\[
H_{zv}(\omega) = \frac{z_1}{F_v} = \frac{1}{m_1 \omega^2 - 2k_1 - 4k_2 - \frac{16k_2^2}{m_2 \omega^2 - 4k_2 - 4k_3 - \frac{16k_3^2}{m_3 \omega^2 - 4k_3}}}. 
\]  \(2-11\)
Figure 2-9. Bode plot of $H_{zu}$.

Figure 2-9 is the Bode plot of the transfer function $H_{zu}$.

2.3.3 Longitudinal and Pitch Dynamics

The longitudinal motion in the $\hat{x}$-direction and the pitch motion are strongly coupled due to the fact that the break-off position of the wires are either above or below the line through the center of the mass. The tilt angle $\theta$ and the longitudinal displacement $x$ will show up together as the variables in the linear differential equations that characterize the dynamics of the pendulum.

First, the dynamic equations of a single pendulum suspended from two wires of length $l$ with a spring constant $k$ are derived. Define $s$ as the half separation between two suspension points and $d$ as the height of the wire breaking off points above the center of mass. Figure 2-10 shows a single pendulum suspended by two wires. Consider the case when the mass is tilted by an angle $\theta$ from the horizontal line and displaced by $x$ in the $\hat{x}$-direction. The wires are affected in opposite ways. Wire 1 is contracted and wire 2 is
stretched as can be seen in Figure 2-11. In Figure 2-11, the unfixed end of wire 1 moves from N to $P'$. MN is the displacement in the $\hat{x}$-direction.

\[ |ON| = |NP| - |OP| = x - s, \]  \hspace{1cm} (2–12)

\[ |MO| = |RT| = |RO'| - |TO'| = s \cos \theta - d \sin \theta, \]  \hspace{1cm} (2–13)

\[ |MN| = |MO| + |ON| = s \cos \theta - d \sin \theta + x - s. \]  \hspace{1cm} (2–14)

For a small angle $\theta$ and a small displacement $x$,

\[ |MN| = x - d\theta. \]  \hspace{1cm} (2–15)

$P'M$ is the displacement in the $\hat{y}$-direction.

\[ |MR| = |OT| = s - s \cos \theta, \]  \hspace{1cm} (2–16)
Figure 2-11. Longitudinal displacement introduced by the pitch motion.

\[ |P'M| = |P'R| - |MR|. \quad (2–17) \]

A small angle approximation gives,

\[ |P'M| \approx s\theta. \quad (2–18) \]

So the length of wire 1 changes from the original length, \( l \), to

\[ l' = \sqrt{(l - s\theta)^2 + (x - d\theta)^2} \approx l - s\theta. \quad (2–19) \]

Hence the change in the length of wire 1 is \( \Delta l \approx s\theta \). We can also give

\[ \sin \alpha \approx \frac{x - d\theta}{l}, \quad (2–20) \]

\[ \cos \alpha \approx 1. \quad (2–21) \]
Similar geometric analyses shows us that wire 2 is stretched by $\Delta l$ and

$$\sin \beta \approx \frac{x - d\theta}{l}, \quad (2-22)$$

$$\cos \beta \approx 1. \quad (2-23)$$

In an equilibrium state, the tension in each wire is

$$T_1 = T_2 = \frac{1}{2}mg. \quad (2-24)$$

Wire 1 is contracted such that in the first order approximation,

$$T_1 = \frac{1}{2}mg - k\Delta l. \quad (2-25)$$

And the tension in wire 2 becomes

$$T_2 = \frac{1}{2}mg + k\Delta l. \quad (2-26)$$

Hence the force on the mass along the $\hat{x}$-direction is

$$F = T_1 \sin \alpha + T_2 \sin \beta. \quad (2-27)$$

Substituting Equation 2–25 and 2–26 into Equation 2–27 gives

$$F = mg \frac{x - d\theta}{l}. \quad (2-28)$$

Finally, the equation for the longitudinal motion of a single pendulum suspended with two wires is

$$m\ddot{x} = -F \approx -mg \frac{x - d\theta}{l}. \quad (2-29)$$

The net torque which tilt the mass can be calculated as

$$Q = T_1 \cos \alpha (s - d\theta) + T_1 \sin \alpha (d + s\theta) - T_2 \cos \beta (s + d\theta) + T_2 \sin \beta (d - s\theta). \quad (2-30)$$

Substituting Equation 2–20, 2–21, 2–22, 2–23, 2–25 and 2–26 into 2–30 gives

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Figure 2-12. Components of restoring forces which act to tilt the mass.

\[ Q \approx mg \frac{d}{l} x - \left( 2ks^2 + mgd + mg \frac{d^2}{l} \right) \theta. \]  

Hence the equation of motion for pitch is

\[ I \ddot{\theta} \approx \frac{mgd}{l} x - \left[ 2ks^2 + mg \left( d + \frac{d^2}{l} \right) \right] \theta. \]  

The equations for the longitudinal/pitch motion of a triple pendulum can be derived based on the mechanical analysis of the single pendulum that is described above, i.e., the equations for each test mass can be given by simply modifying Equation 2–30 and 2–32. \( \theta_i \) and \( x_i \) are used to define the tilt angle and the linear displacement of each mass of the triple pendulum (i = 1, 2, 3 - upper mass, intermediate mass and lower mass). The variables \( x \) and \( \theta \) are replaced with the relative displacement of each mass with respect to the suspension points and the tilt angles between two adjacent masses. Hence for the
lower mass, the equation for the longitudinal motion is

$$\ddot{x}_3 = -\frac{g}{l_3} [(x_3 - x_2) - (d_4\theta_3 - d_3\theta_2)], \quad (2-33)$$

The equation for the pitch motion is

$$I_{3y}\ddot{\theta}_3 = \frac{m_3gd_4}{l_3} (x_3 - x_2) - 4k_3s_2^2 (\theta_3 - \theta_2) - m_3gd_4 \left( \theta_3 + \frac{d_4\theta_3 - d_3\theta_2}{l_3} \right). \quad (2-34)$$

It is noteworthy to mention here that the length of the pendulum is the vertical distance between the suspension points of two stages. The intermediate mass will be stretched by both the lower wires and the intermediate wires. The intermediate wires stretch the intermediate mass the same way that the lower wires do the lower mass while the lower wires affects the intermediate mass the opposite way according to Newton’s third law.

Now the tension along intermediate wires have to balance the gravity force due to total mass of the intermediate mass and the lower mass. The dynamic equations are

$$m_2\ddot{x}_2 = -\frac{(m_2 + m_3)g}{l_2} [(x_2 - x_1) - (d_2\theta_2 - d_1\theta_1)] + \frac{m_3g}{l_3} [(x_3 - x_2) - (d_4\theta_3 - d_3\theta_2)],$$

$$I_{2y}\ddot{\theta}_2 = \frac{(m_2 + m_3)gd_2}{l_2} (x_2 - x_1) - 4k_3s_1^2 (\theta_2 - \theta_1) - (m_2 + m_3)gd_2 \left( \theta_2 + \frac{d_2\theta_2 - d_1\theta_1}{l_2} \right),$$

$$-\frac{m_3g}{l_3} (x_3 - x_2) + \left( 4k_3 + \frac{m_3g}{l_3} \right) s_2^2 (\theta_3 - \theta_2) + m_3gd_3 \left( \theta_3 + \frac{d_4\theta_3 - d_3\theta_2}{l_3} \right). \quad (2-36)$$

There are only two wires above the upper mass. The tension in the wire is

$$T = \frac{1}{2}(m_1 + m_2 + m_3)g. \quad (2-37)$$

Both its vertical and horizontal components act to tilt the upper mass and the net torque is

$$Q = \frac{(m_1 + m_2 + m_3)gd_0}{l_1} (x_1 - d_0\theta_1) - (m_1 + m_2 + m_3)gd_0\theta_1. \quad (2-38)$$
Figure 2-13. Side view of the top mass suspended with two wires.

The differential equations for the upper mass have to include the external control force and the control torque. The force along the \( \hat{x} \)-direction \( F_l \) and the torque that controls the pitch motion, \( Q_p \) are applied via the OSEMs. In this case the equations become

\[
I_{1y} \ddot{\theta}_1 = \frac{(m_1 + m_2 + m_3) g d_0}{l_{t1}} (x_1 - d_0 \theta_1) - (m_1 + m_2 + m_3) g d_0 \theta_1 - \frac{(m_2 + m_3) g d_1}{l_{t2}} (x_2 - x_1) + 4 k_2 s_1^2 (\theta_2 - \theta_1) + (m_2 + m_3) g d_1 \left( \theta_2 + \frac{d_2 \theta_2 - d_1 \theta_1}{l_{t2}} \right) + Q_p, \tag{2–39}
\]

\[
m_1 \ddot{x}_1 = -\frac{(m_1 + m_2 + m_3) g}{l_{t1}} (x_1 - d_0 \theta_1) + \frac{(m_2 + m_3) g}{l_{t2}} [(x_2 - x_1) - (d_2 \theta_2 - d_1 \theta_1)] + F_l. \tag{2–40}
\]

The Fourier transformation of Equation 2–33, 2–34, 2–35, 2–36, 2–39 and 2–40 gives:

\[
x_2 - \left(1 - \omega^2 \frac{l_{t3}}{g} \right) x_3 - d_3 \theta_2 + d_4 \theta_3 = 0, \tag{2–41}
\]
\[-\frac{m_3g d_4}{l_{t_3}} x_2 + \frac{m_3g d_4}{l_{t_3}} x_3 + \left[ 4k_3 s_{2}^2 + \frac{m_3g d_4}{l_{t_3}} \right] \theta_2 + \left[ \omega^2 I_{3y} - 4k_3 s_{2}^2 - m_3g d_4 - \frac{m_3g}{l_{t_3}} \right] \] 

\[(s_{2}^2 + d_4^2) \theta_3 = 0,\]

\[
\frac{(m_2 + m_3) g}{l_{t_2}} x_1 + \left[ \frac{m_2 \omega^2 - (m_2 + m_3) g}{l_{t_2}} - \frac{m_3g}{l_{t_3}} \right] x_2 + \frac{m_3g}{l_{t_2}} x_3 - \frac{(m_2 + m_3) g}{l_{t_2}} d_1 \theta_1 + \left[ \frac{m_3g}{l_{t_3}} \right] \]

\[d_3 + \frac{(m_2 + m_3) g}{l_{t_2}} d_2 \theta_2 - \frac{m_3g}{l_{t_3}} d_4 \theta_3 = 0,\]

\[-\frac{(m_2 + m_3) g d_2}{l_{t_2}} x_1 + \left[ \frac{(m_2 + m_3) g d_2 + m_3g d_3}{l_{t_2}} \right] x_2 - \frac{m_3g d_3}{l_{t_2}} x_3 + \left[ \frac{4k_2 s_{1}^2 + (m_2 + m_3) g}{l_{t_2}} \right] \]

\[d_1 d_2 \theta_1 + \left[ \omega^2 I_{2y} - 4k_2 s_{1}^2 - \frac{(m_2 + m_3) g d_2}{l_{t_2}} - (m_2 + m_3) g d_2 - 4k_3 s_{2}^2 - \frac{m_3g}{l_{t_3}} \right] \]

\[d_3^2 \theta_2 + \left[ 4k_3 s_{2}^2 + \frac{m_3g}{l_{t_3}} d_3 d_4 + m_3g d_3 \right] \theta_3 = 0,\]

\[
\left[ \frac{m_1 \omega^2 - (m_2 + m_3) g}{l_{t_2}} - \frac{(m_1 + m_2 + m_3) g}{l_{t_2}} \right] x_1 + \frac{(m_2 + m_3) g}{l_{t_2}} x_2 + \left[ \frac{(m_1 + m_2 + m_3) g}{l_{t_1}} \right] d_0 \]

\[
+ \frac{(m_2 + m_3) g}{l_{t_2}} d_1 \theta_1 - \frac{(m_2 + m_3) g}{l_{t_2}} d_2 \theta_2 = F_l,\]

and

\[
\left[ \frac{(m_1 + m_2 + m_3) g d_0 + (m_2 + m_3) g d_1}{l_{t_1}} \right] x_1 - \frac{(m_2 + m_3) g d_1}{l_{t_2}} x_2 + \left[ \omega^2 I_{1y} - 4k_2 s_{1}^2 - \frac{gd_0^2}{l_{t_1}} \right] \theta_1 + \left[ \frac{(m_2 + m_3) g d_1}{l_{t_2}} \right] \theta_2 = Q_p.\]

These equations can be written in a matrix form:

\[
\begin{pmatrix}
\Gamma_{11} & \Gamma_{12} & 0 & \Gamma_{14} & \Gamma_{15} & 0 \\
\Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & \Gamma_{25} & \Gamma_{26} \\
0 & \Gamma_{32} & \Gamma_{33} & 0 & \Gamma_{35} & \Gamma_{36} \\
\Gamma_{41} & \Gamma_{42} & 0 & \Gamma_{44} & \Gamma_{45} & 0 \\
\Gamma_{51} & \Gamma_{52} & \Gamma_{53} & \Gamma_{54} & \Gamma_{55} & \Gamma_{56} \\
0 & \Gamma_{62} & \Gamma_{63} & 0 & \Gamma_{65} & \Gamma_{66}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix} = \begin{pmatrix}
F_l \\
0 \\
0 \\
0 \\
Q_p \\
0
\end{pmatrix}.
\]
Here,
\[ \Gamma_{11} = m_1 \omega^2 - \frac{(m_2 + m_3) g}{l_{t2}} - \frac{(m_1 + m_2 + m_3) g}{l_{t1}}, \]  
(2–48)
\[ \Gamma_{12} = \frac{(m_2 + m_3) g}{l_{t2}}, \]  
(2–49)
\[ \Gamma_{14} = \frac{(m_1 + m_2 + m_3) g}{l_{t1}} d_0 + \frac{(m_2 + m_3) g}{l_{t2}} d_1, \]  
(2–50)
\[ \Gamma_{15} = -\frac{(m_2 + m_3) g}{l_{t2}} d_2, \]  
(2–51)
\[ \Gamma_{21} = \frac{(m_2 + m_3) g}{l_{t2}}, \]  
(2–52)
\[ \Gamma_{22} = m_2 \omega^2 - \frac{(m_2 + m_3) g}{l_{t2}} - \frac{m_3 g}{l_{t3}}, \]  
(2–53)
\[ \Gamma_{23} = \frac{m_3 g}{l_{t3}}, \]  
(2–54)
\[ \Gamma_{24} = -\frac{(m_2 + m_3) g}{l_{t2}} d_1, \]  
(2–55)
\[ \Gamma_{25} = \frac{(m_2 + m_3) g}{l_{t2}} d_2 + \frac{m_3 g}{l_{t3}} d_3, \]  
(2–56)
\[ \Gamma_{26} = -\frac{m_3 g}{l_{t3}} d_4, \]  
(2–57)
\[ \Gamma_{32} = 1, \]  
(2–58)
\[ \Gamma_{33} = \omega^2 \frac{l_{t3}}{g} - 1, \]  
(2–59)
\[ \Gamma_{35} = -d_3, \]  
(2–60)
\[ \Gamma_{36} = d_4, \]  
(2–61)
\[ \Gamma_{41} = \frac{(m_1 + m_2 + m_3) gd_0}{l_{t1}} + \frac{(m_2 + m_3) gd_1}{l_{t2}}, \]  
(2–62)
\[ \Gamma_{42} = -\frac{(m_2 + m_3) gd_1}{l_{t2}}, \]  
(2–63)
\[ \Gamma_{44} = \omega^2 I_{y} - 4k_2 s_1^2 - \frac{(m_1 + m_2 + m_3) gd_0}{l_{t1}} d_0 - (m_1 + m_2 + m_3) gd_0 - \frac{(m_2 + m_3) g^2}{l_{t2}} d_1, \]  
(2–64)
\[ \Gamma_{45} = 4k_2 s_1^2 + (m_2 + m_3) gd_1 + \frac{(m_2 + m_3) g}{l_{t2}} d_1 d_2, \]  
(2–65)
\[ \Gamma_{51} = -\frac{(m_2 + m_3) gd_2}{l_{t2}}, \]  
(2–66)
\begin{align*}
\Gamma_{52} &= \frac{(m_2 + m_3) gd_2}{l_{t2}} + \frac{m_3gd_3}{l_{t3}}, \\
\Gamma_{53} &= -\frac{m_3gd_3}{l_{t3}}, \\
\Gamma_{54} &= 4k_2s_1^2 + \frac{(m_2 + m_3) g}{l_{t2}} d_1d_2, \\
\Gamma_{55} &= \omega^2 I_{2y} - 4k_2s_1^2 \frac{(m_2 + m_3) g}{l_{t2}} d_2^2 - (m_2 + m_3) gd_2 - 4k_3s_2^2 - \frac{m_3g}{l_{t3}} d_3^2, \\
\Gamma_{56} &= 4k_3s_2^2 + \frac{m_3g}{l_{t3}} d_3d_4 + m_3gd_3, \\
\Gamma_{62} &= -\frac{m_3gd_4}{l_{t3}}, \\
\Gamma_{63} &= \frac{m_3gd_4}{l_{t3}}, \\
\Gamma_{65} &= 4k_3s_2^2 + \frac{m_3g}{l_{t3}} d_4d_3, \\
\Gamma_{66} &= \omega^2 I_{3y} - 4k_3s_2^2 - m_3gd_4 - \frac{m_3g}{l_{t3}} d_4^2.
\end{align*}

Thus the response of the triple pendulum to the external force applied in the longitudinal direction and the torque that tilts the top mass can be calculated as

\begin{align*}
\begin{pmatrix}
{x_1} \\
{x_2} \\
{x_3} \\
{\theta_1} \\
{\theta_2} \\
{\theta_3}
\end{pmatrix} &=
\begin{pmatrix}
F_t \\
0 \\
0 \\
Q_p \\
0 \\
0
\end{pmatrix},
\end{align*}

where $T = \Gamma^{-1}$. The response of the top mass to $F_t$ and $Q_p$ can be analytically calculated as

\begin{align*}
H_{xt} &= \frac{x_1}{F_t} = T_{11}, \\
H_{xp} &= \frac{x_1}{Q_p} = T_{14}.
\end{align*}
Figure 2-14. Bode plot of $H_{xl}$.

$$H_{xl} = \frac{\theta_1}{F_l} = T_{41}, \quad (2-79)$$

and

$$H_{\theta p} = \frac{\theta_1}{Q_p} = T_{44}. \quad (2-80)$$

These transfer functions are plotted in Figure 2-14, Figure 2-15, Figure 2-16, and Figure 2-17 respectively.

2.3.4 Yaw Motion

First, we consider the case of a mass suspended by four wires of length $l$, which are all at the same angle $\Omega$ with respect to the vertical direction as is shown in Figure 2-18. The tension $T$ in one wire can be projected along two directions. The vertical component $T_v$ balances the gravity force while the horizontal component $T_h$ along the direction $\overrightarrow{BD}$ is balanced by the horizontal force introduced from another wire. However, when the mass rotates through an angle $\phi$, the net torque by the horizontal components of the tension
Figure 2-15. Bode plot of $H_{xp}$.

Figure 2-16. Bode plot of $H_{bl}$. 
force of four wires will not keep the mass in a balanced state any longer. In Figure 2-19 we can clearly see that it is the force $F$ that acts to rotate the mass. $F$ is the projection of $T_h$ in the direction perpendicular to the line connecting the suspension point and the center of mass,

$$ F = T_h \cos \gamma $$  \hspace{1cm} (2–81)

where

$$ T_h = T_v \tan \Omega_t = \frac{1}{4} mg \frac{|BD|}{|CD|} $$  \hspace{1cm} (2–82)

and

$$ \cos \gamma = \frac{|AB|^2 + |BD|^2 - |AD|^2}{2 |AB| |BD|} $$  \hspace{1cm} (2–83)

$|AB|$ and $|AD|$ are determined by the parameters given by the dimensions of the mass:

$$ |AB| = \sqrt{(s^2 + n_i^2)\phi}, $$  \hspace{1cm} (2–84)
Figure 2-18. Yaw motion of a single pendulum. The upper part is the view from above. And the low part is the geometric plot of the effect on one wire when the mass is rotated through an angle $\phi$. 
Figure 2-19. Projection of the tension onto \( \hat{x}-\hat{y} \) plane which produces the restoring torque. The upper plot and the lower plot are associated with two different effects on wires when the mass rotates.
\[ |AD| = n_i - n_j. \]  \hfill (2–85)

\[ |BD| \] is calculated from \(|AB|\) and \(|AD|\),

\[ |BD| = \sqrt{|AB|^2 + |AD|^2 - 2|AB||AD| \cos \angle BAD. \]  \hfill (2–86)

Here \( \angle BAD \) can be \(90^\circ + \theta\) or \(90^\circ - \theta\), depending on whether a wire stretches or contracts when the mass rotates. In Figure 2-18, wires 1 and 3 stretch while wires 2 and 4 contract when the mass rotates in a clockwise direction. The stretch and the contraction changes \(|BD|\) such that

\[ |BD| \pm = \sqrt{(s^2 + n_i^2) \phi^2 + (n_i - n_j)^2 \pm 2s^2 + n_i^2 (n_i - n_j) \sin \theta \phi} \approx n_i - n_j \pm s \phi \]  \hfill (2–87)

Therefore, we can write \( T_h \) and \( \cos \gamma \) as

\[
\cos \gamma_{\pm} = \frac{(s^2 + n_i^2) \phi^2 + (s^2 + n_i^2) \phi^2 + (n_i - n_j)^2 \pm 2s^2 + n_i^2 (n_i - n_j) \sin \theta \phi} {2s^2 + n_i^2 \phi} \left( n_i - n_j \pm \sqrt{s^2 + n_i^2 \sin \theta \phi} \right)
\]

\[
= \frac{\sqrt{s^2 + n_i^2} \phi \pm (n_i - n_j) \sin \theta} {n_i - n_j} \left( 1 \pm \frac{\sqrt{s^2 + n_i^2} \sin \theta \phi} {n_i - n_j} \right)
\]

\[
\approx \frac{\sqrt{s^2 + n_i^2} \cos^2 \theta \phi \pm \sin \theta} {n_i - n_j} = \frac{n_i \cos \theta \phi \pm \sin \theta} {n_i - n_j}
\]  \hfill (2–88)

\[ T_{h_{\pm}} = \frac{1}{4} mg \frac{n_i - n_j \pm s \phi} {\sqrt{l^2 - (n_i - n_j)^2}}. \]  \hfill (2–89)

Now the total torque from four wires is

\[
Q = 2T_{h+} \cos \gamma_{+}\sqrt{s^2 + n_i^2} + 2T_{h-} \cos \gamma_{-}\sqrt{s^2 + n_i^2} \approx \frac{1}{2} mg \frac{n_i - n_j + s \phi} {\sqrt{l^2 - (n_i - n_j)^2}} \left( \frac{n_i^2 \phi + s} {n_i - n_j} \right)
\]

\[
+ \frac{1}{2} mg \frac{n_i - n_j - s \phi} {\sqrt{l^2 - (n_i - n_j)^2}} \left( \frac{n_i^2 \phi - s} {n_i - n_j} \right) = \frac{mg (n_i^2 + s^2)} {\sqrt{l^2 - (n_i - n_j)^2}} \phi.
\]  \hfill (2–90)

Hence the equation of motion for a four wire suspension is

\[ I \ddot{\phi} = - \frac{mg (n_i^2 + s^2)} {\sqrt{l^2 - (n_i - n_j)^2}} \phi. \]  \hfill (2–91)
Similar to what is outlined in the longitudinal and pith case, the differential equation can be extended to describe the yaw motion of a triple pendulum in terms of the rotation angle of each mass with respect to the suspension points of each stage. The tension in the wires of each stage is proportional to the total weight of the masses below the wires. The equations of the yaw motion for the lower mass and the intermediate mass are therefore

\[ I_3 \ddot{\phi}_3 = -\frac{m_3 g (n_5^2 + s_2^2)}{\sqrt{l^2 - (n_5 - n_4)^2}} (\phi_3 - \phi_2), \quad (2-92) \]

and

\[ I_2 \ddot{\phi}_2 = -\frac{(m_2 + m_3) g (n_3^2 + s_1^2)}{l_{t_2}} (\phi_2 - \phi_1) \phi_2 + \frac{m_3 g (n_4^2 + s_2^2)}{l_{t_3}} (\phi_3 - \phi_2). \quad (2-93) \]

The line determined by the two upper suspension points on the top mass passes through the center of mass. So the horizontal components of tensions in two upper suspension wire can be written in a relatively simpler form

\[ T_h = \frac{1}{2} (m_1 + m_2 + m_3) g \frac{n_1 - n_0}{l_{t_1}}. \quad (2-94) \]

This produces a torque on the top mass when the mass is rotated by a small angle \( \phi \),

\[ Q = (m_1 + m_2 + m_3) g \frac{n_1^2}{l_{t_1}^2} \phi. \quad (2-95) \]

Therefore the differential equation which describes the yaw motion of the upper mass becomes

\[ I_1 \ddot{\phi}_1 = -\frac{(m_1 + m_2 + m_3) g n_1^2}{l_{t_1}} \phi_1 + \frac{(m_2 + m_3) g (n_3^2 + s_1^2)}{l_{t_2}} (\phi_2 - \phi_1) + Q_y, \quad (2-96) \]

where \( Q_y \) is the external feedback torque used to stabilize the yaw motion of the top mass. The Fourier transform of Equation 2-92, 2-93 and 2-96 are listed below,

\[ \omega^2 I_3 \phi_3 = \frac{m_3 g (n_5^2 + s_2^2)}{l_{t_3}} (\phi_3 - \phi_2), \quad (2-97) \]
Figure 2-20. Bode plot of $H_{\phi y}$.

\[
\begin{align*}
\omega^2 I_{2z} \phi_2 &= \frac{(m_2 + m_3) g (n_2^2 + s_1^2)}{l_{t2}} (\phi_2 - \phi_1) - \frac{m_3 g (n_2^2 + s_2^2)}{l_{t3}} (\phi_3 - \phi_2), \quad (2-98) \\
\omega^2 I_{1z} \phi_1 &= \frac{(m_1 + m_2 + m_3) g n_1^2}{l_{t1}} \phi_1 - \frac{(m_2 + m_3) g (n_2^2 + s_1^2)}{l_{t2}} (\phi_2 - \phi_1) - Q_y. \quad (2-99)
\end{align*}
\]

From Equation 2.3.4, 2-98 and 2-99, the transfer function that relates the external torque $Q_y$ to the induced rotation of the top mass $\phi_1$ is derived as

\[
H_{\phi y} = \frac{\phi_1}{Q_y} = \left\{ \frac{(m_1 + m_2 + m_3) g n_1^2}{l_{t1}} - \omega^2 I_{1z} + \frac{(m_2 + m_3) g (n_2^2 + s_1^2)}{l_{t2}} \right\} \\
\left( \frac{m_3 g (n_2^2 + s_1^2)}{l_{t2}} \left( 1 - \frac{m_3 g (n_2^2 + s_1^2)}{m_3 g (n_2^2 + s_2^2) - \omega^2 l_{t3} I_{3z}} \right) - \omega^2 I_{2z} \right) \\
\left( \frac{(m_2 + m_3) g (n_2^2 + s_1^2)}{l_{t2}} + \frac{m_3 g (n_2^2 + s_1^2)}{l_{t3}} \left( 1 - \frac{m_3 g (n_2^2 + s_2^2)}{m_3 g (n_2^2 + s_2^2) - \omega^2 l_{t3} I_{3z}} \right) - \omega^2 I_{2z} \right)
\right\}.
\]

The transfer function $H_{\phi y}$ is plotted in Figure 2-20.
2.3.5 Sideways and Roll Motion

The coupling between the sideways and the roll motion is similar to the coupling between the longitudinal and the pitch motion. However, sideways and roll coupling is more complicated. The wires that are used to suspend the masses are angled in the $\hat{y}$-direction. The sideways motion of the mass will create a torque that can excite the roll motion while the longitudinal displacement does not tilt the mass. The details will be discussed below.
Consider a single pendulum suspended with two wires as is plotted in Figure 2-21. The two wires have the same length $l$ in the equilibrium state. And the tensions in both wires are,

$$T_1 = T_2 = \frac{1}{2}mg \frac{l}{l_t}$$  \hspace{1cm} (2–101)

After the mass is displaced by a small distance $y$ in the $\hat{y}$-direction, the center of mass moves from O to P. This is followed by a mass rolling by a small angle $\varphi$. The suspension points now move from G and J to E and F. Thus the new lengths of the two wires can be derived. From Figure 2-21 and Figure 2-22,

$$|AE|^2 = |KH|^2 + l_t^2,$$

$$|AE| = \sqrt{(n_j - n_i + y - d\varphi)^2 + (l_t - n_j \varphi)^2} \approx \sqrt{(n_j - n_i)^2 + 2 (n_j - n_i) (y - d\varphi) + l_t^2 - 2n_jl_t\varphi}$$

$$= \sqrt{l^2 + 2 (n_j - n_i) y - 2 (n_jl_t + n_jd - n_id) \varphi} \approx l + \frac{n_j - n_i}{l} y - \frac{n_jl_t + n_jd - n_id}{l} \varphi.$$  \hspace{1cm} (2–103)

Hence wire 1 is stretched by

$$\Delta l = \frac{n_j - n_i}{l} y - \frac{n_jl_t + n_jd - n_id}{l} \varphi.$$  \hspace{1cm} (2–104)
And for wire 2,
\[
|BF|^2 = |LI|^2 + |l_t + |FL||^2, \tag{2–105}
\]
\[
|BF| = \sqrt{(y - (n_j - n_i) - d\varphi)^2 + (l_t + n_j\varphi)^2} \approx \sqrt{(n_j - n_i)^2 - 2(n_j - n_i)(y - d\varphi) + l_t^2 + 2n_j\varphi}
\]
\[
= \sqrt{l^2 - 2(n_j - n_i)y + 2(n_jl_t + n_jd - n_id)d\varphi} \approx l - \frac{n_j - n_i}{l}y + \frac{n_jl_t + n_jd - n_id}{l}d\varphi. \tag{2–106}
\]

Wire 2 is then contracted by \( \Delta l \). Now the tensions in wire 1 and wire 2 become
\[
T_1 = \frac{1}{2}mg\frac{l}{l_t} + k\Delta l, \tag{2–107}
\]
\[
T_2 = \frac{1}{2}mg\frac{l}{l_t} - k\Delta l. \tag{2–108}
\]

The angle of wire 1 with respect to the vertical direction \( \Omega_1 \) satisfies
\[
\sin \Omega_1 = \frac{n_j - n_i}{l} + \left[ \frac{1}{l} - \frac{(n_j - n_i)^2}{l^3} \right] y - \left[ \frac{d}{l} - \frac{(n_j - n_i)(n_jl_t + n_jd - n_id)}{l^3} \right] \varphi, \tag{2–109}
\]
and
\[
\cos \Omega_1 = \frac{l_t}{l} - \frac{(n_j - n_i)l_t}{l^3} y - \left[ \frac{n_j}{l} - \frac{l_t(n_jl_t + n_jd - n_id)}{l^3} \right] \varphi. \tag{2–110}
\]

The sloping angle of wire 2 satisfies
\[
\sin \Omega_2 = -\frac{n_j - n_i}{l} + \left[ \frac{1}{l} - \frac{(n_j - n_i)^2}{l^3} \right] y - \left[ \frac{d}{l} - \frac{(n_j - n_i)(n_jl_t + n_jd - n_id)}{l^3} \right] \varphi, \tag{2–111}
\]
and
\[
\cos \Omega_2 = \frac{l_t}{l} + \frac{(n_j - n_i)l_t}{l^3} y + \left[ \frac{n_j}{l} - \frac{l_t(n_jl_t + n_jd - n_id)}{l^3} \right] \varphi. \tag{2–112}
\]

The equation of motion for the displacement of the center of mass, \( y \), is
\[
m\ddot{y} = -T_1 \sin \Omega_1 - T_2 \sin \Omega_2. \tag{2–113}
\]

Substituting for \( T_1, T_2 \) from Equation \( 2–107 \) and \( 2–108 \), for \( \Omega_1 \) and \( \Omega_2 \) from Equation \( 2–109, 2–110, 2–111, \) and \( 2–112 \) and using the first order approximation for small
displacement $y$ and small angle $\varphi$ gives

$$m\ddot{y} = -\left\{ 2k \left( \frac{(n_j - n_i)^2}{l^2} + \frac{mg}{l_t} \left[ 1 - \left( \frac{(n_j - n_i)^2}{l^2} \right)^2 \right] \right\} y +$$

$$\left\{ 2k \left( \frac{(n_j - n_i)(n_j l_t + n_j d - n_i d)}{l^2} + \frac{mg}{l_t} \left[ d - \left( \frac{(n_j - n_i)(n_j l_t + n_j d - n_i d)}{l^2} \right) \right] \right\} \varphi. \tag{2–114}$$

With reference to Figure 2-23, we can see that the component of force which rolls the mass for wire 1 is $T_1 \sin \alpha$ and for wire 2 is $T_2 \cos \beta$ where

$$\alpha = 90^\circ - (\sigma + \varphi) + \Omega_1, \tag{2–115}$$

and

$$\beta = \sigma - \varphi + \Omega_2. \tag{2–116}$$

The angle $\sigma$ is defined by the parameters of the pendulum. Using Equation 2–109, 2–110, 2–111 and 2–112, $\sin \alpha$ and $\cos \beta$ are calculated to be
\[
\sin \alpha \approx \frac{n_j l_t + d (n_j - n_i)}{l \sqrt{n_j^2 + d^2}} + \frac{1}{\sqrt{n_j^2 + d^2}} \left\{ \frac{d}{l} \left[ \frac{1}{l} - \frac{(n_j - n_i)^2}{l^3} \right] - n_j \frac{(n_j - n_i) l_t}{l^3} \right\} y
- \frac{1}{\sqrt{n_j^2 + d^2}} \left\{ \frac{n_j}{l} - \frac{l_t (n_j l_t + n_j d - n_i d)}{l^3} \right\} + d \left[ \frac{d}{l} - \frac{(n_j - n_i) (n_j l_t + n_j d - n_i d)}{l^3} \right] \right\} \varphi,
\]

\[
\cos \beta \approx \frac{n_j l_t + d (n_j - n_i)}{l \sqrt{n_j^2 + d^2}} + \frac{1}{\sqrt{n_j^2 + d^2}} \left\{ \frac{d}{l} \left[ \frac{1}{l} - \frac{(n_j - n_i)^2}{l^3} \right] + \frac{n_j}{\sqrt{n_j^2 + d^2}} \frac{(n_j - n_i) l_t}{l^3} \right\} y + \right\} \varphi,
\]

Now the torque for the roll motion is

\[
Q = \sqrt{n_j^2 + d^2} (T_1 \sin \alpha - T_2 \cos \beta) = \left\{ \begin{array}{l}
mgd \left[ \frac{1}{l_t} - \frac{(n_j - n_i)^2}{l^2 l_t} \right] - mgn_j \frac{(n_j - n_i)}{l^2} + \\
2k \frac{n_j l_t (n_j - n_i) + d (n_j - n_i)^2}{l^2} \\
- \frac{(n_j - n_i) (n_j l_t + n_j d - n_i d)}{l^2 l_t} \right\} y \right\} \varphi.
\]

And the equation of roll motion around the center of mass is

\[
I_x \ddot{\phi} = \left\{ \begin{array}{l}
mgd \left[ \frac{1}{l_t} - \frac{(n_j - n_i)^2}{l^2 l_t} \right] - mgn_j \frac{(n_j - n_i)}{l^2} + 2k \frac{n_j l_t (n_j - n_i) + d (n_j - n_i)^2}{l^2} \\
\left\{ mgn_j \frac{n_j}{l_t} - \frac{(n_j l_t + n_j d - n_i d)}{l^2} \right\} + mgd \left[ \frac{d}{l_t} - \frac{(n_j - n_i) (n_j l_t + n_j d - n_i d)}{l^2 l_t} \right] + mg \left( \begin{array}{l}
\right\} \varphi.
\right\}
\]
Figure 2-24. Relative motion between two adjacent masses.

The extension to a triple pendulum is outlined below. Although the lower mass and the intermediate mass are both suspended with four wires, the way the wires introduce the sideways and the roll motion is not different from the case where they are suspended with two wires in the manner described above. However, since what we need to consider here is the relative motion between two adjacent masses, the equations of motion become complicated. In Figure 2-24, the length of the suspension wires can be calculated by substituting $y$ with $y_m - y_n$, $d\varphi$ with $d_l\varphi_m - d_u\varphi_n$, and $n_j\varphi$ with $n_j\varphi_m - n_i\varphi_n$ in Equation
The angle of wire 1 with respect to the vertical direction $\Omega_1$ now satisfies

$$
sin \Omega_1 \approx \frac{n_j - n_i + y_m - y_n - d_t \varphi_m + d_u \varphi_n}{l + \frac{n_j - n_i}{l} (y_m - y_n) - \frac{(n_j \varphi_m - n_i \varphi_n) l_t}{l} - \frac{(n_j - n_i) (d_t \varphi_m - d_u \varphi_n)}{l}} \approx \frac{n_j - n_i}{l}.
$$

$$
cos \Omega_1 \approx \frac{l_t - n_j \varphi_m + n_i \varphi_n}{l + \frac{n_j - n_i}{l} (y_m - y_n) - \frac{(n_j \varphi_m - n_i \varphi_n) l_t}{l} - \frac{(n_j - n_i) (d_t \varphi_m - d_u \varphi_n)}{l}} \approx \frac{l_t}{l}.
$$

The sin and cos value of the sloping angle of wire 2 now become

$$
sin \Omega_2 \approx -\frac{n_j - n_i - y_m + y_n + d_t \varphi_m - d_u \varphi_n}{l - \frac{n_j - n_i}{l} (y_m - y_n) + \frac{(n_j \varphi_m - n_i \varphi_n) l_t}{l} + \frac{(n_j - n_i) (d_t \varphi_m - d_u \varphi_n)}{l}} \approx -\frac{n_j - n_i}{l},
$$

$$
cos \Omega_2 \approx \frac{-d_t \varphi_m - d_u \varphi_n}{l - \frac{n_j - n_i}{l} (y_m - y_n) + \frac{(n_j \varphi_m - n_i \varphi_n) l_t}{l} + \frac{(n_j - n_i) (d_t \varphi_m - d_u \varphi_n)}{l}} \approx \frac{-d_t \varphi_m - d_u \varphi_n}{l}.
$$
\[
\begin{aligned}
\cos \Omega_2 & \approx \frac{l_t + n_j \varphi_m - n_i \varphi_n}{l - \frac{n_j - n_i}{l} (y_m - y_n) + \frac{(n_j \varphi_m - n_i \varphi_n) l_t}{l} + \frac{(n_j - n_i) (d_l \varphi_m - d_u \varphi_n)}{l}} \\
& \approx \frac{l_t}{l} \left[ 1 + \frac{(n_j \varphi_m - n_i \varphi_n)}{l_t} \right] \left[ 1 + \frac{n_j - n_i}{l^2} (y_m - y_n) - \frac{l_t}{l^2} (n_j \varphi_m - n_i \varphi_n) - \frac{(n_j - n_i)}{l^2} (d_l \varphi_m - d_u \varphi_n) \right] \\
& - \frac{l_t}{l} + \frac{(n_j - n_i) l_t}{l^3} (y_m - y_n) + \frac{i^2 - l^2}{l^3} (n_j \varphi_m - n_i \varphi_n) - \frac{l_t (n_j - n_i)}{l^3} (d_l \varphi_m - d_u \varphi_n). \\
\end{aligned}
\]

The force on the lower mass in Figure 2-24 becomes

\[
F = -\frac{1}{2} \frac{mg}{l_t} (\sin \Omega_1 + \sin \Omega_2) - k \Delta l (\sin \Omega_1 - \sin \Omega_2) \approx \left\{ mg \left[ \frac{1}{l_t} - \frac{(n_j - n_i)^2}{l^2 l_t} \right] - 2k \right\} \\
\left\{ \frac{(n_j - n_i)^2}{l^2} \right\} (y_m - y_n) + \left\{ mg \left[ \frac{1}{l_t} - \frac{2k n_j - n_i l_t}{l} \right] \right\} (n_j \varphi_m - n_i \varphi_n) + \left\{ mg \left[ \frac{(n_j - n_i)^2}{l^2 l_t} \right] \right\} \\
- \frac{1}{l_t} + 2k \frac{(n_j - n_i)^2}{l^2} \right\} (d_l \varphi_m - d_u \varphi_n). \\
\]

And the equation of motion for the displacement of the lower mass \( m_m \) is

\[
m_m \ddot{y}_m = R_m (y_m - y_n) + (S_m n_j - R_m d_l) \varphi_m - (S_m n_i - R_m d_u) \varphi_n. \\
\]

Here,

\[
R_m = -M_m g \left[ \frac{1}{l_{tm}} - \frac{(n_j - n_i)^2}{l^2_m l_{tm}} \right] + 2k_m \frac{(n_j - n_i)^2}{l^2_m}, \\
S_m = -M_m g \frac{(n_j - n_i)}{l^2_m l_{tm}} - 2k_m \frac{n_j - n_i}{l_m} \frac{l_{tm}}{l_m}, \\
\]

where \( M_m \) represents the total mass suspended on the wire. If there is no other mass suspended below \( m_m \), \( M_m = m_m \).

The angles \( \alpha_l \), \( \beta_l \), \( \alpha_u \) and \( \beta_u \) as specified in Figure 2-24 can be given using simple geometric analysis.

\[
\begin{aligned}
\alpha_l &= 90^\circ - (\sigma_{mu} + \varphi_m) + \Omega_1, \\
\beta_l &= \sigma_{mu} - \varphi_m + \Omega_2, \\
\alpha_u &= 90^\circ - \sigma_{nl} + \varphi_n - \Omega_1, \\
\beta_u &= \sigma_{nl} - \varphi_n + \Omega_2.
\end{aligned}
\]
\[ \beta_u = \sigma_{nt} + \varphi_n - \Omega_2. \] (2–133)

Therefore, the torque which rolls the lower mass is

\[
Q_m^+ = \sqrt{n_j^2 + d_i^2} \left( T_1 \sin \alpha_i - T_2 \cos \beta_i \right) = \sqrt{n_j^2 + d_i^2} \left[ \frac{1}{2} M_m g \frac{l_m}{l_{tm}} (\sin \alpha_i - \cos \beta_i) + k_m \Delta l (\sin \alpha_i + \cos \beta_i) \right] = A_m (y_m - y_n) + B_m (n_j \varphi_m - n_i \varphi_n) + C_m (d_i \varphi_m - d_u \varphi_n) + M_m g \left( n_j - \frac{n_i}{l_{tm}} - d_i \right) \varphi_m,
\] (2–134)

where

\[
A_m = M_m g d_i \left[ \frac{1}{l_{tm}} - \left( \frac{n_j - n_i}{l_{tm}} \right)^2 \right] - M_m g n_i \left( \frac{n_j - n_i}{l_{tm}} \right)^2 + 2k_m \frac{n_j l_{tm} + d_i (n_j - n_i) n_j - n_i}{l_{tm}} l_m,
\] (2–135)

\[
B_m = M_m g \frac{n_j l_{tm}}{l_{tm}^2} - M_m g n_i + M_m g \frac{d_i (n_j - n_i)}{l_{tm}^2} - 2k_m \frac{l_{tm} n_j l_{tm} + d_i (n_j - n_i)}{l_{tm}} l_m,
\] (2–136)

\[
C_m = M_m g \frac{n_j (n_j - n_i)}{l_{tm}^2} + M_m g d_i \left[ \frac{(n_j - n_i)^2}{l_{tm}^2} - \frac{1}{l_{tm}^2} \right] - 2k_m \frac{n_j l_{tm} + d_i (n_j - n_i)}{l_{tm}} l_m.
\] (2–137)

And the torque which rolls the upper mass is

\[
Q_m^- = D_n (y_m - y_n) + E_n (n_j \varphi_m - n_i \varphi_n) + G_n (d_i \varphi_m - d_u \varphi_n) + M_m g \left\{ n_i \frac{n_j - n_i}{l_{tm}} - d_u \right\} \varphi_n,
\] (2–138)

where

\[
D_n = M_m g d_u \left[ \frac{1}{l_{tm}} - \left( \frac{n_j - n_i}{l_{tm}} \right)^2 \right] - M_m g n_i \left( \frac{n_j - n_i}{l_{tm}} \right)^2 - 2k_m \frac{n_j l_{tm} + d_u (n_j - n_i) n_j - n_i}{l_{tm}} l_m,
\] (2–139)

\[
E_n = M_m g \frac{n_i l_{tm}}{l_{tm}^2} - M_m g \frac{n_j}{l_{tm}} + M_m g \frac{d_u (n_j - n_i)}{l_{tm}^2} + 2k_m \frac{l_{tm} n_i l_{tm} + d_u (n_j - n_i)}{l_{tm}} l_m,
\] (2–140)

\[
G_n = M_m g \frac{n_i (n_j - n_i)}{l_{tm}^2} + M_m g d_u \left[ \frac{(n_j - n_i)^2}{l_{tm}^2} - \frac{1}{l_{tm}^2} \right] + 2k_m \frac{n_j l_{tm} + d_u (n_j - n_i)}{l_{tm}} l_m.
\] (2–141)

Equation 2–127, 2–134, 2–138 can be used as general equations to describe the roll and sideways motion of all three masses of the pendulum. The functions \( R, S, V A, B, C, D, \)
The equations for the intermediate mass are

\[ m_3 \ddot{y}_3 = R_3 (y_3 - y_2) + (S_3 n_5 - R_3 d_4) \varphi_3 - (S_3 n_4 - R_3 d_3) \varphi_2, \tag{2–142} \]

\[ I_{3x} \ddot{\varphi}_3 = -A_3 y_2 + A_3 y_3 - (B_3 n_4 + C_3 d_3) \varphi_2 + \left[ B_3 n_5 + C_3 d_4 + m_3 g \left( \frac{n_5 n_5 - n_4}{l_3} - d_4 \right) \right] \varphi_3, \tag{2–143} \]

where

\[ R_3 = -m_3 g \left[ \frac{1}{l_3^2} + \frac{(n_5 - n_4)^2}{l_3^2 l_3^2} \right] - 4k_3 \left( \frac{n_5 - n_4}{l_3} \right)^2, \tag{2–144} \]

\[ S_3 = -m_3 g \frac{(n_5 - n_4)}{l_3^3} - 4k_3 \left( \frac{n_5 - n_4}{l_3 l_3} \right), \tag{2–145} \]

\[ A_3 = m_3 g d_4 \left[ \frac{1}{l_3^2} - \frac{(n_5 - n_4)^2}{l_3^2 l_3^2} \right] - m_3 g n_5 \frac{\left( n_5 - n_4 \right)}{l_3^2} + 4k_3 \frac{\left( n_5 - n_4 \right) n_5 - n_4}{l_3}, \tag{2–146} \]

\[ B_3 = m_3 g n_5 l_3 - m_3 g n_5 \frac{n_5}{l_3} + m_3 g \frac{d_4 (n_5 - n_4)}{l_3^2} - 4k_3 \frac{l_3 n_5 l_3 + d_4 (n_5 - n_4)}{l_3}, \tag{2–147} \]

\[ C_3 = m_3 g \frac{n_5 (n_5 - n_4)}{l_3^2} + m_3 g d_4 \left[ \frac{(n_5 - n_4)^2}{l_3^2 l_3^2} - \frac{1}{l_3^2} \right] - 4k_3 \frac{(n_5 - n_4) n_5 l_3 + d_4 (n_5 - n_4)}{l_3}. \tag{2–148} \]

The equations for the intermediate mass are

\[ m_2 \ddot{y}_2 = R_2 (y_2 - y_1) + (S_2 n_3 - R_2 d_2) \varphi_2 - (S_2 n_2 - R_2 d_1) \varphi_1 - R_3 (y_3 - y_2) - \]

\[ (S_3 n_5 - R_3 d_4) \varphi_3 - (S_3 n_4 - R_3 d_3) \varphi_2 = -R_2 y_1 + (R_2 + R_3) y_2 - R_3 y_3 - \]

\[ (S_2 n_2 - R_2 d_1) \varphi_1 + (S_2 n_3 - R_2 d_2 + S_3 n_4 - R_3 d_3) \varphi_2 - (S_3 n_5 - R_3 d_4) \varphi_3, \tag{2–149} \]
\[ I_{2x} = Q^+_2 - Q^-_2 = A_2 (y_2 - y_1) + B_2 (n_3 \varphi_2 - n_2 \varphi_1) + C_2 (d_2 \varphi_2 - d_1 \varphi_1) + (m_2 + m_3) g \left( \frac{n_3 - n_2}{l_{t2}} - d_2 \right) \varphi_2 - D_2 (y_3 - y_2) - E_2 (n_5 \varphi_3 - n_4 \varphi_2) - G_2 (d_4 \varphi_3 - d_3 \varphi_2) - m_3 g \left( \frac{n_4 - n_4}{l_{t3}} - d_3 \right) \varphi_2 = -A_2 y_1 + (A_2 + D_2) y_2 - D_2 y_3 - (n_2 B_2 + d_1 C_2) \varphi_1 + \left[ n_3 B_2 + d_2 C_2 + n_4 E_2 + d_3 G_2 + (m_2 + m_3) g \left( \frac{n_3 - n_2}{l_{t2}} - d_2 \right) - m_3 g \left( \frac{n_4 - n_4}{l_{t3}} - d_3 \right) \right] \varphi_2 - (n_5 E_2 + d_4 G_2) \varphi_3. \]  

(2–150)

Here

\[ R_2 = (m_2 + m_3) g \left( \frac{1}{l_{t2}} - \frac{(n_3 - n_2)^2}{l_{t2}^2} \right) - 4k_2 \left( \frac{n_3 - n_2}{l_{t2}} \right)^2, \]  

(2–151)

\[ S_2 = (m_2 + m_3) g \left( \frac{n_3 - n_2}{l_{t2}^2} \right) + 4k_2 \left( \frac{n_3 - n_2}{l_{t2}} \right) l_{t2}, \]  

(2–152)

\[ A_2 = (m_2 + m_3) g d_2 \left[ \frac{1}{l_{t2}} - \frac{(n_3 - n_2)^2}{l_{t2}^2} \right] - (m_2 + m_3) g n_3 \left( \frac{n_3 - n_3}{l_{t2}} \right) + 4k_2 \frac{n_3 l_{t2} + d_2 (n_3 - n_2) n_3 - n_2}{l_{t2}}, \]  

(2–153)

\[ B_2 = (m_2 + m_3) g \frac{d_2 n_3 l_{t2}}{l_{t2}^2} - (m_2 + m_3) g \frac{n_3}{l_{t2}} + (m_2 + m_3) g \frac{d_2 (n_3 - n_2)}{l_{t2}^2} - 4k_2 \frac{l_{t2}}{l_{t2}}, \]  

(2–154)

\[ C_2 = (m_2 + m_3) g \frac{n_3 (n_3 - n_2)}{l_{t2}^2} + (m_2 + m_3) g d_2 \left[ \frac{(n_3 - n_2)^2}{l_{t2}^2} \right] - \frac{1}{l_{t2}} \]  

(2–155)

\[ D_2 = m_3 g d_3 \left[ \frac{1}{l_{t3}} - \frac{(n_5 - n_4)^2}{l_{t3}^2} \right] - m_3 g n_4 \left( \frac{n_5 - n_4}{l_{t3}^2} \right) - 4k_3 \frac{n_4 l_{t3} + d_3 (n_5 - n_4) n_5 - n_4}{l_{t3}}, \]  

(2–156)

\[ E_2 = m_3 g \frac{n_4 l_{t3}}{l_{t3}^2} - m_3 g \frac{n_5}{l_{t3}} + m_3 g \frac{d_3 (n_5 - n_4)}{l_{t3}^2} + 4k_3 \frac{l_{t3}}{l_{t3}} \]  

(2–157)

\[ G_2 = m_3 g \left( \frac{n_5 - n_4}{l_{t3}} \right) + m_3 g d_3 \left[ \frac{(n_5 - n_4)^2}{l_{t3}^2} \right] + \frac{1}{l_{t3}} + 4k_3 \frac{(n_5 - n_4) n_4 l_{t3} + d_3 (n_5 - n_4)}{l_{t3}}, \]  

(2–158)
And for the top mass, the equations for the sideways and roll motion are

\[
m_1 \ddot{y}_1 = R_1 y_1 + (S_1 n_1 - R_1 d_0) \varphi_1 - R_2 (y_2 - y_1) - (S_2 n_3 - R_2 d_2) \varphi_2 + (S_2 n_3 - R_2 d_2) \varphi_2, \tag{2–159}
\]

\[
\begin{align*}
I_{1x} \ddot{\varphi}_1 &= Q_1 - Q^- = A_1 y_1 + B_1 n_1 \varphi_1 + C_1 d_0 \varphi_1 + (m_1 + m_2 + m_3) g \left( n_1 \frac{n_1 - n_0}{l_{t1}} - d_0 \right) \varphi_1 - D_1 (y_2 - y_1) - E_1 (n_3 \varphi_2 - n_2 \varphi_1) - G_1 (d_2 \varphi_2 - d_1 \varphi_1) - (m_2 + m_3) g (n_3 \frac{n_3 - n_2}{l_{t2}} - d_1) \varphi_2 - d_1) \varphi_2 = (A_1 + D_1) y_1 - D_1 y_2 + B_1 n_1 + n_2 E_1 + d_1 G_1 + (m_1 + m_2 + m_3) g \left( n_1 \frac{n_1 - n_0}{l_{t1}} - d_0 \right) \varphi_1 - [n_3 E_1 + d_2 G_1 + (m_2 + m_3) g \left( n_3 \frac{n_3 - n_2}{l_{t2}} - d_1 \right) \varphi_2], \tag{2–160}
\end{align*}
\]

with

\[
R_1 = -(m_1 + m_2 + m_3) g \left[ \frac{1}{l_{t1}} - \frac{(n_1 - n_0)^2}{l_{t1}^2} \right] + 2k_1 \left( \frac{n_1 - n_0}{l_{t1}} \right)^2, \tag{2–161}
\]

\[
S_1 = -(m_1 + m_2 + m_3) g \frac{(n_1 - n_0)}{l_{t1}^2} - 2k_1 \frac{(n_1 - n_0)}{l_{t1}^2}, \tag{2–162}
\]

\[
A_1 = (m_1 + m_2 + m_3) g d_0 \left[ \frac{1}{l_{t1}} - \frac{(n_1 - n_0)^2}{l_{t1}^2} \right] - (m_1 + m_2 + m_3) g n_1 \frac{(n_1 - n_0)}{l_{t1}^2} + 2k_1 \frac{n_1 - n_0}{l_{t1}}, \tag{2–163}
\]

\[
B_1 = (m_1 + m_2 + m_3) g \frac{n_1 l_{t1}}{l_{t1}^2} - (m_1 + m_2 + m_3) g \frac{n_1}{l_{t1}} + (m_1 + m_2 + m_3) g \frac{d_0 (n_1 - n_0)}{l_{t1}^2} - 2k_1 \frac{l_{t1} n_1 l_{t1} + d_0 (n_1 - n_0)}{l_{t1}}, \tag{2–164}
\]

\[
C_1 = (m_1 + m_2 + m_3) g \frac{n_1 (n_1 - n_0)}{l_{t1}^2} + (m_1 + m_2 + m_3) g d_0 \left[ \frac{(n_1 - n_0)^2}{l_{t1}^2} - \frac{1}{l_{t1}} \right] - 2k_1 \frac{(n_1 - n_0) n_1 l_{t1} + d_0 (n_1 - n_0)}{l_{t1}}, \tag{2–165}
\]

\[
D_1 = (m_2 + m_3) g d_1 \left[ \frac{1}{l_{t2}} - \frac{(n_3 - n_2)^2}{l_{t2}^2} \right] - (m_2 + m_3) g n_2 \frac{(n_3 - n_2)}{l_{t2}} - 4k_2 \frac{n_2 l_{t2} + d_1 (n_3 - n_2)}{l_{t2}} \frac{n_3 - n_2}{l_{t2}}, \tag{2–166}
\]

\[\text{79}\]
\[ E_1 = (m_2 + m_3) g \frac{n_2 l_{t2}}{l_2} - (m_2 + m_3) g \frac{n_3}{l_{t2}} + (m_2 + m_3) g \frac{d_1 (n_3 - n_2)}{l_2^3} + 4k_2 \frac{l_{t2} n_2 l_{t2} + d_1 (n_3 - n_2)}{l_2}, \]

and

\[ G_1 = (m_2 + m_3) g \frac{n_2 (n_3 - n_2)}{l_2^2} + (m_2 + m_3) g d_1 \left[ \frac{(n_3 - n_2)^2}{l_2^3 l_{t2}} - \frac{1}{l_{t2}} \right] + 4k_2 \frac{n_2 l_{t2} + d_1 (n_3 - n_2)}{l_2} \]

The Equation 2–142, 2–149, 2–159, 2–143, 2–150, 2–160 are first Fourier-transformed and then rewritten in matrix form

\[
\begin{pmatrix}
\Lambda_{11} & \Lambda_{12} & 0 & \Lambda_{14} & \Lambda_{15} & 0 \\
\Lambda_{21} & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} & \Lambda_{25} & \Lambda_{26} \\
0 & \Lambda_{32} & \Lambda_{33} & 0 & \Lambda_{35} & \Lambda_{36} \\
\Lambda_{41} & \Lambda_{42} & 0 & \Lambda_{44} & \Lambda_{45} & 0 \\
\Lambda_{51} & \Lambda_{52} & \Lambda_{53} & \Lambda_{54} & \Lambda_{55} & \Lambda_{56} \\
0 & \Lambda_{62} & \Lambda_{63} & 0 & \Lambda_{65} & \Lambda_{66}
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
\varphi_1 \\
\varphi_2 \\
\varphi_3
\end{pmatrix} =
\begin{pmatrix}
F_s \\
0 \\
0 \\
Q_r \\
0 \\
0
\end{pmatrix}.
\]

Here,

\[
\Lambda_{11} = R_1 + R_2 + m_1 \omega^2, \quad (2–170)
\]

\[
\Lambda_{12} = -R_2, \quad (2–171)
\]

\[
\Lambda_{14} = S_1 n_1 - R_1 d_0 + S_2 n_2 - R_2 d_1, \quad (2–172)
\]

\[
\Lambda_{15} = -S_2 n_3 + R_2 d_2, \quad (2–173)
\]

\[
\Lambda_{21} = -R_2, \quad (2–174)
\]

\[
\Lambda_{22} = R_2 + R_3 + m_2 \omega^2, \quad (2–175)
\]

\[
\Lambda_{23} = -R_3, \quad (2–176)
\]
\[ \Lambda_{24} = -S_2 n_2 + R_2 d_1, \quad (2-177) \]
\[ \Lambda_{25} = S_2 n_3 - R_2 d_2 + S_3 n_4 - R_3 d_3, \quad (2-178) \]
\[ \Lambda_{26} = -S_3 n_5 + R_3 d_4, \quad (2-179) \]
\[ \Lambda_{32} = -R_3, \quad (2-180) \]
\[ \Lambda_{33} = R_3 + m_3 \omega^2, \quad (2-181) \]
\[ \Lambda_{34} = -S_3 n_4 + R_3 d_3, \quad (2-182) \]
\[ \Lambda_{36} = S_3 n_5 - R_3 d_4, \quad (2-183) \]
\[ \Lambda_{41} = A_1 + D_1, \quad (2-184) \]
\[ \Lambda_{42} = -D_1, \quad (2-185) \]
\[ \Lambda_{44} = B_1 n_1 + n_2 E_1 + d_1 G_1 + (m_1 + m_2 + m_3) g \left( n_1 \frac{n_1 - n_0}{l_{t1}} - d_0 \right) + I_1 \omega^2, \quad (2-186) \]
\[ \Lambda_{45} = n_3 E_1 + d_2 G_1 + (m_2 + m_3) g \left( n_3 \frac{n_3 - n_2}{l_{t2}} - d_1 \right), \quad (2-187) \]
\[ \Lambda_{51} = -A_2, \quad (2-188) \]
\[ \Lambda_{52} = A_2 + D_2, \quad (2-189) \]
\[ \Lambda_{53} = -D_2, \quad (2-190) \]
\[ \Lambda_{54} = -n_2 B_2 - d_1 C_2, \quad (2-191) \]
\[ \Lambda_{55} = n_3 B_2 + d_2 C_2 + n_4 E_2 + d_3 G_2 + (m_2 + m_3) g \left( n_3 \frac{n_3 - n_2}{l_{t2}} - d_2 \right) - m_3 g \left( n_4 \frac{n_5 - n_4}{l_{t3}} - d_3 \right) + I_2 \omega^2, \quad (2-192) \]
\[ \Lambda_{56} = -n_5 E_2 - d_4 G_2, \quad (2-193) \]
\[ \Lambda_{62} = -A_3, \quad (2-194) \]
\[ \Lambda_{63} = A_3, \quad (2-195) \]
\[ \Lambda_{65} = -B_3 n_4 - C_3 d_3, \quad (2-196) \]
\[ \Lambda_{66} = B_3 n_5 + C_3 d_4 + m_3 g \left( n_5 \frac{n_5 - n_4}{l_3} - d_4 \right) + I_3 \omega^2. \]  

(2–197)

Hence the sideways motion induced by \( F_s \) and the roll motion induced by \( Q_r \) can be described as

\[
\begin{pmatrix}
    y_1 \\
    y_2 \\
    y_3 \\
    \varphi_1 \\
    \varphi_2 \\
    \varphi_3
\end{pmatrix} = \Xi
\begin{pmatrix}
    F_s \\
    0 \\
    0 \\
    Q_r \\
    0 \\
    0
\end{pmatrix},
\]

(2–198)

where \( \Xi = \Lambda^{-1} \). The transfer functions that characterize the response of the top mass to \( F_s \) and \( Q_r \) are

\[
H_{ys} = \frac{y_1}{F_s} = \Xi_{11},
\]

(2–199)

\[
H_{yr} = \frac{y_1}{Q_r} = \Xi_{14},
\]

(2–200)

\[
H_{\varphi s} = \frac{\varphi_1}{F_s} = \Xi_{41},
\]

(2–201)

\[
H_{\varphi r} = \frac{\varphi_1}{Q_r} = \Xi_{44}.
\]

(2–202)

The Bode plots for the above transfer functions are shown in Figure 2-25, Figure 2-26, Figure 2-27, and Figure 2-28 respectively.

### 2.4 Local Control of the Triple Pendulum

Figure 2-29 shows the control loop of the triple pendulum system. Here, \( C \) represents the control channels while \( S \) represents the sensing channels. The control signals, which is the change of the current in coils, introduce forces on the magnets attached on the top mass. \( T_1 \) is the actuator matrix that describes the conversion from the current in the coils to the forces and torques on the top mass,
Figure 2-25. Bode plot of $H_{ys}$.

Figure 2-26. Bode plot of $H_{yr}$. 
Figure 2-27. Bode plot of $H_{\varphi s}$.

Figure 2-28. Bode plot of $H_{\varphi r}$.
Figure 2-29. Feedback control block diagram of the triple pendulum.

\[
\begin{pmatrix}
F_l \\
Q_p \\
F_v \\
Q_y \\
F_s \\
Q_r \\
\end{pmatrix}
= T_1
\begin{pmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4 \\
C_5 \\
C_6 \\
\end{pmatrix},
\]

(2–203)

\(T_1\) can be represented by a matrix,
The transfer functions from the forces and torques to the motions are included in $H$. The resulting motions of the top mass in six DOF are sensed. $T_2$ is the sensor matrix that describes the conversion from these motions to the change of the positions of magnets,

\[
T_2 = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
1_s & l_s & l_p & -l_p & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
l_y & -l_y & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & l_r & l_r & -l_r & l_s
\end{pmatrix}.
\]

The induced current variations are fed back as the error signals to adjust the current in coils. $G$ is the feedback controller.

Only the local control on the top mass has been considered in this model for the triple pendulum system so far, but the model can be extended to include the global control on
all three masses. The model can be used to figure out the mode frequencies of the triple pendulum. It is also going to be a straightforward practice to add the damping force terms in the model. Understanding the dynamics of the triple pendulum is essential to, for example, study the cross coupling effect which is the most serious problem involving a multistage isolation system, and model the effect of damping the pendulum mode frequencies.

2.5 Measurement Result

To confirm the mechanical analysis described above and test the modeling for the system including the local control part, measurements of transfer functions between various control channels on the upper mass were carried out. The measurement for the system in Figure 2-29 is shown in Figure 2-30. White noise input signals generated by an SR785 model signal analyzer were injected into specified channels. By choosing different input points, motions of the pendulum of different degrees of freedom were excited while
the feed-back control forces brought the pendulum back to the balanced position. Signals which represented motions of top mass were splitted out from the control loops and measured by the signal analyzer. Part of the measurement results are described below. Table 2-1 lists three sets of measurements with specified input and output points. And the transfer functions are plotted and compared with the modeling results.

Figure 2-31 shows the close loop transfer function with channel 1 and channel 2 as the input points and channel 1 as the output point. This transfer function was used to confirm the mode frequencies of the longitudinal motion. There is a reasonably good match between the measurement result (the red curve) and the modeling result (the blue curve). We can excited longitudinal mode at 6 Hz, 1.3 Hz and 2.3 Hz, which has been shown in the pendulum transfer function plot (Figure 2-14). The extra dips at around 1 Hz and 2.1 Hz shown in the measurement result are believed to be associated with the yaw motion that was excited due to the difference between the electrical gain for two coils. These two mode frequencies are conformed in Figure 2-20. The small dips located between 7 Hz and 8 Hz in the measurement curve represent motions of the supporting stack which are used to hold the suspension pendulum.

Vertical modes can be seen in the transfer functions plotted in Figure 2-32. Mismatches between the mode frequencies shown in the measurement result and the calculated ones are around 0.4 Hz. This is possibly due to the inaccuracy of the spring constants of the wires used in the model.

The transfer functions shown in Figure 2-33 were obtained with the input and output points connected in channel 6. Both the measurement and modeling results show sideways

Table 2-1. Input and output points for different transfer function measurements.

<table>
<thead>
<tr>
<th>Motion</th>
<th>Input point</th>
<th>Output point</th>
</tr>
</thead>
<tbody>
<tr>
<td>longitudinal</td>
<td>CH1 + CH2</td>
<td>CH1</td>
</tr>
<tr>
<td>vertical</td>
<td>CH3 + CH4</td>
<td>CH5</td>
</tr>
<tr>
<td>sideway</td>
<td>CH6</td>
<td>CH6</td>
</tr>
</tbody>
</table>
Figure 2-31. Close loop transfer function with the input signal being injected to coil 1 and coil 2 and the output signal from channel 1.
Figure 2-32. Close loop transfer function with the input signal being injected to coils 3, 4 and 5 and the output signal from channel 3.
Figure 2-33. Close loop transfer function with the input signal being injected to coil 6 and the output signal from channel 6.

modes around 0.59 Hz and 1.3 Hz. There is a 2.1 Hz dip in the measurement curve while the modeling result shows a 2.2 Hz dip.
CHAPTER 3
ELECTRO-OPTIC MODULATOR (EOM)

3.1 Input Optics Subsystem (IO) of Advanced LIGO

The IO works as the bridge between the pre-stabilize laser (PSL) and the main interferometer (IFO). Figure 3.1 shows the modified schematic view of the IO system designed for the Advanced LIGO (the original figure is taken from Ref. [28]). The light is conditioned in the IO to meet the primary scientific requirements for Advanced LIGO.

The laser beam from the PSL is first modulated using radio frequency (RF) electro-optic modulators (EOMs) to generate the frequency components (sidebands) for controlling the interferometer. The working mechanism of an EOM will be discussed in Section 3.2. The mode matching telescope (MMT) after the EOMs modifies the mode of the light to match the mode defined by the mode cleaner. The power control part is based on the conventional design of using a polarizer and a half wave plate. A motorized rotational stepper stage will be used to rotate the half wave plate. The optical power incremental step needs to be small and the power changing rate needs to be slow to ensure that the length-control system of the mode cleaner will be able to track the disturbance on the mirrors due to the change of the radiation pressure.

The input mode cleaner stabilizes the frequency and suppresses spatial fluctuations of the laser beam. The length of the mode cleaner cavity determines that the modulation frequencies need to be integer multiples of the free spectral range (FSR) of the mode cleaner. That is

\[ \Omega = N \frac{c}{2L}, \]

(3–1)

where \( \Omega \) is the modulation frequency, \( N \) is an integer, \( c \) is the speed of light and \( L \) is the length of the mode cleaner cavity. In this way, both carrier light and sidebands will resonate inside the mode cleaner and will be transmitted.

The Faraday isolator (FI) performs the conventional role of rejecting the back-reflected light from propagating towards the input port. The FI designed for the Advanced LIGO
Figure 3-1. Overall IO schematic.
uses a birefringence-compensated Faraday rotator (consisting of two TGG crystals and a quartz rotator) and a thermal lens compensation material so that it is suitable for high power laser applications.

The MMT after the FI matches the beam to the proper mode to satisfy the resonant condition of the main interferometer. Currently, there are two conceptual designs for the MMT, according to which two possible designs of the power recycling cavity (PRC) will be used: either a marginally stable power recycling cavity (MSPRC) or a stable power recycling cavity (SPRC) [29]. MSPRC will allow many spatial modes of RF sidebands to be resonant inside the PRC and introduce losses to the $TEM_{00}$ mode of the sidebands. SPRC can reduce the loss in the fundamental mode, but has the potential drawback of increasing the complexity of the alignment sensing and control (ASC) system and might contribute to the parametric instabilities [30].

The IO subsystem of Advanced LIGO requires higher quality optical components in comparison with LIGO. The electro-optic phase modulator (EOM), one of the key components in the IO, is of interest in this thesis.

### 3.2 Application of EOMs

When light traverses through an EOM, the phase of the light field which accumulated along the optical path in the EOM crystal is proportional to the refractive index. A variation of the refractive index is introduced by the applied electric field via the electro-optic effect. Since the phase of a laser field is proportional to the light path, the phase modulation (PM) of the light can be realized by applying a time-dependent voltage signal across the crystal inside an EOM. In the frequency domain, a phase-modulated laser field can be viewed as being split into a series of frequency components. The laser field after the EOM can be described as a mixture of the carrier field, a pair of sidebands separated from the central frequency component (the carrier) by the modulation frequency $\Omega$, and high order sidebands with angular frequencies $N \times \Omega$ away from the central
frequency. The light field can be written as

$$E = E_0 e^{i(\omega t + m \sin \Omega t)} = J_0 (m) E_0 e^{i\omega t} + J_1 (m) E_0 e^{i(\omega + \Omega) t} - J_1 (m) E_0 e^{i(\omega - \Omega) t} + \cdots.$$  (3-2)

Here $E_0$ is the field amplitude, $\omega$ is the angular frequency of the original laser field, $\Omega$ is the modulation frequency, and $m$ is the modulation index. The higher order harmonics are omitted in the equation.

All laser field components, including carrier and sidebands, are delivered to the interferometer. The reflected, transmitted, and internal pickoff fields which are sensed by interferometer control photodetectors, which are placed at various detection ports, can be generally described by frequency dependent transfer functions,

$$E_{out} = E_0 \left\{ J_0 (m) T_0 e^{i\omega t} + J_1 (m) T_+ e^{i(\omega + \Omega) t} - J_1 (m) T_- e^{i(\omega - \Omega) t} \right\} + \cdots.$$  (3-3)

$T_0$, $T_+$, $T_-$ represent the transfer functions for the carrier, the upper sideband and the lower sideband to the specific photodetectors. These transfer functions are trivially dependent on the location of the photodetectors. They are also functions of the relative positions of the recycling mirrors, beam splitter and test masses. The motions of these optical components affect the laser field at the photodetector as in different regions of the interferometer different field components are in different resonance conditions. Figure 3-3 shows the possible locations of the photodetectors used for the length sensing and control.
Figure 3-3. Possible locations of photodetectors in Advanced LIGO.

in Advanced LIGO. As has been briefly mentioned in Chapter 1, demodulating a specified beat signal between two frequency components of the laser field will yield an error signal that reveals a change in certain longitudinal degrees of freedom of certain mirrors.

The proposed control schemes [31, 32] for Advanced LIGO both employ one pair of sidebands in order to sense the length of the mode cleaner and two pairs of sidebands to sense the longitudinal degrees of freedom of the interferometer. However, when two phase modulators are used in series, the sidebands created by the first EOM will be phase modulated by the following one. This generates ‘sidebands on sidebands’ which will beat with the carrier field. The frequencies of the beat signals will be identical to the frequencies of the beat signals between the sidebands. Beat notes between sidebands will be used to generate the length sensing signals of Advanced LIGO. Hence the existence of ‘sidebands on sidebands’ will produce offsets in the real length sensing signals and degrade the diagonalization of the locking matrix.
There are two techniques proposed as the solution to the sidebands on sidebands problem. One solution is to split the incoming beam using a Mach-Zehnder interferometer and place an EOM in each arm [33]. In this Mach-Zehnder interferometer, the ‘sidebands on sidebands’ do not exist anymore because the sidebands generated by one EOM are not phase modulated by the other EOM. Another solution, termed complex modulation, involves the use of one amplitude modulator and one phase modulator to apply specified (non-sinusoidal) amplitude and phase modulation, resulting in an electro-optically modulated field which is comprised of desired frequency components without mixing terms [34]. The target modulation state is reached by adjusting only the electrical fields which drive both modulators. In general, the relationship between the original light field $E$ and the amplitude and the phase modulated light field $E'$ can be described as

$$E' = E e^{A(t) + i\phi(t)},$$  \hspace{1cm} (3-4)

where $A(t)$ is the amplitude modulation function, and $\phi(t)$ is the phase modulation function. Given $E$ and a desired modulated field $E'$, these two functions can be calculated:

$$A(t) = \ln \left| \frac{E'}{E} \right|,$$  \hspace{1cm} (3-5)
\[
\phi(t) = \tan^{-1} \left( \frac{\text{Im}\{\frac{E'}{E}\}}{\text{Re}\{\frac{E'}{E}\}} \right).
\]  

(3-6)

Detailed discussion of the length sensing and control of the dual-recycling interferometer is beyond the scope of this thesis. What we focus on are the technical issues related to the application of EOMs. In general, the rules of thumb are:

i) The implementation of the EOMs should not affect the function of other units of the interferometer;

ii) The EOMs should not produce noise that could deteriorate the sensitivity of Advanced LIGO.

3.3 RTP Crystal EOMs

EOMs which are currently used in LIGO are commercially available EOMs made of LiNbO\(_3\) crystals. LiNbO\(_3\) crystals have been shown to have a relatively large optical absorption coefficient at 1064 nm, the wavelength of the laser used in LIGO. The proposed application of 180 W laser power in Advanced LIGO will cause unacceptable thermal lensing problems if these EOMs are used. In order to meet the requirements of Advanced LIGO, we have developed EOMs based on RbTiO\(_2\)PO\(_4\) (RTP) crystals instead of LiNbO\(_3\).

3.3.1 Physical Properties of RTP Crystals

Our RTP crystals are provided by Raicol. Raicol has claimed that the optical absorption coefficient of RTP crystals at 1064 nm wavelength is as low as 50 ppm/cm. RTP crystals also have a low electrical loss angle and a high laser damage threshold. Table 3-1 lists some physical properties of RTP crystals.

RTP crystals have an orthorhombic crystal structure and their EO-coefficient matrix (linear electro-optic tensor) is:
Table 3-1. Physical properties of RTP crystals.

<table>
<thead>
<tr>
<th>properties (units)</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>$dn/dT , (\times 10^{-6}/K)$ (^a)</td>
<td>-2.79</td>
<td>0.24</td>
<td></td>
</tr>
<tr>
<td>$\kappa , (\text{W/K/m})$ (^b)</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$\epsilon , (\times 10^{-5}/K)$ (^c)</td>
<td>-1.2733</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>$n , 1064 \text{ nm} , ^d$</td>
<td>2.15</td>
<td>2.38</td>
<td>2.27</td>
</tr>
<tr>
<td>laser damage threshold (MW/cm(^2), 10 ns @ 1064nm (^e))</td>
<td>600</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(^a\) The $dn/dT$ value for RTP was obtained from [35].
\(^b\) The thermal conductivity data was obtained from [36].
\(^c\) The thermal expansion coefficient was obtained from [35].
\(^d\) The refractive indices data come from [37].
\(^e\) The laser damage threshold data was provided by Raicol.

\[ \mathbf{r} = \begin{bmatrix} 0 & 0 & 12.5 \\ 0 & 0 & 17.1 \\ 0 & 0 & 39.6 \\ 0 & r_{42} & 0 \\ r_{51} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ pm/V}^1 \right] . \tag{3-7} \]

3.3.2 EOM Configuration

The largest EO-coefficient of the RTP crystal is $r_{33}$. Therefore, the optimal configuration is to apply voltage along the $\hat{z}$ direction and let the $z$-polarized light propagate along the $\hat{y}$ direction. If the incoming light’s polarization direction is not perfectly parallel to the $\hat{z}$ direction, the light beam is separated into two beams when it transmits the front surface of the wedged crystal (see Figure 3-5). The difference between the refractive index of the p-polarized light and that of the s-light results in two different refraction angles according to Snell’s law. Both the front and the end surfaces of the crystal have the same wedge angle. In Figure 3-5, the light path of the p-polarized light

\(^1\) $r_{42}$ and $r_{51}$ are currently unknown.
Figure 3-5. RTP crystals (wedged and non-wedged) mounted between two electrodes.

parallels the electrodes so that the deflection angle of the p-polarized light, $\varphi_{\text{out}(p)}$, equals the incident angle, $\varphi_{\text{in}}$. The s-light and p-polarized light no longer overlap each other after transmitting the wedged-crystal.

The modulation index of the modulator for the p-polarized light is

$$m_p = \frac{\pi L}{\lambda} r_{333} n_z^3 \frac{V_z}{d}, \quad (3-8)$$

and that for the s-light is

$$m_s = \frac{\pi L}{\lambda} r_{133} n_x^3 \frac{V_z}{d}, \quad (3-9)$$

where $L$ is the length of the crystal, $d$ is the thickness in the $\hat{z}$-direction, $\lambda$ is the wavelength of the light, $n_z$ and $n_x$ are the indices of refraction of the crystal along the $z$-axis and the $x$-axis, and $V_z$ is the applied voltage. Our non-wedged RTP crystals have a dimension of $4 \text{ mm} \times 4 \text{ mm} \times 15 \text{ mm}$, thus the half-wave voltage $V_\pi$ for the p-polarized light is 612.5 V.
Because electro-optic crystals perform like a capacitor in electronic circuits, the simplest way to build up the voltage across the crystal is to connect it in series with an inductor to form an L-C resonant circuit. The impedance of the resonant circuit needs to be matched to the output impedance of the EOM driver (usually 50 Ω) at the resonant frequency to reduce the power loss due to standing waves. This is simply realized by connecting an adjunct capacitor in parallel with the $L$ and $C_{\text{crystal}}$. The circuit diagram is shown in Figure 3-6. The inductor $L$ and the capacitor $C$ can be tuned to find a balance between the consideration for both the voltage built up and the power loss. As compared with the widely adopted technique of using a resonant transformer to drive an EOM, this design gained an advantage that the input impedance at low frequencies is very high.

### 3.4 Technical Features

Technical concerns associated with the application of EOMs in the laser interferometric interferometer include piezo-resonances, thermal lensing, residual amplitude modulation, and potential amplitude and phase noise imposed on the light.

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2 Heinzl discussed this design in his thesis [38].
3.4.1 Piezo-resonances

3.4.1.1 Theory

As we mentioned earlier, the electro-optic effect induces the change of the refractive index of the crystal inside an EOM. Besides this primary effect, the refractive index can also be changed via a secondary effect due to piezoelectricity and photo-elasticity of the crystal. An electric field will cause a strain inside the crystal via the converse piezoelectric effect, and this in turn will create a change in refractive index through the photo-elastic effect. Coupling between the electric field applied across the crystal and its mechanical strain is maximized at the piezoelectric resonance frequencies. Piezo-resonances of the crystal near the modulation frequency interfere with the modulation and alter the amplitude and the phase of the sidebands. This becomes a practical limitation for the application of EOMs especially in cases when the stability of the phase modulation is demanded.

In general, the piezoelectric resonance frequencies are determined by the velocity of the surface acoustic wave (SAW), $v_{\text{saw}}$, and the dimensions of the crystal $L_x$, $L_y$ and $L_z$. It can be calculated as

$$f = \frac{v_{\text{saw}}}{N L_i}, \quad (3-10)$$

where $i = x, y, z$ and $N = 1, 2, 3 \cdots$. A SAW is an acoustic wave traveling along the surface of a material having some elasticity, with an amplitude that typically decays away from the surface. The physical properties of the SAW and its crystal orientation determine the SAW velocity. $v_{\text{saw}}$ can be calculated as,

$$v_{\text{saw}} = \sqrt{\frac{c}{\rho}}, \quad (3-11)$$

where $\rho$ is the mass density of the material, $c$ is the effective elastic constant which depends on the elastic and piezoelectric tensors and their temperature coefficients. Further discussion of the effective elastic constant can be found in Ref. [39] and examples for calculating $c$ are given in Ref. [40].
3.4.1.2 Measurement

Figure 3-7 shows the experimental set up to measure the piezo-resonance frequencies of the RTP crystal. We shined a 1064nm Nd:YAG laser through two cross polarizers and align the laser beam on a photodetector (Thorlabs PDA255). The RTP crystal was placed between the polarizers. The crystal with both ends anti-reflection coated was mounted between two electrodes which were connected to the source of a spectrum analyzer. The signal, in swept sine mode from the spectrum analyzer, was applied across the electrodes to modulate the crystal.

The polarization direction of the incoming light, which was determined by the first polarizer, was misaligned with the optical axis of the crystal. Applying an electric field to the crystal induced a change in the indices of refraction (both ordinary and extraordinary) giving rise to an electric field dependent birefringence. The crystal acted as a variable wave plate with the change of the phase retardation linearly dependent on the
applied electric field. The second polarizer converted the phase change into an amplitude modulated signal. The way that this kind of set up converts the phase retardation into an amplitude modulation signal which was detected by the photodetector is going to be described in Subsection 3.4.3. The amplitude modulation signal was maximized by rotating the polarization direction of the polarizers. At the piezo-resonance frequencies of the crystal, the response of the light to the driving signal was greatly enhanced and appears as resonant peaks in the spectrum measured via the spectrum analyzer.

From the transfer function plotted in Figure 3-8, the largest piezo resonance is at 680 kHz. The resonance with the highest frequencies are seen around 6 MHz. Although the modulation scheme of Advanced LIGO has not been finalized, there is no proposal of using a resonant modulator below 9 MHz. Consequently, piezo-resonances will not be a technical problem with the application of the new RTP crystal modulator in Advanced LIGO.
3.4.2 Thermal Lensing

3.4.2.1 Thermal effects in crystals

Thermal lensing in the EOM could affect the spatial mode of the laser field. The mechanism occurs under the following way. The EOM crystal’s optical absorption of the laser beam traveling through it will create a temperature gradient $\Delta T$ in the crystal. If the laser beam propagates through the center of the crystal along the $\hat{z}$ direction and the heat conducts from the center to the edge of the crystal along radial direction $\hat{r}$, the temperature gradient can be found by solving the thermal diffusion equation

$$\nabla^2 T(r, z) + \frac{A(r, z)}{\kappa} = 0,$$

(3–12)

where $T(r, z)$ is the spatial temperature distribution inside the crystal. $A(r, z)$ accounts for the deposition of heat due to the optical absorption of the laser beam. $\kappa$ is the thermal conductivity. The solution can be obtained analytically or numerically once the boundary conditions for Equation 3–12 are given. The boundary conditions are determined by the shape of the crystal and the way the heat is extracted from the crystal. In our EOMs, the ground electrode below the crystal which directly contacts the aluminum packaging case plays the role of a heat sink. Without giving an explicit expression for the temperature profile $T(r, z)$, we know that the center of the crystal is at a higher temperature than the edges due to the Gaussian distribution of the light intensity. The temperature gradient $\Delta T(r, z)$ changes the refractive index of the crystal along the axes perpendicular to the propagation axis according to the size of $dn/dT$. So the laser beam traversing through the crystal experiences a change in the optical path length due to the position dependent refractive index. Thermo-elastic deformation and thermally dependent elasto-optic effects will change the optical path length. In our case, these two effects are smaller than the $dn/dT$ effect. So we can roughly estimate the spatial dependence of the differential optical
path length as:

$$\Delta OPL(r) = \frac{dn}{dT} \int_0^L \Delta T(r,z) \, dz. \quad (3–13)$$

This additional optical path length put an additional spatial phase on the fundamental TEM$_{00}$ mode of the beam transmitting an EOM. This effect is known as ‘thermal lensing’ because it alters the modal properties of a beam in a way similar to an ordinary lens. The thermally induced modal distortion was characterized by monitoring the change of the beam divergence due to the ‘thermal lens’ created in the EOM crystal.

### 3.4.2.2 Experiment

Figure 3-9 shows the experimental arrangement used to characterize the thermal lensing effect. We used a single mode CW laser to probe the ‘thermal lens’ formed inside a RTP crystal. The output power of the laser was 100 W. First, the laser beam was focused by a lens and the beam divergence was measured using a beam profiler. The radius of the beam waist after the lens was about 0.4 mm. Then the RTP crystal was placed in the beam path about 0.25 m away from the waist where the beam radius was about 600 µm. This location was within the Rayleigh range of the beam. The beam divergence after the
crystal was measured again. The change in the beam divergence is plotted in Figure 3-10. The beam divergence can be fitted into the profiles with a $M^2$ value of about 1.75, which is due to the laser modal distortion created by the other optical components heated by the high power laser. Using simple geometric optics, the focal length $f$ of the thermal lens can be derived from

$$\frac{w}{f} = \tan \alpha - \tan \beta.$$  \hspace{1cm} (3–14)

Here, $w$ is the beam radius at the crystal, and the angles $\alpha$ and $\beta$ represent the far-field divergence of the laser beam with and without the crystal in the beam path. Hence, we calculated a lower limit on the ‘thermal lens’ of about 9 m. This 9 m ‘thermal lens’ corresponded to a change in the laser beam’s wavefront sagittal depth, $\delta s$, of about 10 nm.
(see Figure 3-11). This was estimated from:

$$\delta s \approx \frac{w^2}{4f}.$$  \hfill (3–15)

The thermal lensing effect is basically determined by the optical absorption coefficient $\alpha$, the thermal conductivity $\kappa$, the temperature dependence of the refractive index $dn/dT$, the length of the crystal $L$ and the laser power $P$. The sag change of the wavefront $\delta s$ can be evaluated using the equation from [41]:

$$\delta s \approx 0.105\frac{\alpha LP}{\kappa} \frac{dn}{dT}.$$ \hfill (3–16)

From the measurement result as described above, the optical absorption coefficient $\alpha$ of the RTP crystal can be estimated to be about 250 ppm/cm using Equation 3–16.

In Advanced LIGO, the laser power will be 180 W. So the thermal lensing magnitude in terms of $\delta s$ can be estimated to be 1.8 times the $\delta s$ introduced by a 100 W laser, which is 18 nm. Supposing that all light can be coupled into the arm cavities when the input laser’s power is low, the change of the wavefront when the laser power is increased to 180 W will cause mode mismatch. The fraction of light that will not be coupled to the arm cavities due to the change of the wavefront can be estimated to be [42]

$$\frac{\delta P}{P} \approx \left(\frac{\pi \delta s}{\lambda}\right)^2.$$ \hfill (3–17)

So the maximum power that could be lost due to the thermal lensing in an RTP crystal EOM is 0.25% assuming that we don’t compensate this in the mode-matching telescope.
It is noteworthy to mention that the 0.25% mode mismatch is derived based on an assumption that the thermally induced optical path length gradient follows a parabolic curve inside the crystal. Hence the ‘thermal lens’ described here is an ideal spherical lens. In principle, the change of the mode matching due to an additional spherical lens in the beam path can be corrected by adjusting the mode-matching telescope. In reality, the non-spherical part of the ‘thermal lens’ is unavoidable, which will create high order modes that cannot be coupled to arm cavities. However, the power loss due to the thermal lensing inside RTP crystal EOMs in Advanced LIGO will be much less than 0.25% and the losses become negligible.

3.4.3 Residual Amplitude Modulation (RAM)

3.4.3.1 Generation mechanism

Misalignment of the crystal axis with respect to the polarization direction of the incoming light causes residual amplitude modulation when an EOM is used to modulate the phase of the light field. This is illustrated in Figure 3-12 which shows the incoming light as a linearly polarized light with a wave vector normal to the front surface of the crystal. The angle between the polarization direction of the incoming light and the $e$ axis of the crystal is $\beta$. The light field can be seen as consisting of two orthogonal components along the principal axes of the crystal. The optical path lengths for the two components are different due to the birefringence of the crystal. The resulting phase retardation causes

Figure 3-12. Residual amplitude modulation due to the misalignment of an EOM.
the light after the EOM to become elliptically polarized. The light is then ‘filtered’ by subsequent polarization sensitive optical components, e.g., a polarizer. Should there be a misalignment between the crystal axis and the direction of the polarizer, the output light field will become both phase- and amplitude-modulated. Given an incoming light field

\[ \vec{E}_0 = E_0 e^{i\omega t}, \]

the output field can be written as

\[ \vec{E} = E_0 \left( \cos \beta \cos \gamma e^{i\omega t + m_1 \sin \Omega t} + \sin \beta \sin \gamma e^{i\omega t + m_2 \sin \Omega t + \Delta \phi} \right). \]  

(3–19)

Here \( \gamma \) is the angle between the crystal axis and the direction of the polarizer. \( \Omega \) is the modulation frequency. \( m_1 \) and \( m_2 \) are two modulation indices along two principal crystallographic axes, which are plotted as the \( e \)-axis and the \( o \)-axis in Figure 3-12 and lay in the plane vertical to the beam propagation direction. \( \Delta \phi \) is the phase retardation between the light field along the \( e \)-axis and the light field along the \( o \)-axis. This is due to the difference between the refractive indices, \( n_o \) and \( n_e \), along these two axes such that,

\[ \Delta \phi = (n_o - n_e) \frac{\omega}{c} L. \]  

(3–20)

Equation 3–19 can be expanded as

\[
\vec{E} \simeq \left( \cos \beta \cos \gamma J_0 (m_1) + \sin \beta \sin \gamma e^{i\Delta \phi} J_0 (m_2) \right) e^{i\omega t} E_0 + \left( \cos \beta \cos \gamma J_1 (m_1) + \sin \beta \sin \gamma e^{i\Delta \phi} J_1 (m_2) \right) e^{i(\omega - \Omega)t} E_0.
\]

(3–21)

Thus the complex amplitude of the carrier field is

\[ E_c = \left( \cos \beta \cos \gamma J_0 (m_1) + \sin \beta \sin \gamma e^{i\Delta \phi} J_0 (m_2) \right) E_0. \]  

(3–22)

Since \( \beta \) and \( \gamma \) can be minimized to be very small through alignment, \( E_c \) can be approximated as \( \left( J_0 (m_1) + \beta \gamma e^{i\Delta \phi} J_0 (m_2) \right) E_0 \). Hence the carrier field can be seen as consisting of two parts: the original field \( J_0 (m_1) E_0 e^{i\omega t} \) and a residual component \( \beta \gamma e^{i\Delta \phi} J_0 (m_2) E_0 e^{i\omega t} \).
Similarly, an RF sideband also consists of a principal component \( J_1 (m_1) E_0 e^{i(\omega+\Omega)t} \) and a residual term \( \beta \gamma e^{i\Delta \phi} J_1 (m_2) E_0 e^{i(\omega+\Omega)t} \).

Equation 3–19 describes RAM as a function of misalignment angles for a simplified case which is shown in Figure 3-12. Consider more general cases when the incoming light is not vertical to end surfaces of the crystal. The way that refractive indices change with an electrical field being applied across the crystal needs to be specified in detail in order to accurately calculate the amount of light that is amplitude modulated by the EOM, knowing the misalignment angles. In Figure 3-13, \( \vec{E} \) is the light field with a wave vector \( \vec{k} \). Before applying voltage across the crystal, the principal crystallographic axes are along the \( X, Y \) and \( Z \) axes, respectively. And the orientation and relative magnitude of refractive indices of the crystal can be described by the index ellipsoid equation,

\[
\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1. \quad (3–23)
\]
$n_x$, $n_y$ and $n_z$ are refractive indices along three axes and $L$ is the distance that the light travels inside the crystal. When an electrical field is built up across the crystal, the principal axes will rotate and the magnitude of the principal refractive indices will change. Consider the specific example of RTP crystals. The index ellipsoid equation in the presence of an electric field $\vec{\varepsilon}$ can be written as

$$\left(\frac{1}{n_x^2} + r_{13}\varepsilon_z\right)x^2 + \left(\frac{1}{n_y^2} + r_{23}\varepsilon_z\right)y^2 + \left(\frac{1}{n_z^2} + r_{33}\varepsilon_z\right)z^2 + 2r_{42}\varepsilon_y y z + 2r_{51}\varepsilon_x x z = 1, \quad (3-24)$$

where $\varepsilon_x$, $\varepsilon_y$ and $\varepsilon_z$ are components of the applied electric field along the $X$, $Y$ and $Z$ directions. This new ellipsoid reduces to the original one (defined by Equation 3–23) when $\vec{\varepsilon} = 0$.

A new set of principal axes ($X'$, $Y'$, $Z'$ axes in Figure 3-13) can be found by a principal-axis transformation of the quadratic form $S$

$$S = \begin{pmatrix}
\frac{1}{n_x^2} + r_{13}\varepsilon_z & 0 & r_{51}\varepsilon_x \\
0 & \frac{1}{n_y^2} + r_{23}\varepsilon_z & r_{51}\varepsilon_y \\
r_{51}\varepsilon_x & r_{51}\varepsilon_y & \frac{1}{n_z^2} + r_{33}\varepsilon_z
\end{pmatrix}. \quad (3-25)$$

The $S$ matrix can be transformed to a diagonal matrix $S'$. The components of $S'$ are eigenvalues of $S$. The magnitudes of the principal axes of the index ellipsoid (i.e., the refractive indices along new principal axes) are

$$n_i = \frac{1}{\sqrt{S_i'}}, \quad (3-26)$$

where $i = X', Y', Z'$. They change as the electric field $\vec{\varepsilon}$ changes. In general, the orientation of the new principal axes also change with $\vec{\varepsilon}$. But in some cases, the new orientation of the ellipsoid could coincide with the unperturbed axes. For example, when $\vec{\varepsilon}$ is applied along $Z$-axis of an RTP crystal, the quadratic form becomes
\[ S = \begin{pmatrix}
\frac{1}{n_x^2} + r_{13} \varepsilon_z & 0 & 0 \\
0 & \frac{1}{n_y^2} + r_{23} \varepsilon_z & 0 \\
0 & 0 & \frac{1}{n_z^2} + r_{33} \varepsilon_z
\end{pmatrix}. \tag{3–27}
\]

The orientation of the principal axes remains unchanged while the new refractive indices are

\[ n'_x = \sqrt{\frac{1}{\frac{1}{n_x^2} + r_{13} \varepsilon_z}} \approx n_x - \frac{1}{2} n_x^3 r_{13} \varepsilon_z, \tag{3–28} \]

\[ n'_y = \sqrt{\frac{1}{\frac{1}{n_y^2} + r_{23} \varepsilon_z}} \approx n_y - \frac{1}{2} n_y^3 r_{23} \varepsilon_z, \tag{3–29} \]

\[ n'_z = \sqrt{\frac{1}{\frac{1}{n_z^2} + r_{33} \varepsilon_z}} \approx n_z - \frac{1}{2} n_z^3 r_{33} \varepsilon_z. \tag{3–30} \]

The orientation and magnitudes of the new principal axes determine the polarization of the light field \( \vec{E}' \) after the EOM. \( \vec{E}' \) can be written as

\[ \vec{E}' = \left( E'_x e^{i\phi_1} \hat{x} + E'_y e^{i\phi_2} \hat{y} + E'_z e^{i\phi_3} \hat{z} \right) e^{i\omega t}, \tag{3–31} \]

where \( \phi_1, \phi_2 \) and \( \phi_3 \), which satisfy \( \phi_i = 2\pi \frac{n_i L}{\lambda}, i = x, y, z \), are phases accumulated by three orthogonal field components \( E'_x, E'_y \) and \( E'_z \) as the light travels through the crystal. This elliptically polarized light field can be converted into an amplitude modulated, linearly polarized light by a polarizer. In Figure 3-13, plane ABCD is the incident plane defined by the polarizer. The field of the transmitted p-polarized light is given by

\[ \vec{E}' = \left( E'_x e^{i\phi_1} \cos \alpha + E'_y e^{i\phi_2} \cos \beta + E'_z e^{i\phi_3} \cos \gamma \right) e^{i\omega t}, \tag{3–32} \]

where \( \alpha, \beta \) and \( \gamma \) are the angles between \( X', Y', Z' \) axes and plane ABCD.

RAM can also be introduced via the Fabry-Perot cavity effect if an un-wedged crystal is used inside an EOM. Back-reflections between the end surfaces of the crystal will alter
the harmonic content of the modulated optical beam and form a measurable amplitude modulation component onto the light field. In Figure 3-14, the relationship between the incident field $E$ and the transmitted field $E'$ can be written as

$$E' = \frac{t_1 t_2 e^{-i\phi}}{1 - r_1 r_2 e^{-i\phi}} E,$$  \hspace{1cm} (3–33)

where $r_1, t_1$ are the amplitude reflectivity and transmissivity of the front surface, $r_2, t_2$ are those of the end surface, $\phi$ is the phase determined by the optical path length

$$\phi = \frac{2\pi L n}{\lambda}.$$  \hspace{1cm} (3–34)

where $\lambda$ is the wavelength of the light field, $L$ is the length of the crystal, and $n$ is the refractive index along the the $\hat{z}$ direction. When a sinusoidal voltage signal $V \sin \Omega t$ is applied across the crystal along the $\hat{z}$ direction, the refractive index becomes

$$n = n_0 - \frac{1}{2} n_0^3 r_{33} \frac{V}{d}$$  \hspace{1cm} (3–35)

according to Equation 3–30, where $n_0$ is the refractive index before applying the driving voltage. Hence the phase $\phi$ can be written as

$$\phi = \phi_0 + m \sin \Omega t,$$  \hspace{1cm} (3–36)

where

$$\phi_0 = \frac{2\pi L n_0}{\lambda},$$  \hspace{1cm} (3–37)

$$m = \frac{\pi L}{\lambda} r_{33} n_0^3 \frac{V}{d}.$$  \hspace{1cm} (3–38)

Hence Equation 3–33 becomes

$$E' = \frac{t_1 t_2 e^{-i(\phi_0 + m \sin \Omega t)}}{1 - r_1 r_2 e^{-i(\phi_0 + m \sin \Omega t)}} E = \frac{t_1 t_2 \left( e^{i(\phi_0 + m \sin \Omega t)} - r_1 r_2 \right)}{1 - 2r_1 r_2 \cos \left( \phi_0 + m \sin \Omega t \right) + r_1^2 r_2^2} E.$$  \hspace{1cm} (3–39)

Equation 3–39 implies that the field $E'$ becomes an amplitude-modulated field. The magnitude of the amplitude modulation increases with the reflectivity $r_1$ and $r_2$.  

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3.4.3.2 RAM in Advanced LIGO

As mentioned in Section 1.5, changes in the amplitude and phase relationship between the carrier field and the sidebands reflect the relative positions of the mirrors in an interferometric GW detector. However, RAM also changes the amplitudes and phases of the light fields. These changes are indistinguishable from the signature due to the motion of the mirrors. RAM results in unwanted offsets in the length sensing and control signals. This is intuitively understandable with a phasor-diagram (see Figure 3-15). The left part of Figure 3-15 shows us the light fields at the dark port of LIGO for an ideal case that RAM is completely eliminated. $C$ represents the carrier field and $S$ represents an RF sideband. When the interferometer is locked, the phase difference between the leaked carrier light and the RF sidebands at the dark port will be exactly $90^\circ$. The deviation of
the interferometer from its operation point will cause the phase difference to be away from $90^\circ$. The beat notes between the carrier and the RF sidebands will be demodulated to generate error signals which are used to adjust the mirrors of the interferometer back to operation positions via the length sensing and control system.

The right phasor-diagram describes how RAM creates a beat note between the carrier light and the RF sidebands when the lengths of the interferometer satisfy the resonant condition. Here $C_A$ and $S_A$ are residual components added on the carrier and an RF sideband respectively. The resulting carrier field $C'$ and RF sideband $S'$ are not orthogonal to each other so that the beat note will lead to the generation of feed-back control forces which act on the mirrors and push the interferometer away from the ideal resonant condition. The amplitude of RAM must be kept below a certain level to ensure that the static length-errors of the interferometer are within the design tolerances. Errors signals due to RAM can also be balanced out by adding electrical voltage offsets in the feed-back control loops.

Possible offsets generated in various control loops depends on the length sensing and control scheme of the interferometric detector. In LIGO, the fractional RAM is required to be less than $10^{-3}$ to ensure that the static error of the differential arm length is less than the residual deviation requirement of $10^{-13}$ [43]. The length sensing and control scheme for Advanced LIGO is not finalized, but will be much more complicated. However, because of the intrinsic asymmetries in the detuned Advanced LIGO interferometer and DC sensing, the requirement of RAM will probably not be tightened.

### 3.4.3.3 Characterization

The EOM crystal will expand or contract when its temperature fluctuates. The refractive indices $n_o$ and $n_e$ are also temperature-dependent. From Equations 3–20 and 3–19, we can see that RAM will vary when the temperature drifts. This could affect the operation of an interferometric detector since the adjustment of laser power will potentially lead to a temperature drift which in turn changes the RAM. So a
Figure 3-16. Experimental setup to characterize the residual amplitude modulation created by an EOM heated by a 1053nm Nd:YLF laser.

Pre-installation test of the power dependent RAM imposed by an EOM is essential for high power laser interferometers, such as Advanced LIGO.

Figure 3-17 is the optical layout of the experiment that is used to measure the RAM generated by an EOM being heated by a laser at different power levels. The pump-probe scheme experiment used a 500 mw non-planar ring oscillator (NPRO) Nd:YAG laser as the probe laser to sense the amplitude modulation signal and a 50 W Nd:YLF laser as the heating source. A Faraday Isolator (FI) was used to protect the probe laser from being affected by the back-reflected light or light from the pump laser. A half wave plate and a polarizing beam splitter were used to control the laser power. The polarizing beam splitter was aligned so that the transmitted light was s-polarized with respect to the optical table. Its divergence was adjusted by a MMT consisting of lenses L1 and L2. The beam was then reflected by a thin film polarizer TFP1. The brewster angle of the thin film polarizer is about 56°. The EOM was aligned as precisely as possible to match the crystal’s principal axis to the vertical direction. The beam diameter of the probe light at the EOM was about 1 mm. The probe light is then reflected by another thin film polarizer TFP2. Part of the light was then split off and sent towards an optical spectrum analyzer (OSA) which
was used to measure the modulation index of the EOM. The rest transmitted another cube polarizer which was aligned in the same way as the first one. The transmitted laser beam was focused on a photodetector. The photodetector has separated DC and AC output ports. The amplitude of the DC signal from the photodetector is proportional to the total intensity of the light detected while the AC signal was used to monitor the amplitude modulation of the laser field. The heating beam from the YLF laser propagated in the opposite direction, with the beam divergence being adjusted by a MMT consisting of lenses L3 and L4. Rotating the half wave plate in front of TFP2 changed the intensity of the p-polarized light that was delivered to the EOM. The output power of the YLF laser was fixed. So we knew the power of the heating beam that transmits the EOM by monitoring the laser power reflected at TFP 2. The diameter of the heating beam was controlled to be about 2 mm at the EOM to avoid clipping. The extinction ration of the thin film polarizer is greater than 1000:1 (Tp:Ts) so that most of the heating beam

Figure 3-17. Correlation between the heating power and RAM.
transmitted TFP1. A small amount of YFL light that was delivered to the probe laser was rejected by the FI.

The amplitude of the AC signal was measured using a spectrum analyzer and the DC signal was measured using an oscilloscope. These signals were continuously recorded. We started with tracking the drift of the residual amplitude modulation for more than 20 minutes. Then we exposed the EOM to a 5 W heating laser and kept on measuring the temporal variation of the RAM. The heating power was then increased to 20 W and finally to 40 W in the following two steps and the time series measurements were repeated.

The modulation index is maintained to be about 0.2 during the measurement. From the measurement result plotted in Figure 3-17, we clearly saw the correlation between the heating power and the RAM created by the EOM. We also noticed that realigning the EOM can bring the RAM back to the original low levels. RAM created by the EOM that we used can be minimized to be below $10^{-5}$ of total light intensity. We believed this is due to the Fabry-Perot cavity effect imposed by the unwedged crystal inside the EOM, since the reflectivity of the AR-coated end surfaces of the crystal was measured to be about 0.2%, which is high enough to create a $10^{-5}$ amplitude modulation.

### 3.4.4 Laser Amplitude and Phase Noise Produced by EOMs

#### 3.4.4.1 Generation mechanism

We have seen that both the amplitude and phase of a laser will be modulated as it traverses through EOMs. Instabilities of the modulation will create amplitude and phase noise on the carrier field and sidebands. Amplitude noise imposed by EOMs, along with the intrinsic amplitude noise of the laser will be coupled to the readout channel such that the strain sensitivity of GW interferometric detectors can be deteriorated. Variation of the phase between the RF sidebands and the carrier field will be converted into laser frequency noise via the length sensing and control loops. Laser noise coupling in Advanced LIGO has been studied by Somiya [44]. The final requirement on the laser noise level is
to be determined, but is expected to be very stringent. The amplitude and phase noise imposed by EOMs on the laser should not exceed these requirements.

The laser noise associated with an EOM is generated through several mechanisms. Noise sources are due to either the instability of the modulation signal, or unstable physical properties of the EOM crystal. Two noise generation mechanisms are addressed here.

First, variant modulations in the index causes amplitude noise of both the carrier field and sidebands. The change of the amplitude of the carrier field as a function of the modulation index can be written as

\[ E_c = J_0(m_0 + \delta m) E_0 \approx J_0(m_0) E_0 + \frac{\partial J_0(m)}{\partial m} |_{m=m_0} \delta m E_0 = J_0(m_0) E_0 - J_1(m_0) \delta m E_0. \] (3–40)

The relative amplitude noise of the carrier due to the variation of the modulation index is

\[ RAN_c = \frac{\delta E_c}{E_c} = \frac{J_1(m_0) \delta m}{J_0(m_0)}. \] (3–41)

The amplitude of an RF sideband changes with the modulation index

\[ E_s = J_1(m_0 + \delta m) E_0 \approx J_1(m_0) E_0 + \frac{\partial J_1(m_0)}{\partial m} \delta m |_{m=m_0} E_0 \approx (J_1(m_0) + \frac{\delta m}{2}) E_0. \] (3–42)

The relative amplitude noise of the sideband is

\[ RAN_s = \frac{\delta m}{2J_1((m_0)). \] (3–43)

Second, variations of RAM also contribute to the amplitude and the phase noise of the light field. Equation 3–21 describes RAM as a function of the modulation index \( m \), misalignment angles \( \alpha \) and \( \beta \) and the phase retardation \( \Delta \phi \). Fluctuations of these parameters at frequencies in the GW bandwidth will lead to additional laser noise that could interfere with the signal.

We can see from Equations 3–8 and 3–9 that the stability of the modulation index of an EOM depends on the stability of the driving signal, refractive indices, electro-optic
coefficients and dimensions of the crystal. The amplitude and phase stability of the
driving voltage is determined by the quality of the oscillator used to drive the EOM.
Ultra-low noise oscillators are demanded. Refractive indices and electro-optic coefficients
are temperature dependent. Temperature fluctuations will also lead to the expansion or
contraction of the crystal. Moreover, the thermal-elasto effect could create strain inside
the crystal, which in turn changes the index ellipsoid via the elasto-optic effect. In sum,
it is oscillator noise and temperature variations that affect the modulation index. These
two noise sources also cause fluctuations of RAM. First, fluctuations of the modulation
index induced by these two noise sources will lead to fluctuations of RAM. Thermally
induced variations of the refractive indices and dimensional changes of the crystal result
in fluctuations of $\Delta \phi$. Variations of the principal axes of the crystal cause variations of
the misalignment angles - $\alpha$ and $\beta$. In sum, we believe two principal noise sources that
perturb the amplitude and the phase of the laser via an EOM are oscillator noise and
temperature variations.

Another potential noise source is acoustic noise. Acoustic waves could change the
dimensions of the crystal via the elasto-acoustic coupling as they pass through the crystal,
perturb the optical path length which a light field experiences inside the EOM. This in
turn leads to the variation of the modulation index and RAM, generating residual laser
noise. A straightforward case is shown in Figure 3-18. The deformation depths of two end
surfaces of the crystal are expected to be different due to the energy dissipation of the

Figure 3-18. Deformation of end surfaces of a crystal when an acoustic wave passes by.
acoustic wave. Uncorrelated vibrations of two surfaces result in fluctuations of the optical path length.

3.4.4.2 Characterization

Noise associated with amplitude and phase variations of an optical signal is most easily studied in the frequency domain. Unfortunately, direct measurements of amplitude or phase noise of a laser field at Advanced LIGO required levels is very difficult because of limitations on the dynamic range and the intrinsic noise of optical spectrum analyzers. We developed a heterodyne technique to measure the residual optical noise imposed by an EOM when it was used to phase modulate a laser beam. The light field to be measured was mixed with another light field to create an optical beat signal. By locking two light fields with a constant frequency offset via a PLL, both the amplitude and the phase could be precisely tracked using a novel phase meter [45]. Figure 3-19 shows the electro-optical layout of the experiment. The two lasers were non-planar ring oscillator (NPRO) Nd:YAG lasers with a 1064 nm wavelength. The laser field of NPRO 1 propagates through a half-wave plate and a polarizer first, allowing the power to be controlled by rotating the half-wave plate. The laser field was then modulated via an EOM which was driven
by a sine wave signal with a frequency $\Omega$. The divergence of the transmitted light was adjusted by a mode-matching telescope consisting of lenses L1 and L2. The laser beam from NPRO 2 also propagates through a half-wave plate and a polarizer, with the polarization direction adjusted to be parallel to the polarizer in front of NPRO 1. Another mode-matching telescope consisting of lenses L3 and L4 is used to control the divergence of this laser beam. Two laser beams were superimposed at a power beam splitter (BS). The mode matching between the two lasers was optimized by adjusting two mode-matching telescopes.

The laser field from NPRO 2 beat with the carrier field and RF sidebands of NPRO 1 at the two photodetectors (PD 1 and PD 2). Demodulating the beat notes as measured by PD 1 with frequency $\Omega_0$ using a function generator and a double balanced mixer created an error signal. The demodulated signal was filtered by the PLL controller and used to phase-lock NPRO 2 with respect to NPRO 1. The constant phase offset between two lasers was maintained by tuning the frequency of NPRO 2. The frequency/phase conversion added a 6 dB/octave slope to the loop transfer function of the phase locking loop (PLL), which was a part of the proportional-integral (PI) control designed for the system. Figure 3-20 shows the loop transfer function of the PLL measured using a network spectrum analyzer. We could see that the unity gain frequency of the PLL was around 30 KHz.

If the field of NPRO 1 is represented by $E_1$ and the field of NPRO 2 is represented by $E_2$, the photocurrent of each photodetector could be written as

$$S = \left( \bar{E}_1 + E_2 \right) \left( \bar{E}_1 + E_2 \right)^* \approx J_0^2(m) E_1^2 + E_2^2 + 2J_0(m) E_1E_2 \cos (\Omega t + \Delta \phi)$$

$$+ 2J_1(m) E_1E_2 \cos [(\Omega - \Omega_0) t + \Delta \phi] - 2J_1(m) E_1E_2 \cos [(\Omega + \Omega_0) t + \Delta \phi]$$  \hspace{1cm} (3-44)

Figure 3-21 shows the frequency intervals between fields of two lasers.

The signal from PD 2 was analyzed by the phase meter developed by the LISA group at the University of Florida. The phase meter is a digital signal processing device that
can be used for lock-in detection of RF signals. It had four synchronized input channels and the analog-to-digital converter (ADC) of each channel digitizes the incoming signal with a 100 MHz sampling rate. The digital filters inside the phase meter can decimate the digital signal by a factor of $2^{10}$. The phase meter provided the amplitude and the phase information of the input signals via the in-phase/quadrature (I/Q) demodulation technique. The schematic of the I/Q demodulation of the digitized signal is shown in Figure 3-22. In order to overcome the dynamic range limitation of $90^\circ$ or $180^\circ$ for phase measurement, the local oscillator (LO), a numerically controlled oscillator (NCO), of the I/Q demodulator was phase locked to the RF signal under test by tuning the frequency of the LO signal around the frequency of the input signal.

The function generators used to drive the EOM and to provide the reference signal for phase-locking, and the phase meter were synchronized to a synthesized clock generator.
In our experiment, we used three channels of the phase meter to measure the amplitude and the phase of the carrier-carrier (C-C) beat note of two lasers, of the beat note between the carrier of NPRO 2 and one RF sideband of NPRO 1 (C-S), and of the modulation signal from the EOM driver simultaneously. The angular frequencies of the NCOs for 3 channels were tuned to be $\Omega_0$, $\Omega - \Omega_0$ and $\Omega$, respectively. These amplitude and phase data were recorded in the form of time series. The linear spectral density (LSD) of the amplitude or the phase could be calculated via the Fourier transformation.

We tried our first test on an unwedged crystal EOM (manufactured by New Focus). The EOM was driven by a 27 MHz sinusoidal signal from a function generator (Fluke model 6062A). The reference signal in the PLL was a 10 MHz sinusoidal signal from another function generator (Stanford research system model DS345). The amplitude noise
Figure 3-23. Amplitude noise spectra of the laser beat notes and the modulation signal from the EOM driver as measured by the phase meter.

and phase noise spectra of the 10 MHz C-C beat note and 17 MHz C-S beat note are shown in Figure 3-23 and Figure 3-24, respectively.

As far as we know, the amplitude noise of beat notes come from several sources besides the amplitude noise imposed by the EOM: intrinsic laser amplitude noise, variation of the overlapping (beam jitter) between two laser beams, and electronic noise associated with the phase meter (e.g. digitization noise). Noticing that the first two effects will impose noise on the C-C and the C-S beat notes with the same relative amplitude, such noise, being labeled as ‘common-mode’ noise, can be eliminated via a common-mode rejection analysis. Specifically, if the relative magnitude of the ‘common-mode’ amplitude noise is represented by $\delta A_c$ while $\delta A_1$ and $\delta A_2$ represent the relative amplitude noise of the C-C and C-S beat notes in ‘differential-mode’, we can calculate the amplitude ratio of the C-S beat notes to the C-C beat note

$$\Gamma = \frac{A_{C-S}}{A_{C-C}} \approx \frac{A_{C-S} (1 + \delta A_c) (1 + \delta A_2)}{A_{C-C} (1 + \delta A_c) (1 + \delta A_1)} \approx \frac{A_{C-S}}{A_{C-C}} (1 + \delta A_2 - \delta A_1). \quad (3-45)$$
Here $A_{c-c}$, $A_{c-s}$ are amplitudes of the C-C and C-S beat notes. According to Equation 3–45, the ‘common-mode’ noise should not show up in the noise spectrum of $\Gamma$. The LSD of $\Gamma$, labeled as ‘c-s/c-c’ (the green curve in Figure 3-23) represents only the amplitude noise in ‘differential-mode.’ As can be seen, the ‘common-mode’ noise below 40 Hz is successfully reduced via this common-mode rejection analysis. We believe that the laser amplitude noise which has been eliminated from the data is largely due to the beam jitter that dominates the low frequency region, since the measured laser intensity noise (represented by the brown curve in Figure 3-23) is more than an order magnitude lower than the measured amplitude noise.

As has been discussed in Section 3.4.4, it is oscillator noise and temperature fluctuations that cause an EOM to produce additional amplitude and phase noise on the laser beam. Our result shows that the oscillator noise (labeled as $V(\Omega)$ in Figure 3-23) only counts for a small part of the amplitude noise floor. The thermally induced noise can be estimated once we know the thermal expansion coefficient, refractive indices and $dn/dT$ values. Thermally induced changes in electro-optic coefficients and the thermal-elasto effect are not considered here. From Equations 3–21 and 3–33, the carrier field after the EOM can be approximated as

$$E'_c \approx \frac{t_1 t_2 e^{-\frac{\omega_0}{c} L}}{1 - r_1 r_2 e^{-\frac{2\omega_0}{c} L}} \left( \cos \beta \cos \gamma J_0 (m_1) + \sin \beta \sin \gamma e^{i\Delta \phi} J_0 (m_2) \right) E_0.$$  \hspace{1cm} (3–46)

The modulation indices are

$$m_1 = \frac{\omega_0}{c} L r_3 n^3_x V_z \frac{V_z}{d},$$  \hspace{1cm} (3–47)

$$m_2 = \frac{\omega_0}{c} L r_3 n^3_x V_z \frac{V_z}{d}.$$  \hspace{1cm} (3–48)

And

$$L = L_0 + \kappa_y \Delta T,$$  \hspace{1cm} (3–49)

$$d = d_0 + \kappa_z \Delta T.$$  \hspace{1cm} (3–50)
The calculation using Equation from 3-46 to 3-52 shows that temperature fluctuations at a level of 0.1 \(mK/\sqrt{Hz}\) could only introduce a \(10^{-9}\) relative amplitude noise in the carrier field. And the electronic noise floor that the phase could introduce in the measurement is not fully understood by the time this thesis is finished, it is highly possible that we are still limited by the measurement sensitivity to see the amplitude noise imposed by the EOM. The amplitude noise curve in Figure 3-23 can be seen as the upper limit of the amplitude noise that is produced by the EOM without stabilizing the temperature of the crystal and applying acoustic isolation methods.

If the electronic noise associated with the phase meter can be omitted, the measured difference between the phase of the C-S beat notes \(\phi_{c-s}\) and that of the C-C beat note \(\phi_{c-c}\) is the residual phase imposed by the EOM. The LSD of \(\phi_{c-s} - \phi_{c-c}\) (green curve in Figure 3-24) is above \(10^{-5}\) cycles/\(\sqrt{Hz}\). This overlaps with the LSD of the phase of the
Figure 3-25. Linear spectral density of phase noise in laser beat notes and the modulation signal from the EOM driver.

The modulation signal $\phi_\Omega$ from the EOM driver (black curve in Figure 3-24) at frequencies below 1000 Hz. Also shown in Figure 3-24 is the LSD of the residual phase given as $\phi_{c-s} - \phi_{c-c} - \phi_\Omega$. The correlation between the phase noise of $\phi_{c-s} - \phi_{c-c}$ and $\phi_\Omega$ is expected to disappear once $\phi_\Omega$ goes below this noise floor. This was confirmed by another measurement where the Fluke function generator was replaced by a better one (Stanford Research System DS 345). The new measurement result is shown in Figure 3-25. In both measurements, the phase noise due to temperature fluctuations or acoustic perturbations should be below the level set by the noise floor.
CHAPTER 4
CONCLUSION

A complete characterization of each mechanical, electrical or optical component that has been planned to be used in Advanced LIGO is an important part of the instrumentation work. The performance of each component must meet the system requirements with consideration given to the improvement of the strain sensitivity and the operational feasibility.

Both the triple pendulum suspension system and the RTP crystal EOM have been planned to be used in the IO subsystem of Advanced LIGO. The triple pendulum described in this thesis is the one that has been installed and tested in the JIF lab at Glasgow University. however, the triple pendulum to be used in Advanced LIGO will be different from this prototype pendulum. The mechanical model that has been developed by the author will be a useful reference model for the design and test of the final version of the triple pendulum suspension system.

The RTP crystal EOM is going to be installed in the 40-m prototype interferometer (at California Institute of Technology) and Enhanced LIGO (a mid-step between LIGO and Advanced LIGO). The characterizations of the EOM has been done at UF is essentially a pre-installation test. The measurement results provide useful information for the final decision of its application in Advanced LIGO.

The modeling of a triple pendulum suspension and the characterization of the new EOM in terms of its practical limitations have been presented in this thesis. The results are summarized below.

4.1 Triple Pendulum Suspension Model

The dynamic model of the triple pendulum can be used to investigate the mode frequencies or transfer function for all degrees of freedom. The model can be used to simulate the response of the pendulum to the external excitation applied through one or multiple local control channels. Based on the mechanical analysis that has been completed in this thesis, the expansion of this model to a model that includes a complete global
control on all three test masses will be a straightforward practice. The cross coupling effect between control channels for motions of different degrees of freedom can also be predicted using this model. As compared with other existing models, it is more complete in details. For example, two OSEMs that are used to adjust the longitudinal and yaw motion of the pendulum are located above the center of the upper mass. A pitch motion will be introduced when the control force is applied via these two OSEMs. The sensing and control matrix described in Chapter 2 has taken this into consideration. The mode frequencies of the vertical, longitudinal, and sideway motions calculated by this model have been confirmed by the measurement results. The errors of theoretical values are less than 0.2 Hz in most part. The biggest disagreement (about 0.4 Hz) shows up in a 5 Hz vertical mode. More characterization measurements are needed to confirm mode frequencies of the pitch, yaw, and roll motions predicted by the model.

The damping constants for the motions of three test masses need to be verified and incorporated in this model.

4.2 EOM

The RTP crystal EOM has been shown to be a quality EOM candidate for Advanced LIGO. It has excellent optical properties and a less than 1000 ppm/cm absorption coefficient at 1064 nm wavelength. The residual amplitude modulation can be maintained below a $10^{-4}$ level without taking any efforts to stabilize the temperature of the crystal. The experimental studies of the amplitude and phase noise imposed by the EOM on the laser field have demonstrated that the relative amplitude noise is below a level of $2 \times 10^{-5}/\sqrt{Hz}$ and the phase noise is below a level of $10^{-5}$ cycles$/\sqrt{Hz}$ in the LIGO bandwidth. The phase noise floor which is not limited by the EOM driver is shown to be around $3 \times 10^{-6}$ cycles$/\sqrt{Hz}$.

The heterodyne technique which has been developed to measure the EOM noise can be improved and used for ultra-sensitive measurement of both the amplitude and phase
of optical signals. This kind of technique is essential for the testing of future electro-optic components to be used in Advanced LIGO.
APPENDIX A
MATLAB MODEL FOR THE TRIPLE PENDULUM

A.1 Close Loop Transfer Function (clp.m)

The file clp.m is used to calculate and plot the close loop transfer functions

\[
T(f) = \frac{1}{1 + T_1 H G T_2},
\]

(A-1)

with G, H, T_1 and T_2 defined in Figure 2-29.

```matlab
% ************************************************************
clear all;
param; % parameter file
matrixT1T2; % control matrix
In = [1, 1, 0, 0, 0, 0]; % input channels
Out = [1, 0, 0, 0, 0, 0]; % output channels
In = transpose(In); fmin = 1.0e-1;
fmax = 12;
bmin = log10(fmin);
bmax = log10(fmax);
N = 1000;
for n = 1:1:N;
    freq = 10^(bmin + (bmax - bmin)*(n-1)/(N-1)); % log span
    w = 2*pi*freq;
s = i*w;
model; % the mechanical model
localCtr; % local control servo
H = T2 * M * T1;
OL = H;
compol = Out * LC * OL * In;
CL = 1/(1 + compol);
```
f(n) = freq;
TF(n) = CL;
end
mag = abs(TF);
mag = 20 * log10(mag);
phi = unwrap(angle(TF)) * 180/pi;
subplot(2,1,1)
semilogx(f, mag)
axis([fmin fmax min(mag) max(mag)])
set(gca, 'FontSize', 14);
xlabel('Frequency (Hz)')
ylabel('Magnitude (dB)')
grid
subplot(2,1,2)
semilogx(f, phi)
axis([fmin fmax min(phi) max(phi)])
set(gca, 'FontSize', 14);
xlabel('Frequency (Hz)')
ylabel('Phase (deg.)')
grid

% ***********************************************************************

A.2 Parameters (parameter.m)

The file (parameter.m) represents the input parameters for the triple pendulum installed in JIF lab. The parameters are defined in Chapter 2.

% ***********************************************************************
g = 9.81;
% X direction separation
su = 0.00; % 1/2 separation of upper wires
si = 0.03; % 1/2 separation of intermediate wires
sl = 0.005; % 1/2 separation of lower wires
s1 = si; s2 = sl;

% arms
lp = 0.04; % half spacing of coils acting on pitch
ly = 0.08; % half spacing of coils acting on yaw
lr = 0.13; % half spacing of coils acting on roll

% masses
m1 = 2.95 + 0.1; m2 = 2.71; m3 = 2.71;
m23 = m2 + m3; m13 = m1 + m2 + m3;

% moments of inertia
I1x = 0.0189; I2x = 0.0066; I3x = 0.0066;
I1y = 0.0035; I2y = 0.00530; I3y = 0.0053;
I1z = 0.0189; I2z = 0.00524; I3z = 0.00524;

% spring constants
k1 = 3.6066e+002;
k2 = 6.9301e+002/2;
k3 = 9.7506e+003/2;

ir = 0.0635; % dimension of intermediate mass (cylinder)
tr = 0.0635; % dimension of lower mass (cylinder)

d0 = 0.001; % height of upper wire break-off (above c.of m. upper mass)
d1 = 0.001; % height of intermediate wire break-off (below c.of m. upper mass)
d2 = 0.001; % height of intermediate wire break-off (above c.of m. of int. mass)
d3 = -0.001; % height of lower wire break-off (below c.of m. intermediate mass)
d4 = 0.001; % height of lower wire break-off (above c.of m. test mass)
n0 = 0.06; % 1/2 separation of upper wires at suspension point
n1 = 0.1; % 1/2 separation of upper wires at upper mass
n2 = 0.03; % 1/2 separation of intermediate wires at upper mass
n3 = ir-0.005+0.01; % 1/2 separation of intermediate wires at intermediate mass
n4 = ir-0.005+0.005; % 1/2 separation of lower wires at intermediate mass
n5 = tr-0.005+0.005; % 1/2 separation of lower wires at test

lt1 = sqrt(l1^2 - (n0-n1)^2);
lt2 = sqrt(l2^2 - (n2-n3)^2);
lt3 = sqrt(l3^2 - (n4-n5)^2);
ls = 0.1; % height of coils 1, 2, and 6 above the center of mass

% gains of six channels
g1 = 1; g2 = 1; g3 = 1; g4 = 1; g5 = 1; g6 = 1;
G = [ g1 0 0 0 0 0
      0 g2 0 0 0 0
      0 0 g3 0 0 0
      0 0 0 g4 0 0
      0 0 0 0 g5 0
      0 0 0 0 0 g6 ];

% **********************************************************************

A.3 Model (model.m)

The file model.m provides the transfer function from the external forces and torques to the induced motion of the upper mass in six degrees of freedom.

% **********************************************************************

% vertical

Hzv = - 1/(2*k1 + 4*k2 + m1 * s^2 - (m3 * s^2 + 4*k3)*16*k2^2 / ( ( 4*k2 + 4*k3 + m2*s^2)*( m3*s^2 + 4*k3 )- 16*k3^2 ) );

% **********************************************************************

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% longitudinal / pitch

\[ T(1,1) = -m1*s^2 - (m2+m3)*g/lt2 - (m1+m2+m3)*g/lt1; \]
\[ T(1,2) = (m2+m3)*g/lt2; T(1,4) = (m1+m2+m3)*g*d0/lt1 + (m2+m3)*g*d1/lt2; \]
\[ T(1,5) = - (m2+m3)*g*d2/lt2; \]
\[ T(2,1) = (m2+m3)*g/lt2; \]
\[ T(2,2) = -m2*s^2 - (m2+m3)*g/lt2-m3*g/lt3; \]
\[ T(2,3) = m3*g/lt3; \]
\[ T(2,4) = -(m2+m3)*g*d1/lt2; \]
\[ T(2,5) = (m2+m3)*g*d2/lt2+m3*g*d3/lt3; \]
\[ T(2,6) = -m3*g*d4/lt3; \]
\[ T(3,2) = 1; T(3,3) = -lt3*s^2/g - 1; \]
\[ T(3,5) = -d3; T(3,6) = d4; \]
\[ T(4,1) = (m1+m2+m3)*g*d0/lt1 + (m2+m3)*g*d1/lt2; \]
\[ T(4,2) = -(m2+m3)*g*d1/lt2; \]
\[ T(4,4) = -s^2*I1y + 4*k2*s1^2-(m1+m2+m3)*g*d0^2/lt1 - (m1+m2+m3)*g*d0 - \]
\[ (m2+m3)*g*d1^2/lt2; \]
\[ T(4,5) =-4*k2*s1^2 + (m2+m3)*g*d1 + (m2+m3)*g*d1*d2/lt2; \]
\[ T(5,1) = -(m2+m3)*g*d2/lt2; \]
\[ T(5,2) = (m2+m3)*g*d2/lt2 + m3*g*d3/lt3; \]
\[ T(5,3) = -m3*g*d3/lt3; \]
\[ T(5,4) = 4*k2*s1^2 + (m2+m3)*g*d1*d2/lt2; \]
\[ T(5,5) = -s^2*I2y - 4*k2*s1^2 - (m2+m3)*g*d2^2/lt2 - (m2+m3)*g*d2 - 4*k3*s2^2 - \]
\[ m3*g*d3^2/lt3 + m3*g*d3; \]
\[ T(5,6) = 4*k3*s2^2 + m3*g*d3*d4/lt3; \]
\[ T(6,2) = -m3*g*d4/lt3; \]
\[ T(6,3) = m3*g*d4/lt3; \]
\[ T(6,5) = 4*k3*s2^2 + m3*g*d4*d3/lt3; \]
\[ T(6,6) = -s^{2}I_{3y} - 4k_{3}s^{2} - m_{3}g*d_{4} - m_{3}g*d_{4}^{2}/lt_{3} \]

\[ \Gamma = \text{inv}(T); \]

\[
\begin{align*}
\% & \text{yaw} \\
H_{xl} &= \Gamma(1,1) ; \\
H_{xp} &= \Gamma(1,4); \\
H_{al} &= \Gamma(4,1); \\
H_{ap} &= \Gamma(4,4); \\
P_{1} &= (m_{1}+m_{2}+m_{3})*g*n_{1}*n_{0}/lt_{1}; \\
P_{2} &= (m_{2}+m_{3})*g*(n_{2}^{2} + s_{1}^{2})/lt_{2}; \\
P_{3} &= (m_{2}+m_{3})*g*(n_{3}^{2} + s_{1}^{2})/lt_{2}; \\
P_{4} &= m_{3}g*(n_{4}^{2} + s_{2}^{2})/lt_{3}; \\
P_{5} &= m_{3}g*(n_{5}^{2} + s_{2}^{2})/lt_{3}; \\
H_{by} &= \frac{1}{P_{1} + P_{2}*( 1 - P_{3}/( P_{3} + P_{4}*( 1 - P_{5}/( P_{5} + s^{2}I_{3z} ) ) ) + s^{2}I_{2z}) ) + s^{2}I_{1z}) ; \\
\% & \text{sideway /roll} \\
A_{1} &= (m_{1}+m_{2}+m_{3})*g*d_{0}*( 1/lt_{1} - ( n_{1} - n_{0} )^{2}/lt_{1}^{2} ) - (m_{1}+m_{2}+m_{3})*g*n_{1}*( n_{1}-n_{0} )/lt_{1}^{2} + 2k_{1}*( n_{1}*lt_{1} + d_{0}*(n_{1}-n_{0}))*(n_{1}-n_{0})/lt_{1}^{2}; \\
B_{1} &= (m_{1}+m_{2}+m_{3})*g*n_{1}*lt_{1}/lt_{1}^{2} - (m_{1}+m_{2}+m_{3})*g*n_{1}/lt_{1} + (m_{1}+m_{2}+m_{3})*g*d_{0}*( n_{1}-n_{0})/lt_{1}^{2} - 2k_{1}*lt_{1}*( n_{1}*lt_{1} + d_{0}*(n_{1}-n_{0}))/lt_{1}^{2}; \\
C_{1} &= (m_{1}+m_{2}+m_{3})*g*n_{1}*(n_{1}-n_{0})/lt_{1}^{2} + (m_{1}+m_{2}+m_{3})*g*d_{0}*( (n_{1}-n_{0})^{2}/lt_{1}^{2} - 1/lt_{1} ) - 2k_{1}*( n_{1} - n_{0} )* ( n_{1}*lt_{1} + d_{0}*(n_{1}-n_{0}))/lt_{1}^{2}; \\
D_{1} &= (m_{2}+m_{3})*g*d_{1}*( 1/lt_{2} - ( n_{3} - n_{2} )^{2}/lt_{2}^{2} ) - (m_{2}+m_{3})*g*n_{2}*( n_{3}-n_{2} )/lt_{2}^{2} - 4k_{2}*( n_{2}*lt_{2} + d_{1}*(n_{3}-n_{2}) )* (n_{3}-n_{2})/lt_{2}^{2}; \\
E_{1} &= (m_{2}+m_{3})*g*n_{2}*lt_{2}/lt_{2}^{2} - (m_{2}+m_{3})*g*n_{3}/lt_{2} - (m_{2}+m_{3})*g*d_{1}*(n_{3}-n_{2})/lt_{2}^{2} + 4k_{2}*lt_{2}/lt_{2}^{2}*( n_{2}*lt_{2} + d_{1}*(n_{3}-n_{2}))/lt_{2}; \\
\end{align*}
\]
\[ G_1 = (m_2+m_3)g(n_3-n_2)/l_2^2 + (m_2+m_3)g(d_1)(n_3-n_2)^2/l_2^2 - 1/l_2 \]
\[ + 4k_2^2(n_2l_2 + d_1(n_3-n_2))(n_3-n_2)/l_2^2; \]
\[ R_1 = -(m_1+m_2+m_3)g(1/l_1 - (n_1-n_0)^2/l_1^2/l_1) + 2k_1(n_1-n_0)^2/l_1^2; \]
\[ S_1 = -(m_1+m_2+m_3)g(n_1)/l_1^2 - 2k_1l_1(n_1-n_0)/l_1^2; \]
\[ A_2 = (m_2+m_3)g(d_2)(1/l_2 - (n_3-n_2)^2/l_2^2/l_2) - (m_1+m_2+m_3)g(n_3^n_2)/l_2^2 + 4k_2(n_3l_2 + d_2(n_3-n_2))(n_3-n_2)/l_2^2; \]
\[ B_2 = (m_2+m_3)g(n_3l_2/l_2^2 - (m_2+m_3)g(n_3/l_2) + (m_2+m_3)g(d_2)(n_3-n_2)/l_2^2 - 4k_2(l_2^2 + d_2(n_3-n_2))(n_3-n_2)/l_2^2; \]
\[ C_2 = (m_2+m_3)g(n_3^n_2)/l_2^2 + (m_2+m_3)g(d_2)(n_3-n_2)^2/l_2^2/l_2 - 1/l_2 - 4k_2(n_3-n_2)(n_3l_2 + d_2(n_3-n_2))/l_2^2; \]
\[ D_2 = m_3^3g(d_3)(1/l_3 - (n_5-n_4)^2/l_3^2/l_3) - m_3^3g(n_5^n_4)/l_3^2 + 4k_3(n_5l_3 + d_3(n_5-n_4))/l_3^2; \]
\[ E_2 = m_3^3g(n_5^n_4/l_3^2 - m_3^3g(n_5l_3 - m_3^3g(d_3)(n_5-n_4)/l_3^2 + 4k_3(n_5l_3 + d_3(n_5-n_4))/l_3^2; \]
\[ G_2 = m_3^3g(n_5^n_4)/l_3^2 + m_3^3g(d_3)(n_5-n_4)^2/l_3^2/l_3 - 1/l_3 + 4k_3(n_5l_3 + d_3(n_5-n_4))/l_3^2; \]
\[ R_2 = -(m_2 + m_3)g(1/l_2 - (n_3 - n_2)^2/l_2^2/l_2) + 4k_2(n_3 - n_2)^2/l_2^2; \]
\[ S_2 = -(m_2 + m_3)g(n_3 - n_2)/l_2^2 - 4k_2l_2(n_3 - n_2)/l_2^2; \]
\[ A_3 = m_3^3g(d_4)(1/l_3 - (n_5-n_4)^2/l_3^2/l_3) - m_3^3g(n_5^n_4)/l_3^2 + 4k_3(n_5l_3 + d_4(n_5-n_4))/l_3^2; \]
\[ B_3 = m_3^3g(n_5^n_4/l_3^2 - m_3^3g(n_5l_3 + m_3^3g(d_4)(n_5-n_4)/l_3^2 - 4k_3(l_3^2 + d_4(n_5-n_4))/l_3^2; \]
\[ C_3 = m_3^3g(n_5^n_4)/l_3^2 + m_3^3g(d_4)(n_5-n_4)^2/l_3^2/l_3 - 1/l_3 - 4k_3(n_5 - n_4)(n_5 - n_4))/l_3^2; \]
\[ R_3 = -m_3^3g(1/l_3 - (n_5 - n_4)^2/l_3^2/l_3) + 4k_3(n_5 - n_4)^2/l_3^2; \]
\[ S_3 = -m_3^3g(n_5 - n_4)/l_3^2 - 4k_3l_3(n_5 - n_4)/l_3^2; \]
\[ Q(1,1) = R_1 + R_2 - m_1s^2; \]
\[ Q(1,2) = -R2; \]
\[ Q(1,4) = S1*n1 - R1*d0 + S2*n2 - R2*d1; \]
\[ Q(1,5) = -S2*n3 + R2*d2; \]
\[ Q(2,1) = -R2; \]
\[ Q(2,2) = R2 + R3 - m2*s^2; \]
\[ Q(2,3) = -R3; \]
\[ Q(2,4) = -S2*n2 + R2*d1; \]
\[ Q(2,5) = S2*n3 - R2*d2 + S3*n4 - R3*d3; \]
\[ Q(2,6) = -S3*n5 + R3*d4; \]
\[ Q(3,2) = -R3; \]
\[ Q(3,3) = R3 - m3*s^2; \]
\[ Q(3,5) = -S3*n4 + R3*d3; \]
\[ Q(3,6) = S3*n5 - R3*d4; \]
\[ Q(4,1) = A1 + D1; \]
\[ Q(4,2) = -D1; \]
\[ Q(4,4) = B1*n1 + n2*E1 + d1*G1 + (m1 + m2 + m3)*g*(n1*(n1 - n0)/lt1 - d0) - I1x*s^2; \]
\[ Q(4,5) = -n3*E1 - d2*G1 - (m2 + m3)*g*(n3*(n3 - n2)/lt2 - d1); \]
\[ Q(5,1) = -A2; \]
\[ Q(5,2) = A2 + D2; \]
\[ Q(5,3) = -D2; \]
\[ Q(5,4) = -n2*B2 - d1*C2; \]
\[ Q(5,5) = n3*B2 + d2*C2 + n4*E2 + d3*G2 + (m2 + m3)*g*(n3*(n3 - n2)/lt2 - d2) - m3*g*(n4*(n5 - n4)/lt3 - d3) - I2x*s^2; \]
\[ Q(5,6) = -n5*E2 - d4*G2; \]
\[ Q(6,2) = -A3; \]
\[ Q(6,3) = A3; \]
\[ Q(6,5) = -B3*n4 - C3*d3; \]
\[ Q(6,6) = B3*n5 + C3*d4 + m3*g* (n5* (n5 - n4)/lt3 - d4)- I3x*s^2; \]
\[ Q_t = \text{inv}(Q); \]
\[ H_{yt} = Q_t(1,1); \]
\[ H_{yr} = Q_t(1,4); \]
\[ H_{ct} = Q_t(4,1); \]
\[ H_{cr} = Q_t(4,4); \]
\[ \text{M} = \begin{bmatrix} H_{xl} & H_{xp} & 0 & 0 & 0 & 0 \\ H_{al} & H_{ap} & 0 & 0 & 0 & 0 \\ 0 & 0 & H_{zv} & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{by} & 0 & 0 \\ 0 & 0 & 0 & 0 & H_{yt} & H_{yr} \\ 0 & 0 & 0 & 0 & H_{ct} & H_{cr} \end{bmatrix}; \]

**A.4 Transformation Matrix (matrixT1T2.m)**

The file `matrixT1T2.m` represents the conversion from the input signals to the six coils to the output forces and torques applied on the top mass.

\[ \text{T1} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ l_{s} & l_{s} & l_{p} & -l_{p} & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ l_{y} & -l_{y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & l_{r} & l_{r} & -l_{r} & l_{s} \end{bmatrix}; \]

\% transformation from 6 degrees of freedom to 6 magnet positions
\begin{verbatim}

% x theta z phi y psi
T2 = [ 1 ls 0 ly 0 0 % 1
     1 ls 0 -ly 0 0 % 2
     0 lp 1 0 0 lr % 3
     0 -lp 1 0 0 lr % 4
     0 0 1 0 0 -lr % 5
     0 0 0 1 ls ]; % 6

% **********************************************************************

A.5 Local Control Servo (LocalCtr.m)

The file LocalCtr.m gives the transfer function of the local control servo. The poles
and zeros are calculated using the matlab model written by Torrie [26].

% **********************************************************************

% the local control servo
num = (s+6.2832e+00)*(s+2.4669e-14)*(s+2.1991e+00);
den = ((s+5.6549e+01)*(s+4.3982e+00)) ^2;
kf = 2.0e5;
servo = kf * num/den;
LC = G * servo;

% **********************************************************************

\end{verbatim}
Figure B-1. Circuit schematic of the phase locking servo.
REFERENCES


BIOGRAPHICAL SKETCH

I was born in China on September 13th, 1979. I became a student when I was 5 years old and have kept this profession ever since. I enrolled at the University of Science and Technology of China in 1996 and got my Bachelor of Science in Materials Science and Engineering in 2001. Since I found that physics classes are more interesting than other classes that I took at college, I decided to go to graduate school to learn more. I became a graduate student at the University of Florida in the fall of 2001 and joined the LIGO group the year after. I got a chance to go to Glasgow University and worked on triple suspension system in the JIF lab. After a 7-month-long interesting and also troublesome experience in Scotland, I went back UF and continued to do some experiments testing the optical components which are designed for the Advanced LIGO Input Optics. I am supposed to end my career as a student at UF.